ANALYSIS OF A BASEBALL SIMULATION GAME USING MARKOV CHAINS

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Abstract. APBA baseball is a sophisticated baseball simulation game. Each major league player is represented by a card, which has numbers on it that reflect his performance in a particular season. Two people play a game by rolling dice and looking on their players' cards to see what is the outcome of the roll of the dice.

In this article we use Markov chains to analyze certain aspects of this game. For example, we can tell whether one player's batting card is more valuable than another's, and we can make informed decisions about strategy in the game.

1. Introduction

APBA baseball is a baseball simulation game invented by Dick Seitz of Lancaster, Pennsylvania, and first marketed in 1951. It has an avid following today([1]), and was featured in a *New York Times* article in August 2009.([5]) A game is played between two ordinary human beings, each of whom uses a set of cards. Each card represents a major league baseball player. A batter's card has numbers on it which are supposed to accurately represent his performance during a specified season. A pitcher's card has a grade (A, B, C, or D) for his out-getting ability and also perhaps letters for his strikeout ability (X and/or Y if he strikes out a lot of batters) and control (W for Wild and Z for good control). All players have a fielding rating and perhaps a speed rating (F for Fast and S for Slow).

In this article, we use Markov chains to analyze certain aspects of this game. For example, we can tell whether one player's batting card is more valuable than another's (Section 2), and we can make informed decisions about strategy in the game (Section 3).

Markov chain analysis has been applied to real baseball in [2], [6], and [7]. For example, two of these discuss how to optimize the batting order, using Markov chains and detailed statistics available from a database. The rigid structure of the game analyzed here lends itself to conclusions of a different nature than those.

Each play of the game begins with a roll of a pair of dice, one red and one white. A red 4 and white 2 is interpreted as 42. There are $6 \cdot 6 = 36$ possible rolls. Each player has a card, which is based on his performance during a particular season. The card associates to each of the 36 possible dice rolls a number from 1 to 41. For example, Hank Aaron's 1962 card is pictured in Figure 1.1.

Figure 1.1. Hank Aaron's 1962 card

Hank Aaron ((F) *OF* (3))		
11–1	31–8	51–8
12–25	32–26	52–27
13–14	33–5	53–16
14–30	34–31	54–32
15–10	35–9	55–8
16–28	36–12	56–29
21–30	41–24	61–9
22–6	42–14	62–14
23–31	43–13	63–34
24–13	44–7	64–40
25–10	45–14	65–35
26–13	46–30	66–1

You look on the batter's card to see what number corresponds to the number rolled. For example, on most players' cards the number corresponding to a roll of 42 will either be 13 or 14. For Aaron, it is 14. Then one looks in a book of outcomes to see what will be the result of this number. The outcome will depend on which bases are occupied, perhaps on the opposing pitcher and fielders, perhaps on the speed of the baserunners, and occasionally on how many are out. Usually 13 is a strikeout and 14 is a walk.

The value of a player's card depends on the 36 numbers on it, since one may assume that each of the 36 numbers is rolled equally often. We can (and will) compute mathematically the expected number of runs that would be scored from any (base,out) situation by a team of average players, and then, using this information, we can determine the average increase or decrease in expected number of runs scored when any number comes up on a player's card. For example, 1 is always a home run. We determine that when a 1 comes up, the team's expected number of runs for that inning is increased by 1.41. On the other hand, 13, which is usually a strikeout, decreases the team's expected number of runs for that inning by 0.23. So we say the value of a 1 is 1.41, while that of a 13 is -0.23.

If you average the 36 values of the numbers on a player's card, you obtain the average amount by which he changes your expected number of runs scored in an inning on a single roll. This is not quite the same as saying that it is the result of a plate appearance, because occasionally the result of a roll will keep the batter at the plate. For example, it might be "Strike. Runner out stealing" or "Ball. W–base on balls." The latter means that if the pitcher has a W, which stands for Wild, the batter walks; otherwise you roll again.

For example, Hank Aaron's 1962 card in Figure 1.1 has two 1's (a home run), a 5 and a 6 (extra-base hits of varying amount), eight numbers (7 to 10) that are often a single, depending on the grade of the opposing pitcher, four 14's (usually a walk), a 16 (which is often "first on error" depending on the opposing team's center fielder), a 40 (which varies but is often an out), and eighteen numbers that are always outs. The total of the values of his 36 numbers is 3.45, and so the average is .096. If you rolled 670 times for him over the course of a season, he would increase the team's expected number of runs by $670 \cdot .096 = 64$. Sabermetricians (aficionados of extremely sophisticated baseball statistics) have a statistic called Batting Runs Above Average (BRAA), which tells how many runs a player increased his team's number of runs during the season compared to the result of an average player. Hank Aaron in 1962 had 58 BRAA, according to [8]. This suggests that the APBA card makers, my analysis, and the sabermetricians are pretty much in synch.

One of the uses that can be made of having these valuations of players' cards is to determine whether one team (determined by a set of cards) is better than another. To do this, you need to be able to have a method for telling the value of pitching ratings, fielding ratings, and speed ratings on a basis comparable to the batting values. This is accomplished in Section 4.

Three strategy aspects which are evaluated are

- When should you Hit and Run? This is an option with a runner on first or runners on first and third.
- When should you "play it safe" with a slow runner on base? There are numbers which say things like "Single, runner to third, S out at third." So, if your runner on first has an S (for slow), you have an option to play it safe on a single. If you play it safe, the runner only goes to second on a single, regardless of whether he would ordinarily have gone to third or been out

at third due to his S. My analysis tells when you should play it safe.

- How should you align your outfielders? Each outfielder has a fielding rating, 1, 2, or 3. We determine, for each combination of these numbers, which alignment into left field, center field, and right field produces optimal results.

The analysis, although totally mathematical once it gets going, depends on input parameters taken from a batch of cards. I use a sample of 350 cards from the period 1956 to 1966, which is when I was actively playing the game. The parameters include what fraction of the time each batting number appears on all the cards[1], what fraction of the pitchers have each grade, A, B, C, and D, how often pitching adornments (W, X, Y, Z) for strikeouts and walks occur, what fraction of the batters are fast (F) or slow (S), and what fraction of the fielders have the various fielding ratings. These fractions have a great effect on the values of the batting numbers. For example, the number 9 is often a single, but is usually an out against a grade A or C pitcher. Since my parameters say that the pitcher will have grade C 46% of the time, this makes 9 have a relatively low value, whereas if there were fewer C pitchers, it would have a higher value.

A very different version of this paper was written for APBA players who may not know any advanced mathematics.([3]) The purpose of the present paper is to explain how matrix methods can be applied to perform an interesting analysis of a game. Although Markov chains are present, no advanced theorems of Markov theory are involved. It is really just matrix manipulation and matrix equations.

Most of the ideas in this analysis were developed by the author in 1964, and a preliminary version of this analysis was performed then. It was limited by inadequate computer access at that time. The computer program used for the current analysis was the computer algebra system `Maple`. The new analysis was inspired by the *New York Times* article.

2. Details of evaluation of batting cards

There are 25 (base,out) states. The 8 possible bases-occupied are 0, 1, 2, 3, 1-2, 1-3, 2-3, and 1-2-3. With none (resp., one, two) out, these comprise states 1 to 8 (resp. 9 to 16, 17 to 24). State 25 is the terminal state of three out. For example, since 1-2 is the fifth bases-occupied situation listed, then situation number $13(= 5 + 8)$ is "runners on first

[1]For example, the most commonly occurring number on batters' cards is 13, which is usually a strikeout; approximately 9% of the numbers on all the cards are 13's.

and second and one out." We write bases occupied as, for example, 1-3 to distinguish "runners on first and third" from situation number 13, just described, and from the strikeout number 13 on a batter's card. We will work with various 25-by-25 transition matrices for the 25 (base,out) situations.

For each integer k from 1 to 41, corresponding to the numbers that may appear on a batter's card, we form the 25-by-25 transition matrix M_k whose entry in the ith row and jth column, denoted $M_k(i,j)$, is the probability of going to state j if you are in state i, and k is rolled. (I find it convenient to use the inaccurate term that the number (from 1 to 41) is "rolled." The dice are rolled and then the number corresponding to the dice is found on the batter's card; it is a consequence of the roll and the batter's card, but I will say it is "rolled.") The sum of the entries of each row of each M_k is 1. The last row of M_k is $[0,\ldots,0,1]$. It is included mainly just so the theory of absorbing Markov chains applies. See, for example [4, §11.2]. The matrices M_k incorporate the various outcomes that can occur when rolling a k, depending upon the distribution of pitching and fielding ratings. They also incorporate the strategies of playing it safe and hit-and-running.

We illustrate how strategy is incorporated into the matrix with one example of a portion of an M_k. If 8 is rolled with a runner on first, it is an out, runner to second, against an A or B pitcher, while against a C or D pitcher it is single, runner to third, S out at third. We determine by methods discussed in Section 3 that, on average, an S runner on first should play it safe on a single against a C or D pitcher. Thus

(2.1)
$$M_8(i,j) = \begin{cases} A+B & \text{if } (i,j) = (2,11),\ (10,19), \text{ or } (18,25) \\ (C+D)S & \text{if } (i,j) = (2,5),\ (10,13), \text{ or } (18,21) \\ (C+D)(1-S) & \text{if } (i,j) = (2,6),\ (10,14), \text{ or } (18,22) \\ 0 & \text{if } i \in \{2,10,18\},\ j \text{ not as above.} \end{cases}$$

Here, for $t = 0$, 1, or 2, state $2+8t$ refers to runner on first and t out, while state $5+8t$ (resp. $6+8t$) refers to runners on first and second (resp. first and third) and t out. Also, A, B, C, and D are the fractions of the time that the pitcher will have each grade. For example $C = .46$ and $D = .17$. Also, $S = .22$ is the fraction of all cards that are S (Slow). Thus, for example, $M_8(2,6) = (.46+.17)(1-.22) = .49$ says that 49% of the time if you roll an 8 with a runner on first and nobody out, you will end up with runners on first and third and nobody out.

Next define a 25-by-25 transition matrix M_0 by

$$M_0 := \sum_{k=1}^{41} p_k M_k, \tag{2.2}$$

where p_k is the fraction of the time that the number k occurs on our batters' cards. The matrix M_0 is the transition matrix for an average batter selected randomly from the set of cards in our sample. It is a Markov chain with one absorbing state, state 25. Let $\widetilde{M}$ denote the 24-by-24 matrix obtained from M_0 by deleting the last row and column. Most of its rows do not sum to 0 because transitions to a third out are not included in $\widetilde{M}$. It is the transient submatrix associated to the absorbing Markov chain M_0. Most of our work will involve $\widetilde{M}$.

In linear algebra, it is common to have the transition probabilities appear in the columns rather than the rows. You can do it either way, but it seems more common in probability texts to put the transition probabilities in rows. The two ways would be equally convenient for our analysis.

First we compute the fraction of the time that a batter is in each of the 24 states. The result is listed in Table 1. Let $\mathbf{q} = [q_1, \dots, q_{24}]$ be a row vector of probabilities of being in the various states prior to a roll, with $\sum_{i=1}^{24} q_i \le 1$. This sum might be less than 1 because it does not include the probability that the inning has already ended. It is necessary to include this in the analysis because we will be dealing with the row consisting of the probabilities of being in the various states after a roll of the dice, and here clearly it is possible that the inning may have ended.

Then $\mathbf{q}\widetilde{M}$ is the row vector of probabilities of being in the various states after the roll, and the sum of its entries may be even smaller than it is for $\mathbf{q}$. Let $\mathbf{e} = [1, 0, \dots, 0]$, a row vector of length 24. It represents the state at the beginning of the inning. Then

$$\mathbf{p} := \mathbf{e} + \mathbf{e}\widetilde{M} + \mathbf{e}\widetilde{M}^2 + \mathbf{e}\widetilde{M}^3 + \cdots \tag{2.3}$$

has as its ith entry the expected number of times that situation i will occur during an inning. The general theory of Markov chains with an absorbing state implies that this infinite series converges to a finite row vector. Let $\mathbf{pr}$ denote the row vector obtained by dividing each entry of $\mathbf{p}$ by the sum of the entries of $\mathbf{p}$. Then $\mathbf{pr}$ gives the probabilities $\mathrm{pr}(i)$ of being in each of the 24 states.

The easiest way to compute $\mathbf{p}$ is derived by first multiplying (2.3) on the right by $\widetilde{M}$, obtaining

$$\mathbf{p}\widetilde{M} = \mathbf{e}\widetilde{M} + \mathbf{e}\widetilde{M}^2 + \mathbf{e}\widetilde{M}^3 + \cdots .$$

Combining this with (2.3) yields that $\mathbf{p}\widetilde{M} = \mathbf{p} - \mathbf{e}$, and so

$$\mathbf{p}(I - \widetilde{M}) = \mathbf{e}, \tag{2.4}$$

where I is the 24-by-24 identity matrix. The matrix equation (2.4) can be solved for $\mathbf{p}$. It is perhaps more commonly thought of as the transposed version, $(I - \widetilde{M})^T\mathbf{p}^T = \mathbf{e}^T$. In this latter version, you are working with column vectors, and, if you are thinking of it as a system of 24 linear equations in 24 unknowns, the equations line up more naturally. At any rate, it is by solving this system of equations, or matrix equation, that $\mathbf{p}$ and then $\mathbf{pr}$ is obtained. This is easily done by `Maple`, yielding that $\mathbf{pr}$ is as in Table 1. A more explicit formula for $\mathbf{p}$ is that it equals the first row of $(I - \widetilde{M})^{-1}$ or the first column of $(I - \widetilde{M^T})^{-1}$, but it is easier just to solve (2.4).

TABLE 1. Fraction of the time a batter is in each situation

		outs		
		0	1	2
	0	.24239	.17284	.13518
	1	.06248	.07462	.07359
Bases	2	.01671	.03318	.04347
	3	.00189	.00728	.01491
	1-2	.00924	.01664	.01895
	1-3	.00584	.01128	.01744
	2-3	.00496	.01013	.01305
	1-2-3	.00233	.00515	.00645

Now we explain how we found the expected number of runs scored in an inning subsequent to being in situation j, which we denote by $E(j)$. Let $\mathbf{E}$ be the column vector of length 24 with entries $E(j)$, yet to be determined, and let $E(25) = 0$. If you are in situation i and roll a k, the number of runs that you expect to score in the remainder of the inning, including on that roll, is

$$\sum_{j=1}^{25} M_k(i,j)(E(j) + r_{k,i,j}),$$

where $r_{k,i,j}$ is the number of runs scored on that roll (rolling k and going from state i to j). Then, for $1 \le i \le 41$,

$$E(i) = \sum_{k=1}^{41} p_k \sum_{j=1}^{25} M_k(i,j)(E(j) + r_{k,i,j}), \tag{2.5}$$

or equivalently

$$\mathbf{E} = \widetilde{M}\mathbf{E} + \mathbf{b}, \tag{2.6}$$

where $\mathbf{b}$ is a column vector of length 24 whose ith entry is

$$\sum_{k=1}^{41} p_k \sum_{j=1}^{25} M_k(i,j) r_{k,i,j}.$$

Note that the ith entry of $\mathbf{b}$ tells the average number of runs scored on a single roll if you are in state i. Equation (2.5) is the key equation, expressing the desired E-values in terms of other E-values, including itself, and the r-values. Once the r-values are known, (2.5) can be solved to find the E-values.

Most of the time, the value $r_{k,i,j}$ does not depend on the roll k and equals

$$r(i,j) := 1 + BR(i) + \left[\tfrac{i-1}{8}\right] - BR(j) - \left[\tfrac{j-1}{8}\right], \tag{2.7}$$

where

$$BR(i) := \begin{cases} 0 & i \equiv 1 \pmod 8 \\ 1 & i \equiv 2,3,4 \pmod 8 \\ 2 & i \equiv 5,6,7 \pmod 8 \\ 3 & i \equiv 0 \pmod 8 \end{cases}$$

is the number of base runners in situation i, and $[\frac{i-1}{8}]$, which denotes the integer part of the fraction, is the number of outs in situation i. The 1 in (2.7) is due to the batter. The formula (2.7) is not valid when $j = 25$, when $r_{k,i,j}$ is usually 0. Let $\mathbf{r}$ be a column vector of length 24 whose ith entry is $\sum_{j=1}^{24} \widetilde{M}(i,j) r(i,j)$. This gives, except for the exceptional situations discussed below, the average number of runs scored on a single roll from situation i. Note that $r(i,j)$ is sometimes negative, which is nonsensical, but this will never happen if $\widetilde{M}(i,j) \ne 0$.

There are two types of exceptions when the formulas of the preceding paragraph do not give the number of runs scored on the play. Those with $j = 25$ are the rare cases, such as 9 with a runner on third and two out against a B or D pitcher[2], in which runs score on the play,

[2] single, runner scores, batter out trying for second

but then a base runner makes the third out. Let $d(k,i)$ denote the expected number of runs which are scored on inning-terminating plays when rolling a k from situation i. For example, $d(9,20) = B+D = .45$, since situation 20 is (base,out)=(3,2). This says that, since 45% of the time a grade B or D pitcher will be pitching, if you roll a 9 in situation 20, then you scored, on average, 0.45 runs on that roll which were not accounted for by the vector $\mathbf{r}$ above.

Let $\mathbf{d}$ be a column vector of length 24 whose ith entry is $d_i := \sum_{k=1}^{41} p_k d(k,i)$, the average number, out of all rolls from situation i, of runs scored on plays which include the third out of the inning. These numbers d_i will be 0 unless $i \geq 20$.

A bigger exception is the cases in which the batter stays up after the roll, such as rolling a 37 with the bases empty[3] if the pitcher does not have a W. Recall that some pitchers have a W adornment if they are "wild," meaning that they walk more batters than an average pitcher. We have $M_{37}(1,1) = 1 - W$, where $W = .052$ is the fraction of all pitchers that have a W. This says that in these cases nothing has happened on the roll. The formula (2.7), which says that $r(1,1) = 1$, would incorrectly say that one runner scored on the play. We define a column vector $\mathbf{No}$ of length 24 which has ith entry $\mathrm{No}_i := \sum_{k=1}^{41} p_k U(k,i)$, where $U(k,i)$ is the probability that the batter stays up if you roll k in situation i. The above example for 37 says $U(37,1) = 1 - W$ and contributes $p_{37}(1-W)$ to No_1.

Now we can write $\mathbf{b} = \mathbf{r} + \mathbf{d} - \mathbf{No}$, and (2.6) becomes

$$(I - \widetilde{M})\mathbf{E} = \mathbf{r} + \mathbf{d} - \mathbf{No}. \tag{2.8}$$

This equation was solved for $\mathbf{E}$ by `Maple` to obtain Table 2.

The most interesting value in $\mathbf{E}$ is the expected number, $E(1)$, of runs scored when there are no runners on base and no outs, since that tells the average number of runs scored in an inning. This value, 0.4327, after being multiplied by 9, yields the average number of runs scored by a team in a 9-inning game. This value, 3.8943, is quite consistent with actual baseball figures during the early 1960's on which this analysis is based. It would be higher if we based our analysis on the cards from the late 1990's.

[3]Strike. W-Base on balls.

Table 2. Expected number of runs from different situations

		outs		
		0	1	2
	0	0.4327	0.2254	0.0792
	1	0.8071	0.4737	0.1892
Bases	2	0.9833	0.6055	0.2862
	3	1.2053	0.8693	0.3481
	1-2	1.3294	0.8288	0.4042
	1-3	1.6274	1.0519	0.4450
	2-3	1.7479	1.1924	0.5279
	1-2-3	2.1726	1.4705	0.7098

The values $V(k)$ of the numbers on a card are now easily obtained as

$$\begin{aligned} V(k) &= \sum_{i=1}^{24} \mathrm{pr}(i)\Bigg(\sum_{j=1}^{24} M_k(i,j)\big(r(i,j) - U(k,i) + E(j) - E(i)\big) \\ &\qquad + M_k(i,25)\big(d(k,i) - E(i)\big)\Bigg). \end{aligned}$$

This is obtained by averaging, over all initial situations i and all subsequent situations j obtained when rolling a k, the number of runs obtained on that roll plus the change in expected number of subsequent runs to be obtained later in the inning. This yields numbers such as $V(1) = 1.4101$ and $V(13) = -0.2317$, which were mentioned in Section 1. They are listed in [3], but we will not list them here, as they would only be of interest to an APBA player.

3. Strategy

In this section, we discuss briefly the way in which the results of Tables 1 and 2 are used in making strategy decisions, so as to maximize our expected number of runs scored (by deciding whether to hit-and-run and whether to play-it-safe) and to minimize the expected number of runs scored by the opposing team by deciding how to align our outfielders.

There is a slight subtlety here in that the strategy decisions are built into the matrices M_k, such as in (2.1). These matrices determine the values of the numbers $E(i)$ which appear in Table 2, and the values $E(i)$ are then used to determine strategy, so it is sort of a feedback loop. The way this was performed was to first make an initial assumption about hit-and-running, playing-it-safe, and aligning your outfielders, then use that assumption to compute numbers pr(i) and $E(i)$ for the two tables, next use these values to determine new strategies, and then finally recompute the numbers pr(i) and $E(i)$ using the new strategy. After this, the process stabilizes because the strategies based on the revised tables are the same as the strategies obtained from the earlier tables.

The biggest difference when hit-and-running as compared to an ordinary at-bat is that rolling a 13, instead of resulting in a strikeout, gives "runner out stealing; if runner has an 11 on card, he steals safely." This refers to having one of the batting numbers on the runner's card be an 11; batters who steal a lot of bases will be given such numbers. Some other batters, such as Hank Aaron in 1962, who steal moderately often will have a 10 (see 15–10 in Figure 1.1) but not an 11.

The effect of some other batting numbers change, too, when hit-and-running, but the changed effect of 13 is the most significant. Because of this, one's intuition is to hit and run if the runner has an 11, and not if he doesn't. The numbers in Table 2 can be used to verify that this is indeed the best strategy. We compare, for the situations in which hit-and-running is allowable, namely $i = 2, 6, 10, 14, 18,$ or 22, i.e., runner on first or runners on first and third, the value of

$$\sum_{k=1}^{41} p_k \sum_{j=1}^{25} M_k(i,j)(E(j) + r_{k,i,j}) \tag{3.1}$$

if $M_k(i,*)$ is the transition matrix[4] for (a) hitting away, (b) hit-and-running with an 11 on the card of the runner on first, and (c) hit-and-running without an 11 on first. Note that the transition matrices M_k will be different depending on which of these three strategies you use, and will lead to different values of the expected number of runs scored as given in (3.1). We obtain the largest value when hit-and-running with a runner with an 11 on first base, and the smallest value when hit-and-running when the runner does not have an 11. This is all based on an average batter. For a specific batter, your strategy might be different, but you can still use Table 2 to help you make your decision.

A similar analysis is performed to decide whether to play it safe if an S-runner is on base in a situation in which "S is out" as in the earlier discussion of 8 with a runner on first, centered around (2.1). For each situation i, one compares (3.1) using two different $M_k(i,*)$'s and the associated $r_{k,i,j}$'s, one with playing it safe and one not playing it safe. In [3], all the conclusions are listed, but they would not be of much interest to anyone except an APBA player. Some of them are quite

[4]and $r_{k,i,j}$ the runs scored on that roll

delicate. For example, with runners on second and third, with an S runner on second, always play it safe on a single against an A pitcher, never play it safe against a B pitcher, and against a C or D pitcher play it safe with less than two outs.

With an S runner on third and less than two out, you may want to play it safe on a fly ball, since some numbers are "fly out, runner scores, S out at home." The analysis of this is performed similarly to that of the preceding pargraph, although it is more complicated. Here again, in practice, an APBA player could use Table 2 together with information about the specific batter, pitcher, and fielders to make a decision. But for determining the transition matrices for the "average" player, we needed to include specific rules about when we were playing it safe, and those are listed in [3], along with a discussion of how they were obtained.

Deciding how to align your outfielders was handled differently. Some APBA players might feel compelled to put Willie Mays in centerfield, but the outfielders' cards allow them to play in any outfield position, and so you, as manager, have the option of putting him in rightfield if doing so will be to your statistical advantage. Willie Mays was a very good fielder; his fielding rating of 3 is the highest an outfielder can have.

Suppose, for example, that your outfield consists of Willie Mays and two other outfielders each of whom has a fielding rating of 2. We consider three versions of the transition matrices M_k, one in which the centerfielder has fielding rating 3 and the other two outfielders have fielding rating 2, one in which the rightfielder has fielding rating 3 and the other two outfielders have fielding rating 2, and one in which the leftfielder is a 3 and the other two are 2's. Note that these fielding ratings affect outcomes. For example, with nobody on base, 17 is an out if the rightfielder has a 3 rating, first on error if he has a 2 rating, and first and second on error if he has a 1 rating. This information would be implemented into the matrix M_{17}. Note that the matrices M_k which were used to find expected number of runs in Section 2 incorporated probabilities of each fielder having the various possible fielding ratings, based on the cards in our sample, but for the analysis here we stipulate exactly what are the fielding ratings of the outfielders. We find that the expected number of runs in an inning is .42463 if your 3 fielder is in leftfield (and the other two outfielders are 2's), .43322 if the 3 is in center, and .42234 if the 3 is in right, so you should put your 3 in right field to minimize the opponent's expected number of runs. A similar analysis is done for each combination of fielding numbers.

4. Values for speed, pitching and fielding

Now all the strategies determined in the preceding section are implemented into the `Maple` program. We estimate that 7% of the time when a runner is on first or first and third the runner will have an 11 and you hit-and-run. Thus the transition matrices in the rows corresponding to these base situations have the usual outcome weighted by 0.93 and the hit-and-run outcome weighted by 0.07. The values of $E(i)$ listed in Table 2 are contingent on this and other strategies, such as playing-it-safe.

The `Maple` program has many lines of the form of (2.1). They are all listed in [3]. In Section 2, the parameter S was set to 0.22, which is the fraction of the runners that have an S (for slow). To determine the value of an S (slow base runner), we compute

$$\sum_{i=1}^{24} \mathrm{pr}(i) \sum_{k=1}^{41} p_k \sum_{j=1}^{25} M_k(i,j)(E(j) + r_{k,i,j}) \tag{4.1}$$

with M_k (and the associated $r_{k,i,j}$) based on having $S = 1$, and then again with $S = 0$, and take the difference. This will give the change in expected runs scored in an average roll by having an S runner on an affected base. This equals $-.01898$. But to compare this with the value of a batting number, such as the previously mentioned $V(1) = 1.41$ which says that rolling a 1 increases your expected number of runs scored in the inning by 1.41, there are several considerations.

One is that the player with the S is not going to always be on base. The more frequently the player is on base, the more disadvantageous his S rating is. But my analysis cannot measure such a fine distinction. We must assume that each player on the team is equally likely to be on base. For a given player with an S, one ninth of the time a specific baserunner (such as the runner on second) would be this player. So the average loss to the team on any roll due to the player's S would be .01898/9. This analysis is happening every play of the game (while your team is at bat). The given S runner could be on an affected base while several batters are up, or he might not be on base at all. The .01898/9 figure takes this into account. It is the average loss caused by the S on each roll of the dice.

If you average the values of the 36 numbers on a batter's card, you obtain the average amount by which he increases the team's expected number of runs on a single roll. On average, a batter will be at bat 4.5 times per game, and so the average of the values of the numbers on his card should be multiplied by 4.5 to give the amount by which his batting numbers increase the team's expected number of runs during a game. The value .01898/9 that a person's S hurts you on every roll of the game should be multiplied by 40.5, for the 40.5 rolls during a game, on average. Since 40.5/9 equals 4.5, over the course of a game the .01898 negative value of an S is exactly comparable to the average of the values of the 36 numbers on the player's card. If comparing it with a single number on the player's card, its .01898 should be multiplied by 36, yielding .683, since the sum of the values of the numbers on the card had to be divided by 36 to form the average. Having an S turns out to be roughly equal to the difference between one of your 36 batting numbers being a 7 (usually a single, although occasionally an out against a good pitcher) rather than a pure out number such as 13 (and all other numbers unchanged), since $V(7) = .47$ and $V(13) = -.23$, and $.47 - (-.23)$ is approximately equal to .683.

A similar analysis can be made for an F rating for a fast runner and for the base-running value of an 11 due to hit-and-running.

To find the value of an A pitcher, one merely needs to run the program with the pitcher parameters set as $A = 1$, $B = 0$, $C = 0$, $D = 0$, which will force the transition matrices to include only the outcomes against an A pitcher. We find that the expected number of runs scored in an inning is .25186 against an A pitcher, which is .18087 less than the value of .43273 against an average pitcher. Thus the A pitcher saves .18087 runs per inning. But how do we compare this with the sum of the values of a batter's 36 numbers? It should be done on a per-game basis. A batter bats 4.5 times per game, and so we multiply the average of the values of his numbers by 4.5 to see how many runs per game do his plate appearances help the team's expected number of runs. The A pitcher pitches roughly 7 innings every fifth game, hence 1.4 innings per game (of his team). Thus the A pitcher's value, per game of his team, is $.18087 \cdot 1.4 = .2532$. A similar analysis shows that if a pitcher has a Z adornment, for good control, this decreases the opponent's number of runs per inning by .04. Thus an A pitcher with a Z has a value per game of his team of $(.18+.04)1.4 = .31$. If this is multiplied by 162, the number of games in a season, this brings his value to the team to about 50 runs during the season, in rough agreement with the sabermetricians' value for the Runs Above Average Pitcher([8]) of a very good pitcher, again establishing a nice compatibility among the APBA card makers, my analysis, and sabermetrics.

If we wish to evaluate a batter's card by the sum of the values of its batting numbers, under the system in which 1 (a home run number) has 1.41, 13 (a strikeout number) $-.23$, and Hank Aaron a total of 3.45, then the value of an A pitcher with a Z would be $.31 \cdot 36/4.5$ (because the sum of the batting numbers didn't take into account the batter's 4.5 at bats per game). So the A pitcher with a Z is worth 2.48 to the team if Hank Aaron's batting is worth 3.44. Aaron's speed and fielding would make him even more valuable.

A similar analysis is made for the other pitching grades and also for fielding. It turns out that the difference between a fielder having the best possible fielding rating for his position and the worst possible, on a scale such as that of the previous paragraph comparable to the sum of the values of the batting numbers on a card, is approximately 1 at each position. Tables for all these appear in [3], as does an annotated version of the `Maple` program which was used to perform the calculations.

In conclusion, the analysis described above, accompanied by the detailed results in [3], should enable APBA players to compare teams and to make informed decisions about strategy. The method of matrix analysis employed here provides a paradigm for analyzing games of a certain type. We might hope that it would motivate some students to learn about matrix equations and ideas related to Markov chains.

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