REAL PROJECTIVE SPACE AS A SPACE OF PLANAR POLYGONS

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ABSTRACT. We describe an explicit homeomorphism between real projective space $\mathbb{RP}^{n-3}$ and the space $\mathcal{M}_{n,n-2}$ of all isometry classes of $n$-gons in the plane with one side of length $n - 2$ and all other sides of length 1. This makes the topological complexity of real projective space more relevant to robotics.

1. Introduction

The topological complexity, $TC(X)$, of a topological space $X$ is, roughly, the number of rules required to specify how to move between any two points of $X$.([4]) This is relevant to robotics if $X$ is the space of all configurations of a robot.

A celebrated theorem in the subject states that, for real projective space $\mathbb{RP}^n$ with $n \neq 1, 3, \text{or } 7$, $TC(\mathbb{RP}^n)$ is 1 greater than the dimension of the smallest Euclidean space in which $\mathbb{RP}^n$ can be immersed.([5]) This is of interest to algebraic topologists because of the huge amount of work that has been invested during the past 60 years in studying this immersion question. See, e.g., [6], [9], [1], and [2]. In the popular article [3], this theorem was highlighted as an unexpected application of algebraic topology.

But, from the definition of $\mathbb{RP}^n$, all that $TC(\mathbb{RP}^n)$ really tells is how hard it is to move efficiently between lines through the origin in $\mathbb{R}^{n+1}$, which is probably not very useful for robotics. Here we show explicitly how $\mathbb{RP}^n$ may be interpreted to be the space of all polygons of a certain type in the plane. The edges of polygons can be thought of as linked arms of a robot, and so $TC(\mathbb{RP}^n)$ can be interpreted as telling how many rules are required to tell such a robot how to move from any configuration to any other.

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Let $M_{n,r}$ denote the moduli space of all oriented $n$-gons in the plane with one side of length $r$ and the rest of length 1, where two such polygons are identified if one can be obtained from the other by an orientation-preserving isometry of the plane. These $n$-gons allow sides to intersect. Since any such $n$-gon can be uniquely rotated so that its $r$-edge is oriented in the negative $x$-direction, we can fix vertices $x_0 = (0, 0)$ and $x_{n-1} = (r, 0)$ and define

$$M_{n,r} = \{ (x_1, \ldots, x_{n-2}) : d(x_{i-1}, x_i) = 1, \ 1 \leq i \leq n-1 \}.$$ 

Here $d$ denotes distance between points in the plane.

Most of our work is devoted to proving the following theorem.

**Theorem 1.2.** If $n-2 \leq r < n-1$, then there is a $\mathbb{Z}/2$-equivariant homeomorphism $\Phi : M_{n,r} \to S^{n-3}$, where the involutions are reflection across the $x$-axis in $M_{n,r}$, and the antipodal action in the sphere.

Taking the quotient of our homeomorphism by the $\mathbb{Z}/2$-action yields our main result. It deals with the space $\overline{M}_{n,r}$ of isometry classes of planar $(1^{n-1}, r)$-polygons. This could be defined as the quotient of (1.1) modulo reflection across the $x$-axis.

**Corollary 1.3.** If $n-2 \leq r < n-1$, then $\overline{M}_{n,r}$ is homeomorphic to $\mathbb{R}P^{n-3}$.

These results are not new. It was pointed out to the author after preparation of this manuscript that the result is explicitly stated in [8, Example 6.5], and proved there, adapting an argument given much earlier in [7]. The result of our Corollary 1.3 was also stated as “well known” in [10]. Nevertheless, we feel that our explicit, elementary homeomorphism may be of some interest.

### 2. Proof of Theorem 1.2

In this section we prove Theorem 1.2. Let $J^m$ denote the $m$-fold Cartesian product of the interval $[-1, 1]$, and $S^0 = \{ \pm 1 \}$. Our model for $S^{n-3}$ is the quotient of $J^{n-3} \times S^0$ by the relation that if any component of $J^{n-3}$ is $\pm 1$, then all subsequent coordinates are irrelevant. That is, if $t_i = \pm 1$, then

$$t_1, \ldots, t_i, t_{i+1}, \ldots, t_{n-2} \sim t_1, \ldots, t_i, t'_{i+1}, \ldots, t'_{n-2}$$

(2.1)
for any $t'_{i+1}, \ldots, t'_{n-2}$. This is just the iterated unreduced suspension of $S^0$, and the antipodal map is negation in all coordinates. An explicit homeomorphism of this model with the standard $S^{n-3}$ is given by

$$(t_1, \ldots, t_{n-2}) \leftrightarrow (x_1, \ldots, x_{n-2}),$$

with

$$x_i = t_i \prod_{j=1}^{i-1} \sqrt{1 - t_j^2}, \quad t_i = \frac{x_i}{\sqrt{1 - x_1^2 - \cdots - x_{i-1}^2}} \text{ if } \sum_{j=1}^{i-1} x_j^2 < 1.$$ 

Then $t_i = \pm 1$ for the smallest $i$ for which $x_1^2 + \cdots + x_i^2 = 1$.

Let $P \in M_{n,r}$ be a polygon with vertices $x_i$ as in (1.1). We will define the coordinates $t_i = \phi_i(P)$ of $\Phi(P)$ under the homeomorphism $\Phi$ of Theorem 1.2.

For $0 \leq i \leq n-2$, we have

$$(2.2) \quad n - 2 - i \leq d(x_i, x_{n-1}) \leq n - 1 - i.$$ 

The first inequality follows by induction on $i$ from the triangle inequality and its validity when $i = 0$. The second inequality also uses the triangle inequality together with the fact that you can get from $x_i$ to $x_{n-1}$ by $n - 1 - i$ unit segments. The second inequality is strict if $i = 0$ and is equality if $i = n - 2$. Let $i_0$ be the minimum value of $i$ such that equality holds in this second inequality. Then the vertices $x_{i_0}, \ldots, x_{n-1}$ must lie evenly spaced along a straight line segment.

Let $C(x, t)$ denote the circle of radius $t$ centered at $x$. The inequalities (2.2) imply that, for $1 \leq i \leq i_0$, $C(x_{n-1}, n - 1 - i)$ cuts off an arc of $C(x_{i-1}, 1)$, consisting of points $x$ on $C(x_{i-1}, 1)$ for which $d(x, x_{n-1}) \leq n - 1 - i$. Parametrize this arc linearly, using parameter values $-1$ to $1$ moving counterclockwise. The vertex $x_i$ lies on this arc. Set $\phi_i(P)$ equal to the parameter value of $x_i$. If $i = i_0$, then $\phi_i(P) = \pm 1$, and conversely.

The following diagram illustrates a polygon with $n = 7$, $r = 5.2$, and $i_0 = 5$. We have denoted the vertices by their subscripts. The circles from left to right are $C(x_i, 1)$ for $i$ from 0 to 4. The arcs centered at $x_6$ have radius 1 to 5 from right to left. We have, roughly, $\Phi(P) = (.7, .6, .5, -.05, 1)$. 
Here is another example, illustrating how the edges of the polygon can intersect one another, and a case with $i_0 < n - 2$. Again we have $n = 7$ and $r = 5.2$. This time, roughly, $\Phi(\mathcal{P}) = (.2, -4, .4, 1, t_5)$, with $t_5$ irrelevant. Because $i_0 = 4$, we did not draw the circle $C(x_4, 1)$.

That $\Phi$ is well defined follows from (2.1); once we have $t_i = \pm 1$, which happens first when $i = i_0$, subsequent vertices are determined and the values of subsequent $t_j$ are irrelevant. Continuity follows from the fact that the unit circles vary continuously with the various $x_i$, hence so do the parameter values along the arcs cut off. Bijectivity follows from the construction; every set of $t_i$’s up to the first $\pm 1$ corresponds to a unique polygon, and $\pm 1$ will always occur. Since it maps from a compact space to a Hausdorff space, $\Phi$ is then a homeomorphism. Equivariance with respect to the
involution is also clear. If you flip the polygon, you flip the whole picture, including
the unit circles, and this just negates all the $t_i$'s.

We elaborate slightly on the surjectivity of $\Phi$. The arc on $C(x_0,1)$ cut off by
$C(x_{n-1},n-2)$ is determined by $n$ and $r$. Given a value of $t_1$ in $[-1,1]$, the vertex
$x_1$ is now determined on this arc. Now the arc on $C(x_1,1)$ cut off by $C(x_{n-1},n-3)$
is determined, and a specified value of $t_2$ determines the vertex $x_2$. All subsequent
vertices of an $n$-gon are determined in this manner.

REFERENCES

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