

January 7, 2007

Jim,

This relates to material that we discussed some time ago. I am not assuming that you recall the discussion.

The question will be “which of the pieces below have appeared in the literature?” If any, then references or hints to references would be nice, and I am willing to search by myself on even the vaguest of hints.

If \mathcal{C} is a category with (functorial) multiplication \otimes , then inside the operad $End_{\mathcal{C}}$ there is a suboperad \otimes derived from the multiplication \otimes and an obvious surjective map of operads $h : \mathcal{T} \rightarrow \otimes$ whose domain is the operad of finite binary trees. This map will take, for example, the tree $\begin{array}{c} \wedge \\ \searrow \end{array}$ to the functor

$$(1) \quad (X, Y, Z) \mapsto X \otimes (Y \otimes Z)$$

in $End_{\mathcal{C}}$ and the tree $\begin{array}{c} \wedge \\ \swarrow \end{array}$ to the functor

$$(2) \quad (X, Y, Z) \mapsto (X \otimes Y) \otimes Z$$

in $End_{\mathcal{C}}$.

If there is a natural isomorphism α given from the functor (1) to the functor (2) in $End_{\mathcal{C}}$, then the isomorphisms generated in the usual way from (composites of expansions of instances of) α and α^{-1} and the identity isomorphisms on the functors in \otimes gives a category structure to \otimes .

There are now two category structures that we can put on the operad \mathcal{T} of finite binary trees. One is a “pullback” category structure that we get from the category structure on \otimes where we use $h : \mathcal{T} \rightarrow \otimes$ to do the pullback. (Morphisms from T_1 to T_2 are just the morphisms from $h(T_1)$ to $h(T_2)$.) The other category structure on \mathcal{T} is the trivial structure in which every pair of trees with the same number of leaves gets a unique (iso)morphism between them in each direction. We let \otimes^h denote the pullback category and reuse the notation \mathcal{T} for the trivial category structure.

There is a forgetful functor from \otimes^h to \mathcal{T} that is the identity on objects.

At this point we probably leave the realm that might exist in the literature you are familiar with. However, I will press on in case the “probably” is wrong, and to tell you what the point of all this is.

Particularly pleasant properties of the operad \mathcal{T} allow one to compute two groups: one $T(\otimes^h)$ from \otimes^h and another F from \mathcal{T} . The second group is well known and is usually referred to as “Thompson’s group F ” so we have kept the letter F for it.

There is a surjective homomorphism (call it a comparison homomorphism) σ from $T(\otimes^h)$ to F . The surjectivity is standard and the arguments are in [1].

Under the assumption that the multiplication \otimes has an identity (an object K in \mathcal{C} with a natural isomorphism ι from the identity on \mathcal{C} to the functor $X \mapsto X \otimes K$ with no further restrictions such as the satisfaction of a coherence property on the isomorphism ι), then one proves easily that the associativity morphism α makes the pentagonal diagrams commute if and only if the comparison homomorphism σ is an isomorphism. In fact, once a certain “non-collapsing” fact is proven from the existence of the identity K , the rest is just a quote of definitions.

Thus \mathcal{C} is a monoidal category if and only if the “identity isomorphism” ι satisfies the usual coherence conditions on identities and the comparison homomorphism σ is an isomorphism.

One can do exactly the same thing with symmetric, monoidal categories (in which case the comparison homomorphism is to a well known group known as Thompson's group V) and braided tensor categories (in which case the comparison homomorphism is to a group BV of mine that I call the braided version of V). In the case of symmetric, monoidal categories, the argument again boils down to a check of definitions once certain basic facts are established. In the case of braided tensor categories, there are real calculations that must be done since the definition of braided tensor categories reads very differently than it does for monoidal and symmetric, monoidal categories.

This ends the summary.

I can clarify my question a bit. I am familiar with the paper of MacLane below. I am familiar with little else. This pretty much identifies the scope of my question.

The language of operads does not appear in MacLane's paper and I am wondering how much of MacLane's results have been reworked to exploit operads and their structures. Referring to the summary above, I am curious about the structures that precede the introduction of the group $T(\otimes)$.

REFERENCES

1. Saunders MacLane, *Natural associativity and commutativity*, Rice Univ. Studies **49** (1963), no. 4, 28–46.

Thank you for anything that you can point me at.

Sincerely,
Matt