ON THE UNORDERED CONFIGURATION SPACE $C(RP^n, 2)$

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Abstract. We prove that, if $n$ is a 2-power, the unordered configuration space $C(RP^n, 2)$ cannot be immersed in $\mathbb{R}^{4n-2}$ nor embedded as a closed subspace of $\mathbb{R}^{4n-1}$, optimal results, while if $n$ is not a 2-power, $C(RP^n, 2)$ can be immersed in $\mathbb{R}^{4n-3}$. We also obtain cohomological lower bounds for the topological complexity of $C(RP^n, 2)$, which are nearly optimal when $n$ is a 2-power. We also give a new description of the mod-2 cohomology algebra of the Grassmann manifold $G_{n+1, 2}$.

1. Nonimmersions, nonembeddings, and immersions of $C(RP^n, 2)$

If $M$ is an $n$-manifold, the unordered configuration space of two points in $M$, $C(M, 2) = (M \times M - \Delta)/\mathbb{Z}_2$, is a noncompact $2n$-manifold, and hence can be immersed in $\mathbb{R}^{4n-1}$ ([17]) and embedded as a closed subspace of $\mathbb{R}^{4n}$.([7]) We prove the following optimal nonimmersion and nonembedding theorem for $C(RP^n, 2)$ when $n$ is a 2-power. Here $RP^n$ denotes $n$-dimensional real projective space.

Theorem 1.1. If $n$ is a 2-power, $C(RP^n, 2)$ cannot be immersed in $\mathbb{R}^{4n-2}$ nor embedded as a closed subspace of $\mathbb{R}^{4n-1}$.

This will be accomplished by showing that the Stiefel-Whitney class $w_{2n-1}$ of its stable normal bundle is nonzero. The implication for embeddings of noncompact manifolds, which is not so well-known as that for immersions, is proved in [12, Cor 11.4].

For contrast, we prove the following immersion theorem.

Theorem 1.2. If $n$ is not a 2-power, then $C(RP^n, 2)$ can be immersed in $\mathbb{R}^{4n-3}$.

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This work was motivated by a question of Mike Harrison. In [10], he defines a totally nonparallel immersion of a manifold in Euclidean space to be one in which tangent vectors at distinct points are never parallel. He proves that if a manifold $M$ admits a totally nonparallel immersion in $\mathbb{R}^k$, then $C(M, 2)$ immerses in $\mathbb{R}^k$. Thus we deduce that if $n$ is a 2-power, then $RP^n$ does not admit a totally nonparallel immersion in $\mathbb{R}^{4n-2}$.

Proof of Theorem 1.1. We denote $C_n = C(RP^n, 2)$, which we think of as the space of unordered pairs of distinct lines through the origin in $\mathbb{R}^{n+1}$. Also, $W_n$ denotes the subspace consisting of unordered pairs of orthogonal lines through the origin in $\mathbb{R}^{n+1}$, and $G_n$ the Grassmann manifold, usually denoted $G_{n+1, 2}$, of 2-planes in $\mathbb{R}^{n+1}$. There is a deformation retraction $C_n \xrightarrow{p_1} W_n$ described in [6, p.324], which we will discuss thoroughly in our proof of Lemma 1.8, and also an obvious map $W_n \xrightarrow{p_2} G_n$, which is an $RP^1$-bundle.

We will work only with $\mathbb{Z}_2$-cohomology. In Section 2, we give a new description of the algebra $H^*(G_n)$. Here we describe just the part needed in this proof, which was first obtained by Feder in [6, Cor 4.1]. The algebra $H^*(G_n)$ is generated by classes $x = w_1$ and $y = w_2$ modulo two relations which cause the top two groups to be $H^{2n-2}(G_n) = \mathbb{Z}_2$ (resp. $H^{2n-3}(G_n) = \mathbb{Z}_2$) with $x^{2t}y^{n-1-i} \neq 0$ (resp. $x^{2t-1}y^{n-1-i} \neq 0$) iff $i = 2^t - 1$ for $t \geq 0$ (resp. $t \geq 1$) and $2^t \leq n$. By [6, Thm 4.3], $p_2^*$ is injective and

$$H^*(W_n) \approx H^*(G_n)[u]/(u^2 = xu),$$

with $|u| = 1$. Also, $Sq^1 y = xy$.

Let $\tau$ denote the tangent bundle, $\eta$ a stable normal bundle, and $w$ the total Stiefel-Whitney class of a bundle. In [15, (3)], it is shown that

$$w(\tau(G_n)) = (1 + x)^{-2}(1 + x + y)^{n+1}. \quad (1.4)$$

The map $p_2$ induces a surjective vector bundle homomorphism $\tau(W_n) \to \tau(G_n)$, and hence a surjective homomorphism

$$\tilde{p}_2 : \tau(W_n) \to p_2^*\tau(G_n)$$

of vector bundles over $W_n$. Then $\ker(\tilde{p}_2)$ is a line-bundle over $W_n$, and there is a vector bundle isomorphism

$$\ker(\tilde{p}_2) \oplus p_2^*\tau(G_n) \approx \tau(W_n).$$
Thus
\[ w(\tau(W_n)) = (1 + w_1(\ker(\bar{p}_2)))(1 + x)^{-2}(1 + x + y)^{n+1}. \] (1.5)

By the Wu formula, \( w_1(\tau(W_n)) \) equals the element \( v_1 \) of \( H^1(W_n) \) for which
\[ \text{Sq}^1 = \cdot v_1 : H^{2n-2}(W_n) \to H^{2n-1}(W_n). \]

Since, for \( j > 0, \text{Sq}^1(x^{2j+1} - 2y^{n-2j}) = 0 \) and
\[ \text{Sq}^1(x^{2j+1} - 3y^{n-2j}) = x^{2j+1} - 2y^{n-2j} u + nx^{2j+1} - 2y^{n-2j} u + x^{2j+1} - 3y^{n-2j} \cdot xu = nx^{2j+1} - 2y^{n-2j} u, \]
we deduce \( w_1(\tau(W_n)) = nx \). From (1.5), we obtain
\[ nx = w_1(\ker(\bar{p}_2)) + (n + 1)x, \]
so \( w_1(\ker(\bar{p}_2)) = x \) and (1.5) becomes
\[ w(\tau(W_n)) = (1 + x)^{-1}(1 + x + y)^{n+1}, \]
and hence
\[ w(\eta(W_n)) = (1 + x)(1 + x + y)^{-n-1}. \]

By Lemma 1.8, we obtain
\[ w(\eta(C_n)) = (1 + x)(1 + x + y)^{-n-1}(1 + x + u)^{-1}. \]

Since \( x^iu^j = u^{i+j} \) for \( j > 0, (1 + x + u)^{-1} = 1 + \sum_{i \geq 1}(x^i + u^i) = (1 + x)^{-1} + u(1 + u)^{-1} \) and
\[ w(\eta(C_n)) = (1 + x + y)^{-n-1} + u(1 + u + y)^{-n-1}. \] (1.6)

By (1.3), \( H^*(C_n) \approx H^*(W_n) \approx H^*(G_n) \oplus uH^*(G_n) \), and the nonzero term in (1.6) of maximal degree must occur in the \( u \)-part. Thus the relevant part of \( w(\eta(C_n)) \) is
\[ \sum_{j,k} \binom{-n-1}{j} \binom{-n-1-j}{k} u^{k+1} y^j. \] (1.7)

The top dimension \( H^{2n-1}(C_n) = \mathbb{Z}_2 \) has as its only nonzero monomials \( u^{2t-1}y^{n-2t-1} \) (all equal), and so
\[
\begin{align*}
  w_{2n-1}(\eta(C_n)) &= \sum_t \binom{-n-1}{n-2t-1} \binom{n-2t-1}{2t-2} \\
  &= \sum_t \binom{2n-2t-1}{n-2t-1} \binom{2n+2t-1-2}{2t-2}.
\end{align*}
\]
Using Lucas’s Theorem, it is easy to see that \( \binom{2n-2^t-1}{n-2^t-1} \) is odd iff \( n \) is a 2-power, and when \( n \) is a 2-power and \( 2^{t-1} \leq n \), \( \binom{2n+2^{t-1}-2}{2^t-2} \) is odd iff \( t = 1 \), proving the theorem.

\[ \square \]

The following lemma was used above.

**Lemma 1.8.** With notation as above, \( w(\tau(C_n)) = (1 + x + u)w(\tau(W_n)) \).

**Proof.** The map \( p_1 : C_n \to W_n \) is defined as follows. For distinct lines \( \ell \) and \( \ell' \), working in their plane, let \( m \) and \( m' \) be the pair of orthogonal lines bisecting the two angles between \( \ell \) and \( \ell' \), and then let \( k \) and \( k' \) be \( 45^\circ \) rotations of \( m \) and \( m' \). Then \( p_1(\{\ell, \ell'\}) = \{k, k'\} \), and the homotopy from the identity map of \( C_n \) to \( i \circ p_1 \) moves \( \ell \) and \( \ell' \) uniformly toward the closer of \( k \) and \( k' \). Here \( i \) is the inclusion of \( W_n \) in \( C_n \).

Two scenarios for this are illustrated in Figure 1.9.

**Figure 1.9.** The map \( C_n \to W_n \)

Let \( Z_n \) be the space of ordered pairs of orthogonal lines in \( \mathbb{R}^{n+1} \), and \( Z_n^+ \) the space of ordered pairs of orthogonal lines in \( \mathbb{R}^{n+1} \) together with an orientation on the plane which they span. Let \( Z_n^+ \xrightarrow{p} W_n \) forget the order and the orientation. This \( p \) is a 4-sheeted covering space. Suppose \( p \) has a section \( s_\alpha \) on an open set \( U_\alpha \) of \( W_n \). If \( p_1(\{\ell, \ell'\}) = \{k, k'\} \in U_\alpha \), then \( s_\alpha \) specifies an order \( (k_1, k_2) \) on \( \{k, k'\} \) and an orientation on the plane containing these vectors. A local trivialization of \( p_1 \) is defined by maps \( h_\alpha : p_1^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R} \) with \( h_\alpha(\{\ell, \ell'\}) = (p_1(\{\ell, \ell'\}), \tan(2\theta)) \), where \( \theta \in (-\frac{\pi}{4}, \frac{\pi}{4}) \) is the angle, with respect to the orientation, through which \( \ell \) or \( \ell' \) was rotated to end at \( k_1 \). Thus \( p_1 \) is a line bundle \( \theta \) over \( W_n \).

Reversing the order of \( (k_1, k_2) \) in \( s_\alpha \) negates \( h_\alpha \), as does reversing the orientation selected by \( s_\alpha \). Thus our line bundle \( \theta \) is \( L_R \otimes L_O \), where \( L_R \) is the line bundle (named
for Reversing) over $W_n$ associated to the double cover $Z_n \to W_n$, and $L_O$ is the line bundle (named for Orientation) over $W_n$ associated to the pullback over $W_n$ of the double cover $G_n^+ \to G_n$ from the oriented Grassmannian to the unoriented one. Thus

$$w_1(\theta) = w_1(L_R) + w_1(L_O).$$

Clearly $w_1(L_O)$ equals $p_2^*\tau$ of the universal $w_1$ of the Grassmannian, and this is our class $x$. That $w_1(L_R) = u$ is proved in [9, Lemma 3.3 and Prop 3.5]. Our map $Z_n \to W_n$ is Handel’s map $Z_n+1, 2 \to SZ_{n+1,2}$. Thus $w_1(\theta) = u + x$, establishing the lemma, since $w(\tau(C_n)) = p_1^*(w(\tau(W_n))) \cdot p_1^*(w(\theta))$. 

The proof of Theorem 1.1 showed that $w_{2n-1}(\eta(C_n))$ is nonzero iff $n$ is a 2-power. We believe that Theorem 1.1 gives all nonimmersion and nonembedding results for spaces $C(RP^n, 2)$ implied by Stiefel-Whitney classes of the normal bundle. Using our description of $H^*(G_n)$ in Section 2 and its implications for $H^*(C_n)$ along with (1.7), we have performed an extensive computer search for other results. Those which we found said that if $n = 2^r + 1$ (resp. $2^r + 2$ or $2^r + 4$), then $w_{2n-5}(\eta(C_n)) \neq 0$ (resp. $w_{2n-9}(\eta(C_n)) \neq 0$ or $w_{2n-17}(\eta(C_n)) \neq 0$), but the nonimmersion and nonembedding results for $C(RP^n, 2)$ implied by these are in the same dimension as the result for $C(RP^{2^r}, 2)$, and so are implied by Theorem 1.1.

Now we prove the existence of immersions in $\mathbb{R}^{4n-3}$ when $n$ is not a 2-power. We continue to denote $C(RP^n, 2)$ as $C_n$.

Proof of Theorem 1.2. We use obstruction theory to show that the map $C_n \to BO$ which classifies the stable normal bundle $\eta(C_n)$ factors through $BO(2n-3)$, which implies the immersion by the well-known theorem of Hirsch.[11] The theory of modified Postnikov towers developed in [8] applies to the fibration $V_k \to BO(k) \to BO$ when $k$ is odd by [14]. The fiber $V_k$ is a union of Stiefel manifolds, and in our case, all we need is

$$\pi_i(V_{2n-3}) = \begin{cases} 0 & i < 2n-3 \\ \mathbb{Z}_2 & i = 2n-3 \\ 0 & i = 2n-2, \text{ n odd} \\ \mathbb{Z}_2 & i = 2n-2, \text{ n even}. \end{cases}$$

Since $H^{2n}(C_n) = 0$, the only possible obstructions are in $H^{2n-2}(C_n; \pi_{2n-3}(V_{2n-3}))$ and $H^{2n-1}(C_n; \pi_{2n-2}(V_{2n-3}))$. The first obstruction is $w_{2n-2}(\eta(C_n))$, which is 0 when $n$ is
not a 2-power by a calculation very similar to that in our proof of Theorem 1.1. This already implies the immersion when \( n \) is odd. When \( n \) is even, we argue similarly to [13, Thm 2.3]. The second and final obstruction has indeterminacy

\[
H^{2n-3}(C_n) \xrightarrow{\text{Sq}^2 + w_2} H^{2n-1}(C_n).
\]

This follows, similarly to the proof in [13, Thm 2.3], from the relation \((\text{Sq}^2 + w_2)w_{2n-2} = 0\) in \( H^*(BO) \). By (1.6), we have, for \( n \) even, \( w_2(\eta(C_n)) = y + u^2 + \left(\frac{n+2}{2}\right)x^2 \). The nonzero element in \( H^{2n-1}(C_n) \) is \( x^2 - 2y^2u \) for an appropriate \( t \). In \( H^{2n-3}(C_n) \) there is a class \( x^2 - 2y^2u \) on which \( \text{Sq}^1 \) is 0, multiplication by \( y \) and \( x^2 \) are 0, but multiplication by \( u^2 \) is nonzero. Therefore the final obstruction can be canceled if it is nonzero. ■

2. Cohomology of \( G_{n+1,2} \)

Descriptions of the cohomology ring (mod 2) of the Grassmann manifold \( G_{n+1,2} \) of 2-planes in \( \mathbb{R}^{n+1} \) were given initially by Chern ([3]) and Borel ([2]). Here we present what we think is a new description that has been useful in our analysis. It is based on the description given by Feder in [6]. As in the proof of Theorem 1.1, we denote \( G_{n+1,2} \) by \( G_n \). In our proof of Theorem 1.1, we used [6, Cor 4.1] which stated that, with \( x = w_1 \) and \( y = w_2 \) the generators, in the top dimension, \( H^{2n-2}(G_n) = \mathbb{Z}_2 \), the nonzero monomials are those \( x^i y^{n-i} \) for which \( i + 1 \) is a 2-power. Working backwards from this, we can prove the following result.

**Theorem 2.1.** In the ring \( H^*(G_n) \), monomials \( x^i y^j \) are independent if \( i + 2j < n \).

For \( \varepsilon \in \{0,1\} \), if \( 2n - 2k - \varepsilon \geq n \), then \( H^{2n-2k-\varepsilon}(G_n) \) has basis \( \beta_1, \ldots, \beta_k \), and \( x^{2i-\varepsilon} y^{n-2k-1} \) equals the sum of those \( \beta_j \) for which \( i + j \) is a 2-power.

**Proof.** That the first relation occurs in grading \( n \) is well-known (e.g., [6, Prop 4.1]). The case \( k = 1, \varepsilon = 0 \) is the result of [6, Cor 4.1] cited above. Multiplication by \( x \) is an isomorphism \( H^{2n-3}(G_n) \to H^{2n-2}(G_n) \) of groups of order 2, implying the result when \( k = 1 \) and \( \varepsilon = 1 \). We will prove the result by induction on \( k \) when \( \varepsilon = 0 \). The induction when \( \varepsilon = 1 \) is identical.
Let $V_k = H^{2n-2k}(G_n)$, a vector space of dimension $k$ by Poincaré duality. Assume the result for $k$. Define
\[ \phi = (\cdot y, \cdot x^2) : V_{k+1} \to V_k \times V_k. \]
In $V_k \times V_k$, let
\[ \gamma_1 = (\beta_1, 0), \gamma_2 = (\beta_2, \beta_1), \ldots, \gamma_k = (\beta_k, \beta_{k-1}), \gamma_{k+1} = (0, \beta_k). \]
By the induction hypothesis,
\[ \phi(x^{2i} y^{n-k-i-1}) = \sum_{i+j \in P} \gamma_j, \]
where $P = \{1, 2, 4, \ldots\}$ denotes the set of 2-powers.

Let $W$ be the subspace of $V_k \times V_k$ spanned by the linearly independent elements $\gamma_1, \ldots, \gamma_{k+1}$. We will show that $\phi$ maps onto $W$. Then since $\dim(V_{k+1}) = \dim(W)$, $\phi$ is injective. Let $\beta_j = \phi^{-1}(\gamma_j)$. Then $\{\beta_1, \ldots, \beta_{k+1}\}$ is a basis for $V_{k+1}$, and
\[ x^{2i} y^{n-k-i-1} = \sum_{i+j \in P} \beta_j, \]
extending the induction and completing the proof, once we establish the surjectivity of $\phi$ onto $W$.

Let $n = 2m + \delta$ with $\delta \in \{0, 1\}$. We first consider the case $k + 1 = m$. Letting $b_i = x^{2i} y^{m+i-1} \in V_{k+1}$ for $1 \leq i \leq m$ (ignoring 1 or 2 monomials not required for the surjectivity), the matrix of $\phi$ with respect to the bases $\{b_1, \ldots, b_m\}$ and $\{\gamma_1, \ldots, \gamma_m\}$ is that of Lemma 2.2, and so $\phi$ is surjective. The cases of smaller values of $k$ have larger domain and smaller codomain, with $\phi$ being an extension of a quotient of the case $k + 1 = m$, and hence is surjective since the case $k + 1 = m$ was.

Lemma 2.2. Let $A_m$ denote the $m$-by-$m$ matrix over $\mathbb{Z}_2$ with
\[ a_{i,j} = \begin{cases} 1 & \text{if } i + j \text{ is a 2-power} \\ 0 & \text{if not} \end{cases} \]
Then $\det(A_m) = 1$.

Proof. The proof is by induction on $m$. Let $m = 2^e + \Delta$ with $0 \leq \Delta < 2^e$. For $0 \leq i \leq \Delta$, row $2^e + i$ contains a single 1, in column $2^e - i$. Subtract this row from other rows which have a 1 in column $2^e - i$. Then do a similar thing with columns.
2^e + j, 0 \leq j \leq \Delta. The result has $A_{2^e - \Delta - 1}$ in the top left, and a $(2\Delta + 1)$-by-$(2\Delta + 1)$ matrix with 1’s along the antidiagonal in the bottom right. All other elements are 0. By the induction hypothesis, this matrix has determinant 1.

In moderately large gradings, there is, for each $j$, a monomial $x^iy^j$ equal to $\beta_j$. For example, in $H^{24}(G_{20})$, the following monomials equal $\beta_1, \ldots, \beta_8$, respectively:

\[
\begin{align*}
x^{14}y^5, & \quad x^{12}y^6, \quad x^{10}y^7, \quad x^{24}, \quad x^{22}y, \quad x^{20}y^2, \quad x^{18}y^3, \quad x^{16}y^4, \\
\end{align*}
\]

and a similar pattern holds in $H^i(G_{20})$ for $23 \leq i \leq 38$. However, in $H^{22}(G_{20})$, $x^{14}y^4 = \beta_1 + \beta_9$, and there is no monomial which equals either $\beta_1$ or $\beta_9$. We can obtain $\beta_1$ as $x^{22} + x^6y^8$, since $x^{22} = \beta_5$ and $x^6y^8 = \beta_1 + \beta_5$.

3. **Topological complexity of $C(RP^n, 2)$**

The topological complexity $TC(X)$ of a topological space $X$ is a homotopy invariant introduced by Farber in [4] which is one less than the number of nice subsets $U_i$ into which $X \times X$ can be partitioned such that there is a continuous map $s_i : U_i \rightarrow X^I$ such that $s_i(x_0, x_1)$ is a path from $x_0$ to $x_1$. This is of interest ([5]) for ordered (resp. unordered) configuration spaces $F(X, n)$ (resp. $C(X, n)$) as it measures how efficiently $n$ distinguishable (resp. indistinguishable) robots can be moved from one set of points in $X$ to another. The determination of $TC(C(X, n))$ has been particularly difficult. ([16],[1])

Farber showed ([4]) that $\zcl(X) \leq TC(X) \leq 2\dim(X)$ if $X$ is a CW complex. Here $\zcl(X)$, the zero-divisor-cup-length, is the largest number of elements of $\ker(\Delta^* : \tilde{H}^*(X \times X) \rightarrow \tilde{H}^*(X))$ with nonzero product, where $\Delta$ is the diagonal map. The main theorem of this section determines $\zcl(C(RP^n, 2))$.

**Theorem 3.1.** If $0 \leq d < 2^e$ and $r = \max\{s \in \mathbb{Z} : 2^s \leq d + \frac{1}{2}\}$, then

\[
\zcl(C(RP^{2^e+d}, 2)) = 2^{e+2} + 2^{r+1} - 4
\]

and $TC(C(RP^{2^e+d}, 2)) \geq 2^{e+2} + 2^{r+1} - 4$.

Since $C(RP^n, 2)$ has the homotopy type of the compact $(2n - 1)$-manifold $W_n$ described in the proof of Theorem 1.1, $TC(C(RP^{2^e+d}, 2)) \leq 2^{e+2} + 4d - 2$. For $d = 0, 1, 2, 3, 4$, the gap between our upper and lower bounds for $TC(C(RP^{2^e+d}, 2))$ is 1, 4, 6, 10, 10, respectively.
Proof. Let \( n = 2^e + d \) and let \( C_n, W_n, \) and \( G_n \) be as in the proof of Theorem 1.1. We identify \( H^*(C_n) \) with \( H^*(W_n) \) and note that the impact of (1.3) is that \( x^i u^j = x^{i+j-1} u \) if \( j > 0 \).

Let \( \overline{x} = x \otimes 1 + 1 \otimes x \), and define \( \overline{y} \) and \( \overline{u} \) similarly. We claim that \( zcl(C_n) \geq 2^{e+2} + 2^{r+1} - 4 \) since

\[
\overline{x}^{2^{e+1}-1} \overline{u}^{2^{e+1}-2} \overline{y}^{2^{r+1}-1} \neq 0. \tag{3.2}
\]

To see this, we first note that the indicated product is, in bigrading \((2^{e+1} + 2d - 1, 2^{e+1} + 2^{r+2} - 2d - 4)\), equal to

\[
\sum_{k,j} x^{2k-1} u^{2^{e+1}+2(d-j-k)} y^j \otimes x^{2^{e+1}-2k} u^{2(j+k-d-1)} y^{2^{r+1}-1-j}.
\]

Since the terms divisible by \( u \) are independent from those not divisible by \( u \), we restrict to terms whose right factor is not divisible by \( u \), and obtain

\[
\sum_{j} x^{2^{e+1}+2(d-j)-2} u y^j \otimes x^{2^{e+1}-2(d-j+1)} y^{2^{r+1}-1-j}. \tag{3.3}
\]

Terms with \( j < d \) (resp. \( j > d \)) have left (resp. right) factor equal to 0 since \( x^{2^{e+1}} = 0 \). Thus (3.3) equals \( x^{2^{e+1}-2} u y^d \otimes x^{2^{e+1}-2} y^{2^{r+1}-1-d} \), which is nonzero by (1.3) and Theorem 2.1.

To see that this bound for \( zcl \) cannot be improved, first note that the exponents of \( \overline{x} \) and \( \overline{y} \) in (3.2) cannot be increased since \( x^{2^{e+1}-1} = 0 \) by [6, Cor 4.2]. If the exponent of \( \overline{u} \) is increased by 1, the top term \( x^{2^{e+1}-2} u \otimes x^{2^{e+1}-2} u \) occurs with even coefficient by symmetry. The only hope of getting a larger nonzero product would be to increase the exponent of \( \overline{y} \). We will use our analysis of \( H^*(C_n) \) to see that this will fail to improve the \( zcl \).

The key observation is that, with \( n = 2^e + d \) and \( \delta \in \{0,1\} \), a nonzero monomial \( x^s u^\delta y^i \) in \( H^*(C_n) \) with \( t > d \) must have \( s \leq 2^e - 2 \). This will follow from Theorem 2.1 once we show that if \( x^s y^j = x^{2i-\varepsilon} y^{n-k-i} \) has \( s \geq 2^e - 1 \) and \( t \geq d + 1 \), and \( 2 \leq 2j \leq 2k \), then \( 2i + 2j \) is not a 2-power. We have \( 2i + 2j \geq 2^e - 1 + \varepsilon + 2 > 2^e \). On the other hand, \( 2i + 2j \leq (2n - 2k - 2d - 2) + 2k = 2^{e+1} - 2 \), implying the claim.

If \( x^{i_1} u^{j_1} \otimes x^{i_2} u^{j_2} \) appears in the expansion of \( x^a u^b y^c \) with maximal exponent sum, it should have \( i_1 = 2^{e+1} - 2, \varepsilon_1 = 1, \) and \( j_1 = d \), as we do not want to sacrifice \( 2^e \) \( x \)-exponents on both sides of the \( \otimes \). To have a monomial \( x^{2^{e+1}-2} u y^d \otimes x^{i_2} u^{j_2} \) whose
exponent sum exceeds our zcl bound would require $i_2 + j_2 + \varepsilon_2 > 2^{e+1} - 3 + 2^{r+1} - d$. If $j_2 > d$, then $i_2 \leq 2^e - 2$, so we would need $j_2 + \varepsilon_2 \geq 2^e + 2^{r+1} - d$ with strict inequality unless $i_2 = 2^e - 2$. We also have $j_2 \leq 2^e + d - 1$, half the dimension of $W_n$. We would also need $\binom{d+j_2}{d} \equiv 1 \mod 2$. But this is impossible by Lemma 3.4 applied to $j = j_2 - 2^e$ unless $i_2 = 2^e - 2$ and $j_2 = 2^e + 2^{r+1} - d - 1$. But then $|x^{2^{e+1} - 2}u^j d \otimes x^{j_2}u^\varepsilon y^j| > 2 \dim(W_n)$. The alternative is $j_2 \leq d$. But, since we need $\binom{d+j_2}{d} \equiv 1 \mod 2$, the largest such $j_2$ was what was used in obtaining our lower bound. ■

**Lemma 3.4.** If $2^r \leq d < 2^{r+1}$ and $2^{r+1} - d - 1 < j \leq d - 1$, then $\binom{d+j}{d} \equiv 0 \mod 2$.

**Proof.** For $\binom{d+j}{d}$ to be odd, the binary expansions of $j$ and $d$ must be disjoint. Since $j \leq 2^{r+1} - 1$, the 1’s in the binary expansion of $j$ would have to be a subset of those of $2^{r+1} - 1 - d$, contradicting $j > 2^{r+1} - d - 1$. ■

**References**


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