ON THE UNORDERED CONFIGURATION SPACE $C(RP^n, 2)$

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Abstract. We prove that, if n is a 2-power, the unordered configuration space $C(RP^n, 2)$ cannot be immersed in $\mathbb{R}^{4n-2}$ nor embedded as a closed subspace of $\mathbb{R}^{4n-1}$, optimal results, while if n is not a 2-power, $C(RP^n, 2)$ can be immersed in $\mathbb{R}^{4n-3}$. We also obtain cohomological lower bounds for the topological complexity of $C(RP^n, 2)$, which are nearly optimal when n is a 2-power. We also give a new description of the mod-2 cohomology algebra of the Grassmann manifold $G_{n+1,2}$.

1. Nonimmersions, nonembeddings, and immersions of $C(RP^n, 2)$

If M is an n-manifold, the unordered configuration space of two points in M, $C(M, 2) = (M\times M-\Delta)/\mathbb{Z}_2$, is a noncompact $2n$-manifold, and hence can be immersed in $\mathbb{R}^{4n-1}$ ([17]) and embedded as a closed subspace of $\mathbb{R}^{4n}$.([7]) We prove the following optimal nonimmersion and nonembedding theorem for $C(RP^n, 2)$ when n is a 2-power. Here RP^n denotes n-dimensional real projective space.

Theorem 1.1. *If n is a 2-power, $C(RP^n, 2)$ cannot be immersed in $\mathbb{R}^{4n-2}$ nor embedded as a closed subspace of $\mathbb{R}^{4n-1}$.*

This will be accomplished by showing that the Stiefel-Whitney class w_{2n-1} of its stable normal bundle is nonzero. The implication for embeddings of noncompact manifolds, which is not so well-known as that for immersions, is proved in [12, Cor 11.4].

For contrast, we prove the following immersion theorem.

Theorem 1.2. *If n is not a 2-power, then $C(RP^n, 2)$ can be immersed in $\mathbb{R}^{4n-3}$.*

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This work was motivated by a question of Mike Harrison. In [10], he defines a totally nonparallel immersion of a manifold in Euclidean space to be one in which tangent vectors at distinct points are never parallel. He proves that if a manifold M admits a totally nonparallel immersion in $\mathbb{R}^k$, then $C(M,2)$ immerses in $\mathbb{R}^k$. Thus we deduce that if n is a 2-power, then RP^n does not admit a totally nonparallel immersion in $\mathbb{R}^{4n-2}$.

Proof of Theorem 1.1. We denote $C_n = C(RP^n, 2)$, which we think of as the space of unordered pairs of distinct lines through the origin in $\mathbb{R}^{n+1}$. Also, W_n denotes the subspace consisting of unordered pairs of orthogonal lines through the origin in $\mathbb{R}^{n+1}$, and G_n the Grassmann manifold, usually denoted $G_{n+1,2}$, of 2-planes in $\mathbb{R}^{n+1}$. There is a deformation retraction $C_n \xrightarrow{p_1} W_n$ described in [6, p.324], which we will discuss thoroughly in our proof of Lemma 1.8, and also an obvious map $W_n \xrightarrow{p_2} G_n$, which is an RP^1-bundle.

We will work only with $\mathbb{Z}_2$-cohomology. In Section 2, we give a new description of the algebra $H^*(G_n)$. Here we describe just the part needed in this proof, which was first obtained by Feder in [6, Cor 4.1]. The algebra $H^*(G_n)$ is generated by classes $x = w_1$ and $y = w_2$ modulo two relations which cause the top two groups to be $H^{2n-2}(G_n) = \mathbb{Z}_2$ (resp. $H^{2n-3}(G_n) = \mathbb{Z}_2$) with $x^{2i}y^{n-1-i} \neq 0$ (resp. $x^{2i-1}y^{n-1-i} \neq 0$) iff $i = 2^t - 1$ for $t \geq 0$ (resp. $t \geq 1$) and $2^t \leq n$. By [6, Thm 4.3], p_2^* is injective and

$$H^*(W_n) \approx H^*(G_n)[u]/(u^2 = xu), \tag{1.3}$$

with $|u| = 1$. Also, $\mathrm{Sq}^1 y = xy$.

Let τ denote the tangent bundle, η a stable normal bundle, and w the total Stiefel-Whitney class of a bundle. In [15, (3)], it is shown that

$$w(\tau(G_n)) = (1+x)^{-2}(1+x+y)^{n+1}. \tag{1.4}$$

The map p_2 induces a surjective vector bundle homomorphism $\tau(W_n) \to \tau(G_n)$, and hence a surjective homomorphism

$$\widetilde{p_2} : \tau(W_n) \to p_2^*\tau(G_n)$$

of vector bundles over W_n. Then $\ker(\widetilde{p_2})$ is a line-bundle over W_n, and there is a vector bundle isomorphism

$$\ker(\widetilde{p_2}) \oplus p_2^*\tau(G_n) \approx \tau(W_n).$$

Thus

$$w(\tau(W_n)) = (1 + w_1(\ker(\widetilde{p_2})))(1 + x)^{-2}(1 + x + y)^{n+1}. \qquad (1.5)$$

By the Wu formula, $w_1(\tau(W_n))$ equals the element v_1 of $H^1(W_n)$ for which

$$\mathrm{Sq}^1 = \cdot v_1 : H^{2n-2}(W_n) \to H^{2n-1}(W_n).$$

Since, for $j > 0$, $\mathrm{Sq}^1(x^{2^{j+1}-2}y^{n-2^j}) = 0$ and

$\mathrm{Sq}^1(x^{2^{j+1}-3}y^{n-2^j}u) = x^{2^{j+1}-2}y^{n-2^j}u + nx^{2^{j+1}-2}y^{n-2^j}u + x^{2^{j+1}-3}y^{n-2^j}\cdot xu = nx^{2^{j+1}-2}y^{n-2^j}u,$

we deduce $w_1(\tau(W_n)) = nx$. From (1.5), we obtain

$$nx = w_1(\ker(\widetilde{p_2})) + (n + 1)x,$$

so $w_1(\ker(\widetilde{p_2})) = x$ and (1.5) becomes

$$w(\tau(W_n)) = (1 + x)^{-1}(1 + x + y)^{n+1},$$

and hence

$$w(\eta(W_n)) = (1 + x)(1 + x + y)^{-n-1}.$$

By Lemma 1.8, we obtain

$$w(\eta(C_n)) = (1 + x)(1 + x + y)^{-n-1}(1 + x + u)^{-1}.$$

Since $x^iu^j = u^{i+j}$ for $j > 0$, $(1+x+u)^{-1} = 1+\sum_{i\ge1}(x^i+u^i) = (1+x)^{-1}+u(1+u)^{-1}$ and

$$w(\eta(C_n)) = (1 + x + y)^{-n-1} + u(1 + u + y)^{-n-1}. \qquad (1.6)$$

By (1.3), $H^*(C_n) \approx H^*(W_n) \approx H^*(G_n) \oplus uH^*(G_n)$, and the nonzero term in (1.6) of maximal degree must occur in the u-part. Thus the relevant part of $w(\eta(C_n))$ is

$$\sum_{j,k} \tbinom{-n-1}{j}\tbinom{-n-1-j}{k} u^{k+1}y^j. \qquad (1.7)$$

The top dimension $H^{2n-1}(C_n) = \mathbb{Z}_2$ has as its only nonzero monomials $u^{2^t-1}y^{n-2^{t-1}}$ (all equal), and so

$$\begin{aligned} w_{2n-1}(\eta(C_n)) &= \sum_t \tbinom{-n-1}{n-2^{t-1}}\tbinom{-2n-1+2^{t-1}}{2^t-2} \\ &= \sum_t \tbinom{2n-2^{t-1}}{n-2^{t-1}}\tbinom{2n+2^{t-1}-2}{2^t-2}. \end{aligned}$$

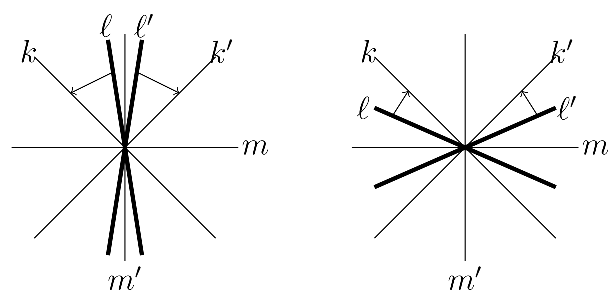
Using Lucas's Theorem, it is easy to see that $\binom{2n-2^{t-1}}{n-2^{t-1}}$ is odd iff n is a 2-power, and when n is a 2-power and $2^{t-1} \le n$, $\binom{2n+2^{t-1}-2}{2^t-2}$ is odd iff $t = 1$, proving the theorem.
■

The following lemma was used above.

Lemma 1.8. *With notation as above,* $w(\tau(C_n)) = (1 + x + u)w(\tau(W_n))$.

Proof. The map $p_1 : C_n \to W_n$ is defined as follows. For distinct lines ℓ and ℓ', working in their plane, let m and m' be the pair of orthogonal lines bisecting the two angles between ℓ and ℓ', and then let k and k' be 45° rotations of m and m'. Then $p_1(\{\ell, \ell'\}) = \{k, k'\}$, and the homotopy from the identity map of C_n to $i \circ p_1$ moves ℓ and ℓ' uniformly toward the closer of k and k'. Here i is the inclusion of W_n in C_n. Two scenarios for this are illustrated in Figure 1.9.

Figure 1.9. The map $C_n \to W_n$



Let Z_n be the space of ordered pairs of orthogonal lines in $\mathbb{R}^{n+1}$, and Z_n^+ the space of ordered pairs of orthogonal lines in $\mathbb{R}^{n+1}$ together with an orientation on the plane which they span. Let $Z_n^+ \xrightarrow{p} W_n$ forget the order and the orientation. This p is a 4-sheeted covering space. Suppose p has a section s_α on an open set U_α of W_n. If $p_1(\{\ell, \ell'\}) = \{k, k'\} \in U_\alpha$, then s_α specifies an order (k_1, k_2) on $\{k, k'\}$ and an orientation on the plane containing these vectors. A local trivialization of p_1 is defined by maps $h_\alpha : p_1^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}$ with $h_\alpha(\{\ell, \ell'\}) = (p_1(\{\ell, \ell'\}), \tan(2\theta))$, where $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$ is the angle, with respect to the orientation, through which ℓ or ℓ' was rotated to end at k_1. Thus p_1 is a line bundle θ over W_n.

Reversing the order of (k_1, k_2) in s_α negates h_α, as does reversing the orientation selected by s_α. Thus our line bundle θ is $L_R \otimes L_O$, where L_R is the line bundle (named for Reversing) over W_n associated to the double cover $Z_n \to W_n$, and L_O is the line bundle (named for Orientation) over W_n associated to the pullback over W_n of the double cover $G_n^+ \to G_n$ from the oriented Grassmannian to the unoriented one. Thus $w_1(\theta) = w_1(L_R) + w_1(L_O)$.

Clearly $w_1(L_O)$ equals p_2^* of the universal w_1 of the Grassmannian, and this is our class x. That $w_1(L_R) = u$ is proved in [9, Lemma 3.3 and Prop 3.5]. Our map $Z_n \to W_n$ is Handel's map $Z_{n+1,2} \to SZ_{n+1,2}$. Thus $w_1(\theta) = u + x$, establishing the lemma, since $w(\tau(C_n)) = p_1^*(w(\tau(W_n))) \cdot p_1^*(w(\theta))$. ■

The proof of Theorem 1.1 showed that $w_{2n-1}(\eta(C_n))$ is nonzero iff n is a 2-power. We believe that Theorem 1.1 gives all nonimmersion and nonembedding results for spaces $C(RP^n, 2)$ implied by Stiefel-Whitney classes of the normal bundle. Using our description of $H^*(G_n)$ in Section 2 and its implications for $H^*(C_n)$ along with (1.7), we have performed an extensive computer search for other results. Those which we found said that if $n = 2^r + 1$ (resp. $2^r + 2$ or $2^r + 4$), then $w_{2n-5}(\eta(C_n)) \neq 0$ (resp. $w_{2n-9}(\eta(C_n)) \neq 0$ or $w_{2n-17}(\eta(C_n)) \neq 0$), but the nonimmersion and nonembedding results for $C(RP^n, 2)$ implied by these are in the same dimension as the result for $C(RP^{2^r}, 2)$, and so are implied by Theorem 1.1.

Now we prove the existence of immersions in $\mathbb{R}^{4n-3}$ when n is not a 2-power. We continue to denote $C(RP^n, 2)$ as C_n.

Proof of Theorem 1.2. We use obstruction theory to show that the map $C_n \to BO$ which classifies the stable normal bundle $\eta(C_n)$ factors through $BO(2n-3)$, which implies the immersion by the well-known theorem of Hirsch.([11]) The theory of modified Postnikov towers developed in [8] applies to the fibration $V_k \to BO(k) \to BO$ when k is odd by [14]. The fiber V_k is a union of Stiefel manifolds, and in our case, all we need is

$$\pi_i(V_{2n-3}) = \begin{cases} 0 & i < 2n-3 \\ \mathbb{Z}_2 & i = 2n-3 \\ 0 & i = 2n-2,\ n \text{ odd} \\ \mathbb{Z}_2 & i = 2n-2,\ n \text{ even.} \end{cases}$$

Since $H^{2n}(C_n) = 0$, the only possible obstructions are in $H^{2n-2}(C_n; \pi_{2n-3}(V_{2n-3}))$ and $H^{2n-1}(C_n; \pi_{2n-2}(V_{2n-3}))$. The first obstruction is $w_{2n-2}(\eta(C_n))$, which is 0 when n is not a 2-power by a calculation very similar to that in our proof of Theorem 1.1. This already implies the immersion when n is odd. When n is even, we argue similarly to [13, Thm 2.3]. The second and final obstruction has indeterminacy

$$H^{2n-3}(C_n) \xrightarrow{\mathrm{Sq}^2 + w_2} H^{2n-1}(C_n).$$

This follows, similarly to the proof in [13, Thm 2.3], from the relation $(\mathrm{Sq}^2 + w_2)w_{2n-2} = 0$ in $H^*(BO)$. By (1.6), we have, for n even, $w_2(\eta(C_n)) = y + u^2 + \binom{n+2}{2}x^2$. The nonzero element in $H^{2n-1}(C_n)$ is $x^{2^t-2}y^{n-2^{t-1}}u$ for an appropriate t. In $H^{2n-3}(C_n)$ there is a class $x^{2^t-3}y^{n-2^{t-1}}$ on which Sq^2 is 0, multiplication by y and x^2 are 0, but multiplication by u^2 is nonzero. Therefore the final obstruction can be canceled if it is nonzero. ■

2. Cohomology of $G_{n+1,2}$

Descriptions of the cohomology ring (mod 2) of the Grassmann manifold $G_{n+1,2}$ of 2-planes in $\mathbb{R}^{n+1}$ were given initially by Chern ([3]) and Borel ([2]). Here we present what we think is a new description that has been useful in our analysis. It is based on the description given by Feder in [6]. As in the proof of Theorem 1.1, we denote $G_{n+1,2}$ by G_n. In our proof of Theorem 1.1, we used [6, Cor 4.1] which stated that, with $x = w_1$ and $y = w_2$ the generators, in the top dimension, $H^{2n-2}(G_n) = \mathbb{Z}_2$, the nonzero monomials are those $x^{2i}y^{n-1-i}$ for which $i + 1$ is a 2-power. Working backwards from this, we can prove the following result.

Theorem 2.1. *In the ring $H^*(G_n)$, monomials x^iy^j are independent if $i + 2j < n$. For $\varepsilon \in \{0, 1\}$, if $2n - 2k - \varepsilon \ge n$, then $H^{2n-2k-\varepsilon}(G_n)$ has basis $\beta_1, \ldots, \beta_k$, and $x^{2i-\varepsilon}y^{n-k-i}$ equals the sum of those β_j for which $i + j$ is a 2-power.*

Proof. That the first relation occurs in grading n is well-known (e.g., [6, Prop 4.1]). The case $k = 1$, $\varepsilon = 0$ is the result of [6, Cor 4.1] cited above. Multiplication by x is an isomorphism $H^{2n-3}(G_n) \to H^{2n-2}(G_n)$ of groups of order 2, implying the result when $k = 1$ and $\varepsilon = 1$. We will prove the result by induction on k when $\varepsilon = 0$. The induction when $\varepsilon = 1$ is identical.

Let $V_k = H^{2n-2k}(G_n)$, a vector space of dimension k by Poincaré duality. Assume the result for k. Define

$$\phi = (\cdot y, \cdot x^2) : V_{k+1} \to V_k \times V_k.$$

In $V_k \times V_k$, let

$$\gamma_1 = (\beta_1, 0),\ \gamma_2 = (\beta_2, \beta_1), \ldots, \gamma_k = (\beta_k, \beta_{k-1}),\ \gamma_{k+1} = (0, \beta_k).$$

By the induction hypothesis,

$$\phi(x^{2i} y^{n-k-i-1}) = \sum_{i+j \in P} \gamma_j,$$

where $P = \{1, 2, 4, \ldots\}$ denotes the set of 2-powers.

Let W be the subspace of $V_k \times V_k$ spanned by the linearly independent elements $\gamma_1, \ldots, \gamma_{k+1}$. We will show that ϕ maps onto W. Then since $\dim(V_{k+1}) = \dim(W)$, ϕ is injective. Let $\beta_j = \phi^{-1}(\gamma_j)$. Then $\{\beta_1, \ldots, \beta_{k+1}\}$ is a basis for V_{k+1}, and

$$x^{2i} y^{n-k-i-1} = \sum_{i+j \in P} \beta_j,$$

extending the induction and completing the proof, once we establish the surjectivity of ϕ onto W.

Let $n = 2m + \delta$ with $\delta \in \{0, 1\}$. We first consider the case $k + 1 = m$. Letting $b_i = x^{2i} y^{m+\delta-i} \in V_{k+1}$ for $1 \le i \le m$ (ignoring 1 or 2 monomials not required for the surjectivity), the matrix of ϕ with respect to the bases $\{b_1, \ldots, b_m\}$ and $\{\gamma_1, \ldots, \gamma_m\}$ is that of Lemma 2.2, and so ϕ is surjective. The cases of smaller values of k have larger domain and smaller codomain, with ϕ being an extension of a quotient of the case $k + 1 = m$, and hence is surjective since the case $k + 1 = m$ was. ■

Lemma 2.2. *Let A_m denote the m-by-m matrix over $\mathbb{Z}_2$ with*

$$a_{i,j} = \begin{cases} 1 & \text{if } i + j \text{ is a 2-power} \\ 0 & \text{if not.} \end{cases}$$

Then $\det(A_m) = 1$.

Proof. The proof is by induction on m. Let $m = 2^e + \Delta$ with $0 \le \Delta < 2^e$. For $0 \le i \le \Delta$, row $2^e + i$ contains a single 1, in column $2^e - i$. Subtract this row from other rows which have a 1 in column $2^e - i$. Then do a similar thing with columns

2^e+j, $0\le j\le\Delta$. The result has $A_{2^e-\Delta-1}$ in the top left, and a $(2\Delta+1)$-by-$(2\Delta+1)$ matrix with 1's along the antidiagonal in the bottom right. All other elements are 0. By the induction hypothesis, this matrix has determinant 1. ■

In moderately large gradings, there is, for each j, a monomial x^iy^ℓ equal to β_j. For example, in $H^{24}(G_{20})$, the following monomials equal $\beta_1,\ldots,\beta_8$, respectively:

$$x^{14}y^5,\ x^{12}y^6,\ x^{10}y^7,\ x^{24},\ x^{22}y,\ x^{20}y^2,\ x^{18}y^3,\ x^{16}y^4,$$

and a similar pattern holds in $H^i(G_{20})$ for $23\le i\le 38$. However, in $H^{22}(G_{20})$, $x^{14}y^4=\beta_1+\beta_9$, and there is no monomial which equals either β_1 or β_9. We can obtain β_1 as $x^{22}+x^6y^8$, since $x^{22}=\beta_5$ and $x^6y^8=\beta_1+\beta_5$.

3. Topological complexity of $C(RP^n,2)$

The topological complexity $\mathrm{TC}(X)$ of a topological space X is a homotopy invariant introduced by Farber in [4] which is one less than the number of nice subsets U_i into which $X\times X$ can be partitioned such that there is a continuous map $s_i:U_i\to X^I$ such that $s_i(x_0,x_1)$ is a path from x_0 to x_1. This is of interest ([5]) for ordered (resp. unordered) configuration spaces $F(X,n)$ (resp. $C(X,n)$) as it measures how efficiently n distinguishable (resp. indistinguishable) robots can be moved from one set of points in X to another. The determination of $\mathrm{TC}(C(X,n))$ has been particularly difficult.([16],[1])

Farber showed ([4]) that $\mathrm{zcl}(X)\le\mathrm{TC}(X)\le 2\dim(X)$ if X is a CW complex. Here $\mathrm{zcl}(X)$, the zero-divisor-cup-length, is the largest number of elements of $\ker(\Delta^*:\widetilde{H}^*(X\times X)\to\widetilde{H}^*(X))$ with nonzero product, where Δ is the diagonal map. The main theorem of this section determines $\mathrm{zcl}(C(RP^n,2))$.

Theorem 3.1. *If* $0\le d<2^e$ *and* $r=\max\{s\in\mathbb{Z}:2^s\le d+\frac12\}$, *then*

$$\mathrm{zcl}(C(RP^{2^e+d},2))=2^{e+2}+2^{r+1}-4$$

and $\mathrm{TC}(C(RP^{2^e+d},2))\ge 2^{e+2}+2^{r+1}-4$.

Since $C(RP^n,2)$ has the homotopy type of the compact $(2n-1)$-manifold W_n described in the proof of Theorem 1.1, $\mathrm{TC}(C(RP^{2^e+d},2))\le 2^{e+2}+4d-2$. For $d=0,1,2,3,4$, the gap between our upper and lower bounds for $\mathrm{TC}(C(RP^{2^e+d},2))$ is $1,4,6,10,10$, respectively.

Proof. Let $n = 2^e + d$ and let C_n, W_n, and G_n be as in the proof of Theorem 1.1. We identify $H^*(C_n)$ with $H^*(W_n)$ and note that the impact of (1.3) is that $x^i u^j = x^{i+j-1}u$ if $j > 0$.

Let $\overline{x} = x \otimes 1 + 1 \otimes x$, and define $\overline{y}$ and $\overline{u}$ similarly. We claim that $\text{zcl}(C_n) \geq 2^{e+2} + 2^{r+1} - 4$ since

$$\overline{x}^{2^{e+1}-1}\,\overline{u}^{2^{e+1}-2}\,\overline{y}^{2^{r+1}-1} \neq 0. \tag{3.2}$$

To see this, we first note that the indicated product is, in bigrading $(2^{e+1} + 2d - 1, 2^{e+1} + 2^{r+2} - 2d - 4)$, equal to

$$\sum_{k,j} x^{2k-1}u^{2^{e+1}+2(d-j-k)}y^j \otimes x^{2^{e+1}-2k}u^{2(j+k-d-1)}y^{2^{r+1}-1-j}.$$

Since the terms divisible by u are independent from those not divisible by u, we restrict to terms whose right factor is not divisible by u, and obtain

$$\sum_{j} x^{2^{e+1}+2(d-j)-2}uy^j \otimes x^{2^{e+1}-2(d-j+1)}y^{2^{r+1}-1-j}. \tag{3.3}$$

Terms with $j < d$ (resp. $j > d$) have left (resp. right) factor equal to 0 since $x^{2^{e+1}} = 0$. Thus (3.3) equals $x^{2^{e+1}-2}uy^d \otimes x^{2^{e+1}-2}y^{2^{r+1}-1-d}$, which is nonzero by (1.3) and Theorem 2.1.

To see that this bound for zcl cannot be improved, first note that the exponents of $\overline{x}$ and $\overline{u}$ in (3.2) cannot be increased since $x^{2^{e+1}-1} = 0$ by [6, Cor 4.2]. If the exponent of $\overline{u}$ is increased by 1, the top term $x^{2^{e+1}-2}u \otimes x^{2^{e+1}-2}u$ occurs with even coefficient by symmetry. The only hope of getting a larger nonzero product would be to increase the exponent of $\overline{y}$. We will use our analysis of $H^*(C_n)$ to see that this will fail to improve the zcl.

The key observation is that, with $n = 2^e + d$ and $\delta \in \{0, 1\}$, a nonzero monomial $x^s u^\delta y^t$ in $H^*(C_n)$ with $t > d$ must have $s \leq 2^e - 2$. This will follow from Theorem 2.1 once we show that if $x^s y^t = x^{2i-\varepsilon}y^{n-k-i}$ has $s \geq 2^e - 1$ and $t \geq d+1$, and $2 \leq 2j \leq 2k$, then $2i + 2j$ is not a 2-power. We have $2i + 2j \geq 2^e - 1 + \varepsilon + 2 > 2^e$. On the other hand, $2i + 2j \leq (2n - 2k - 2d - 2) + 2k = 2^{e+1} - 2$, implying the claim.

If $x^{i_1}u^{\varepsilon_1}y^{j_1} \otimes x^{i_2}u^{\varepsilon_2}y^{j_2}$ appears in the expansion of $\overline{x}^a\,\overline{u}^b\,\overline{y}^c$ with maximal exponent sum, it should have $i_1 = 2^{e+1} - 2$, $\varepsilon_1 = 1$, and $j_1 = d$, as we do not want to sacrifice 2^e x-exponents on both sides of the $\otimes$. To have a monomial $x^{2^{e+1}-2}uy^d \otimes x^{i_2}u^{\varepsilon_2}y^{j_2}$ whose exponent sum exceeds our zcl bound would require $i_2+j_2+\varepsilon_2>2^{e+1}-3+2^{r+1}-d$. If $j_2>d$, then $i_2\le 2^e-2$, so we would need $j_2+\varepsilon_2\ge 2^e+2^{r+1}-d$ with strict inequality unless $i_2=2^e-2$. We also have $j_2\le 2^e+d-1$, half the dimension of W_n. We would also need $\binom{d+j_2}{d}\equiv 1$ mod 2. But this is impossible by Lemma 3.4 applied to $j=j_2-2^e$ unless $i_2=2^e-2$ and $j_2=2^e+2^{r+1}-d-1$. But then $|x^{2^{e+1}-2}uy^d\otimes x^{i_2}u^{\varepsilon_2}y^{j_2}|>2\dim(W_n)$. The alternative is $j_2\le d$. But, since we need $\binom{d+j_2}{d}\equiv 1$ mod 2, the largest such j_2 was what was used in obtaining our lower bound. ■

Lemma 3.4. *If* $2^r\le d<2^{r+1}$ *and* $2^{r+1}-d-1<j\le d-1$, *then* $\binom{d+j}{d}\equiv 0\ (2)$.

Proof. For $\binom{d+j}{d}$ to be odd, the binary expansions of j and d must be disjoint. Since $j\le 2^{r+1}-1$, the 1's in the binary expansion of j would have to be a subset of those of $2^{r+1}-1-d$, contradicting $j>2^{r+1}-d-1$. ■

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