Some Sharp Isoperimetric Theorems for Riemannian Manifolds

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ABSTRACT. We prove that a region of small prescribed volume in a smooth, compact Riemannian manifold has at least as much perimeter as a round ball in the model space form, using differential inequalities and the Gauss-Bonnet-Chern theorem with boundary term. First we show that a minimizer is a nearly round sphere. We also provide some new isoperimetric inequalities in surfaces.

1. INTRODUCTION

1.1 Sharp lower bounds on perimeter. Let M be an $(n+1)$-dimensional, smooth, compact, connected Riemannian manifold. Our main Theorem 4.4 says that if for example the sectional curvature K is less than K_0, then an enclosure of small volume V has at least as much perimeter P as a round sphere of the same volume in the model space form of curvature K_0. The proof follows Kleiner's use of the Gauss-Bonnet theorem in dimension three [K], and depends on knowing that minimizers are nearly round spheres (see 1.2 below). First one estimates the mean curvature H of a minimizer by an application of the Gauss-Bonnet-Chern theorem with boundary. Since dP/dV is a multiple of H, integration yields the desired result.

For a very convex region, the primary term of the Gauss-Bonnet-Chern boundary integrand Φ is just the product $\kappa_1 \cdots \kappa_n$ of the principal curvatures, which is majorized by a power H^n of the mean curvature. The secondary term helpfully involves the sectional curvature. Further terms unfortunately involve general components of the Riemannian curvature tensor, which are controlled by Lemma 4.2.

By way of comparison, Aubin [Au2, Conj. 1] conjectures for example that in a *simply connected* (complete) manifold *of nonpositive curvature* $K \leq K_0 \leq 0$, a least-perimeter enclosure of *any volume* has at least as much perimeter as in the space form of curvature K_0. This has been proved only in ambients of dimension two (as follows from (1.1) below) and three (Kleiner [K], 1992). For the special case of $K_0 = 0$, the conjecture has been proved for ambients of dimension four (Croke [Cr], 1984). Our restriction to small volume is of course necessary for $K_0 > 0$, as shown by a long cylinder, feeding into a large sphere at one or both ends.

The best previous lower bound on least perimeter P for small volume V seems to be that of Bérard and Meyer [BéM, Lemma, p. 514]:

$$P \geq (1 - CV^{2/(n+1)(n+4)})P^*,$$

where P^* is the perimeter of the Euclidean ball of volume V. Our results imply for example that

$$P \geq (1 - C'K_0V^{2/(n+1)})P^*.$$

1.2 Small isoperimetric regions are spheres. Theorem 2.2 proves that a least-perimeter enclosure S of small volume is a (nearly round) sphere. (This result was known only in the relatively trivial case when the ambient M is two dimensional.) This is of course well known at the infinitesimal level, i.e., in Euclidean space. Our proof is a compactness argument. The main difficulties are to show that S lies in a small ball and to show that the convergence to the limit is smooth. These results follow by so-called "monotonicity of mass ratio" and the Allard regularity theorem [A], given bounds on the mean curvature. Such bounds follow from the Heintze-Karcher [HK] estimate on enclosed volume in terms of mean curvature.

1.3 Sharp upper bounds on least perimeter. If the sectional curvature K of the ambient manifold M satisfies an opposite inequality $K \geq K_0$, Theorem 3.4 shows that a least-perimeter enclosure of volume V has at most as much perimeter P as a round sphere of the same volume in the model space form of curvature K_0, with equality only if $K = K_0$. It comes from integrating a second order differential inequality [BP]:

$$P(V)P''(V) \leq \frac{-P'(V)^2}{n} - nK_0,$$

as for example in Bray [Br, Section 2.1].

The hypothesis $K \geq K_0$ may be relaxed to the hypothesis that the Ricci curvature is at least nK_0.

Actually, Theorem 3.5 deduces from a strong form of Bishop's theorem that any metric ball is isoperimetrically superior to the model's.

1.4 Isoperimetric inequalities for surfaces. The standard Bol-Fiala inequality [Os2, 4.25] for a smooth Riemannian surface of Gauss curvature $K \leq K_0$ says that the perimeter P and area A of a disc satisfy

$$P^2 \geq 4\pi A - K_0 A^2. \tag{1.1}$$

Proposition 5.2 proves the corresponding inequality for the area of some region bounded by a given curve of perimeter P:

$$A \leq \frac{2\pi - \sqrt{4\pi^2 - K_0 P^2}}{K_0} \quad (K_0 \neq 0),$$

if the ambient surface is compact or convex at infinity, P is less than the length of any closed geodesic, and $K_0 P^2 \leq 4\pi^2$. These additional hypotheses are necessary. The curve may, however, have several components.

For a simply connected surface with some boundary convexity, Theorem 5.3 proves the following sharp generalization of the Bol-Fiala inequality (1.1) from discs to regions of any topological type:

$$P^2 \geq \min\{(2L_0)^2,\ 4\pi A - K_0 A^2\},$$

where L_0 is the infimum of lengths of simple closed geodesics.

The proofs use Grayson's curve shortening [Gr], as applied to isoperimetric estimates by Benjamini and Cao [BeC].

1.5 References. There has been much recent work on the isoperimetric problem. See for example [BaP], [K], [Pan], [PeR], [RR]. Surveys are provided by [Os2], [BuZ], and [HHM].

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2. SMALL ISOPERIMETRIC REGIONS ARE SPHERES

2.1 Second variation formula. We will need a second variation formula [BaP, Section 7] for the area of a smooth, compact hypersurface S of constant mean curvature enclosing volume V in a smooth Riemannian manifold under a unit normal variation ν:

$$P''(V) = -P^{-2} \int_S |II|^2 + \mathrm{Ric}(\nu, \nu). \tag{2.1}$$

This holds essentially because $P'(V) = P^{-1} \int nH$ and

$$n \frac{dH}{dt} = -|II|^2 - \mathrm{Ric}(\nu, \nu),$$

where $n = \dim S$ and H is the inward mean curvature $(\kappa_1 + \kappa_2 + \cdots + \kappa_n)/n$; see [Br, Section 2.1].

If S has a compact singular set S_0 of k-dimensional Hausdorff measure 0, with $k = \dim S - 2$, then there are variations vanishing on a neighborhood of S_0 with $P''(V)$ close to (2.1) (implying a similar result attributed in [BaP, Section 7] to Bérard, Besson, and Gallot). Indeed, S_0 may be covered by finitely many balls $\mathbb{B}(p_i, r_i)$ with $\sum r_i^k$ small. Choose smooth functions $\varphi_i : S \to [0,1]$ such that φ_i vanishes on $\mathbb{B}(p_i, 2r_i)$, $\varphi_i = 1$ on $S \setminus \mathbb{B}(p_i, 3r_i)$, $|\nabla \varphi_i| \le C r_i^{-1}$, $|D^2 \varphi_i| \le C r_i^{-2}$. Then $(\Pi \varphi_i)\nu$ gives the desired variation, with negligible contribution to $P''(V)$ from the small set where $\varphi = \Pi \varphi_i \ne 1$. (Actually some care is required to control $|D^2 \varphi|$ where many balls overlap. For details see [MR, Lemma 3.1].)

Theorem 2.2. *Let M be a smooth, compact Riemannian manifold. Then a least-perimeter enclosure of small volume is a* (*nearly round*) *sphere.*

By "nearly round," we mean that rescalings to unit volume are smoothly close to the Euclidean sphere of unit volume.

Proof. Let $n = \dim M - 1$. We may assume that M is connected. For given volume $0 < V < \mathrm{vol}(M)$, an enclosure S of least perimeter $P(V)$ exists and is a smooth hypersurface of constant inward mean curvature $H = (\kappa_1 + \kappa_2 + \cdots + \kappa_n)/n$, except possibly for a compact singular set of Hausdorff dimension at most $n - 7$ [M2, 8.6]. We will use the Heintze-Karcher inequality for the volume V^* inside a smooth separating hypersurface S of mean curvature at least H^* [BuZ, 34.1.10(11)], [HK, Theorem 2.1]:

$$V^* \le \int_0^r [c(t) - H^* s(t)]^n \, dt \quad \mathrm{area}\,(S), \tag{2.2}$$

where r is the radius of the largest ball enclosed by S and $c(t)$ and $s(t)$ are bounded nonnegative functions depending only on a lower bound on the Ricci curvature of M. This formula and its derivation applies to our minimizer S even if S has singularities, because for any point off S, the nearest point on S is a regular point, because the tangent cone lies in a halfspace and hence must be a plane.

For V small, applying (2.2) to the complementary region with $V^* = \operatorname{vol}(M) - V$ and $H^* = -H$ shows that the integral blows up and hence

(2.3) $$H \text{ must be very large.}$$

Now applying (2.2) to the original region with $V^* = V$ and $H^* = H$ shows that

$$r \geq \frac{C_1^{-1} V}{P(V)}$$

for some constant C_1. Since $r \leq C_2/H$, it follows that for small V,

(2.4) $$H \leq \frac{C_3 P(V)}{V}.$$

Also by (2.3) the second variation (2.1) is negative for small V. It follows that for small V a minimizer has just one component. Otherwise expanding one and shrinking the other so as to preserve volume would reduce perimeter.

For V small, by the standard isoperimetric inequality [M2, p. 117] and comparison with small, nearly Euclidean balls,

(2.5) $$C^{-1} V^{n/(n+1)} \leq P(V) \leq C V^{n/(n+1)}.$$

Now consider a sequence of minimizers of volume $V_i \to 0$, perimeter P_i, and mean curvature H_i. Scale up M to M_i by a factor $V_i^{-1/(n+1)}$ to yield minimizers of unit volume, perimeter $P_i V_i^{-n/(n+1)} \leq C$ by (2.5), and mean curvature

(2.6) $$h_i = H_i V_i^{1/(n+1)} \leq \frac{C_3 P_i}{V_i} V_i^{1/(n+1)} \leq C_3 C$$

by (2.4) and (2.5). By the Nash embedding theorem [N], we may assume that M is a smooth submanifold of some Euclidean space $\mathbb{R}^N$. Since all scalings of M have a uniform bound on the second fundamental tensor, by (2.6) the mean curvatures in $\mathbb{R}^N$ are bounded. Since the perimeters are bounded, by "monotonicity of the mass ratio" [A, 5.1(3)], each minimizer lies in a ball B_i of fixed radius r_0. We may assume that all these balls are the ball about the origin in $\mathbb{R}^N$ and that each scaling of M is tangent to $\mathbb{R}^{n+1} \times \{0\}$ there. By compactness [M2, 5.5] we may assume that the minimizers converge weakly to a solution in $\mathbb{R}^{n+1} \times \{0\}$, namely a round sphere of unit volume. Since mean curvature is bounded, $C^{1,\alpha}$ convergence follows by Allard's regularity theorem [A, Section 8]. Hence eventually the minimizers are nearly round spheres.

To obtain higher order convergence, locally view each minimizer and the limit as graphs of functions with difference u_i. Since each surface has constant mean curvature h_i in its ambient M_i, each u_i satisfies a system of second order partial differential equations of the form

$$a_{ijk}D_{jk}u_i + b_{ijk} = 0. \tag{2.7}$$

The variable coefficients a_{ijk} and b_{ijk} depend on u_i, Du_i, M_i, and h_i. (The derivation of [Os3, Theorems 2.1 and 2.2], for the case of mean curvature 0, generalizes immediately to prescribed mean curvature. Mean curvature in $\mathbb{R}^N$ differs from mean curvature in the ambient by an expression involving Du_i and the second fundamental tensor of M_i in $\mathbb{R}^N$.) The system is uniformly elliptic; indeed a_{ijk} can be approximately δ_{jk}. The coefficients a_{ijk} and b_{ijk} are uniformly Hölder continuous. Since the Du_i approach 0, it follows by Schauder estimates (e.g. [LU, Chapter 3, 1.12 and 1.13]) that in a shrunken domain the second derivatives D^2u_i are Hölder continuous and approach 0.

Technically such estimates require that u_i vanish on the boundary of the domain. So just alter u_i by a harmonic function φ_i. Since the boundary values are Hölder continuous and small, $D^2\varphi_i$ is Hölder continuous and small and does not disturb the argument.

Finally, a standard bootstrap argument, differentiating (2.7) and reapplying the Schauder estimates, shows that all derivatives of u_i of all orders approach 0. □

Remark. Theorem 2.2 does not generalize to connected constant-mean-curvature hypersurfaces of small area, as shown by two nearly tangent spheres joined by a thin neck, as in surfaces of Delaunay. G. Huisken suggests it may generalize to *stable* connected constant-mean-curvature hypersurfaces.

3. SHARP UPPER BOUNDS ON LEAST PERIMETER

Proposition 3.3 establishes some properties of the isoperimetric profile which we will need in Section 4. As an application, Theorem 3.4 provides a sharp upper bound on least perimeter in general dimension, although the stronger Theorem 3.5 follows alternatively from a strong form of Bishop's theorem. We begin with some useful facts about the perimeter of geodesic balls.

3.1 Perimeter of geodesic balls. *Let p be a point of scalar curvature $R > R_0$ in a smooth Riemannian manifold. Then for small prescribed volume, a geodesic ball about p has less perimeter than a round ball in the model space form of scalar curvature R_0.* This follows from the asymptotic formulas for the perimeter P and volume V of geodesic balls about a point p of scalar curvature R (e.g. [M3, (6.10)]):

$$\begin{aligned}
P &= (n+1)\alpha_{n+1} r^n \left(1 - \frac{R}{6(n+1)} r^2 + O(r^4)\right), \\
V &= \alpha_{n+1} r^{n+1} \left(1 - \frac{R}{6(n+3)} r^2 O(r^4)\right), \\
\frac{P^{n+1}}{V^n} &= (n+1)^{n+1}\alpha_{n+1} \left(1 - \frac{R}{2(n+3)} r^2 + O(r^4)\right),
\end{aligned} \tag{3.1}$$

where $n + 1$ is the dimension and α_{n+1} is the volume of the unit ball in $\mathbb{R}^{n+1}$. (We remark that $R \geq R_0$ does not imply that a small geodesic ball has no more perimeter than the round model, as shown by examples, such as $\mathbb{S}^m \times (\mathbb{S}^1)^m$ for $m \geq 3$, easily computable by (3.3) below.)

Thus the inequality $R > R_0$ gives an upper bound on the least perimeter for small volume. We do not know whether the opposite inequality $R < R_0$ yields a lower bound on least perimeter, because perimeter minimizers are generally better than geodesic balls, though probably not by much. For example, even in the singular case of two unit Euclidean discs identified along their boundaries, the isoperimetric ratio satisfies

$$\frac{L^2}{A} = 4\pi \left(1 - \frac{2}{3\pi^2} L + BL^2 + \cdots\right),$$

where $B = -11/36\pi^4$ for a small geodesic circle centered on the seam and $B = -5/9\pi^4$ for two circular arcs perpendicular at the seam (the presumptive minimizer).

We note for future reference that for geodesic balls

$$\lim_{P \to 0} P^{2/n} P'^2 = n^2 |\mathbb{S}^n|^{2/n}, \tag{3.2}$$

where $|\mathbb{S}^n|$ denotes the area of the unit Euclidean sphere.

Actually the coefficient of the next, r^4, term in (3.1) is given in orthonormal coordinates for the tangent space by

$$\frac{5R^2 + 8\sum R_{ij}^2 - 3\sum R_{ijk\ell}^2 - 18\Delta R}{360(n+1)(n+3)}, \tag{3.3}$$

where the sums in the second and third terms are just the squares of the L^2 norms of the Ricci and Riemannian curvature tensors. Aubin [Au1, Lemma 1, p. 270] elegantly derives two such formulas, with the help of the first and contracted second Bianchi identities, from an asymptotic expression for the determinant of the metric [Ptr, Section 7, (7.17) and Exercise 3]; cf. Lee and Parker [LeP, Lemma 5.5 and p. 68]. Alternatively, the coefficient is given by the average over all directions v of

$$\frac{5\mathrm{Ric}^2(v,v) - 2\sum K_i^2 - 9\mathrm{Ric}''}{360}, \tag{3.4}$$

where K_i are "principal" sectional curvatures of sections containing v (maximizing $\sum K_i^2$). Because of the occurrence of the "principal" curvatures, it is hard to verify directly that (3.3) and (3.4) are equivalent. The derivative terms can be shown equal with the help of the contracted second Bianchi identity, but the rest do not correspond term by term. We have checked directly that they agree on all two-dimensional surfaces and on $(n+1)$-dimension surfaces of revolution. The derivation of (3.4) is quite short.

Derivation of (3.4)**.** The stretch f in direction w of the exponential map of the unit Euclidean sphere to the geodesic sphere in the manifold along a ray in the direction of a unit vector v depends on the sectional curvature K of $w \wedge v$. Indeed, it satisfies Jacobi's equation [Cha, (2.43)] $f'' + Kf = 0$. Since $f(0) = 0$ and $f'(0) = 1$,

$$f = r\left(1 - K\frac{r^2}{3!} - 2K'\frac{r^3}{4!} + (K^2 - 3K'')\frac{r^4}{5!} + O(r^6)\right).$$

Hence the Jacobian in terms of eigendirections $w = e_i$ with curvatures K_i of sections $e_i \wedge v$ is

$$r^n\left(1 - \sum K_i\frac{r^2}{3!} - 2\sum K_i'\frac{r^3}{4!} + \sum K_iK_j\frac{r^4}{36} + \left(\sum K_i^2 - 3K_i''\right)\frac{r^4}{5!} + O(r^5)\right).$$

Thus the area equals

$$|\mathbb{S}^n|r^n\left(1 - R\frac{r^2}{6(n+1)} + Br^4 + O(r^5)\right),$$

where B is the average over all v of

$$\frac{10\sum K_iK_j + 3\sum K_i^2 - 9\sum K_i''}{360}, \tag{3.5}$$

which is equal to (3.4).

Lemma 3.2. *Let $f : (a, b) \to \mathbb{R}$ be continuous. Then f is convex if and only if for every $x_0 \in (a, b)$ there is a smooth convex function $g \le f$ with $g(x_0) = f(x_0)$.*

Proof. If f is convex, just take g to be linear. Suppose f is not convex. Choose $\epsilon > 0$ such that $f_\epsilon(x) = f(x) + \epsilon x^2$ is not convex. Choose $x_1 < x_3$ such that the graph of f_ϵ over $[x_1, x_3]$ goes above the line $\ell(x)$ from $(x_1, f_\epsilon(x_1))$ to $(x_3, f_\epsilon(x_3))$. Choose $x_1 < x_2 < x_3$ to maximize $f_\epsilon(x_2) - \ell(x_2)$. By hypothesis, there is a smooth convex function $g \le f$ with $g(x_2) = f(x_2)$. Then $g_\epsilon(x) = g(x) + \epsilon x^2 \le f_\epsilon(x)$, $g_\epsilon(x_2) = f_\epsilon(x_2)$, and $g''_\epsilon(x_2) \ge 2\epsilon$, a contradiction. □

Proposition 3.3. *Let M be a smooth, compact, connected $(n + 1)$-dimensional Riemannian manifold. Then the least perimeter P as a function of prescribed volume V is absolutely continuous and twice differentiable almost everywhere. The left and right derivatives $P'_L \ge P'_R$ exist everywhere and*

(3.6) their singular parts are nonincreasing.

Indeed, locally there is a constant C_0 such that $P(V) - C_0V^2$ is concave.

Moreover, if nK_0 is a lower bound on the Ricci curvature of M, then almost everywhere

$$PP'' \le \frac{P^2}{n} - nK_0, \tag{3.7}$$

with equality in the simply connected space form of constant sectional curvature K_0. If equality holds, then a perimeter minimizer is totally umbilic.

Proof. For given volume $0 < V_0 < \mathrm{vol}\,(M)$, an enclosure S of least perimeter $P(V_0)$ exists and is a smooth hypersurface of constant mean curvature $H = (\kappa_1 + \cdots + \kappa_n)/n$, except possibly for a singular set of Hausdorff dimension at most $n - 7$ [M2, p. 87]. Of course if $P_0(V)$ denotes the perimeter from smooth, one-sided perturbations of S, then $P(V) \le P_0(V)$, with equality at V_0. $P'_0(V_0) = H$, and by (2.1),

$$P''_0(V_0) \le -\frac{P'_0(V_0)^2/n + nK_0}{P(V_0)}$$

because $|II_0|^2 \ge nH^2$. If equality holds, then S is totally umbilic. By Lemma 3.2, for $0 < V_1 \le V \le V_2 < \mathrm{vol}\,(M)$, there is a constant $C_0(V_1, V_2)$ such that $P(V) - C_0V^2$ is concave. The proposition follows, with the additional observation that $P'_L \ge P'_0 \ge P'_R$. □

As a first application of Proposition 3.3, Theorem 3.4 after Bray [Br, Sections 2.1, 3.3] provides a sharp upper bound for a least-perimeter enclosure in terms of the Ricci curvature of the ambient. It depends on Bishop's theorem (which Bray's more sophisticated argument reproves). Actually, Theorem 3.4 and more follow immediately from strong forms of Bishop's theorem, as proved in Theorem 3.5 below.

The case of two-dimensional ambients appears in [MHH, Theorem 2.7].

Theorem 3.4. *Let M be a smooth, compact, connected $(n+1)$-dimensional Riemannian manifold with Ricci curvature* $\mathrm{Ric} \geq nK_0$. *For given volume V, let R be a region of minimum perimeter $P(V)$. Then $P(V)$ is no greater than the perimeter $P_0(V)$ of a ball B_V of volume V in the simply connected space form M_0 of constant curvature K_0.*

Suppose that for some $V_0 > 0$, $P(V_0) = P_0(V_0)$. Then for $V < V_0$, $P(V) = P(V_0)$ and minimizers are totally umbilic. Moreover, M has constant Ricci curvature nK_0.

Remarks. As in Bray [Br, Section 2.1] and suggested to us by M. Ritoré, when $n = 2$, the hypothesis $\mathrm{Ric} \geq nK_0$ on the Ricci curvature may be relaxed to a hypothesis $R \geq n(n+1)K_0$ on the scalar curvature together with $\mathrm{Ric} > 0$, with the added restriction for $K_0 > 0$ that $V \leq (\mathrm{vol}\,(M_0))/2$. (In the conclusion for the case of equality, M has constant *scalar* curvature.) The assumption of positive Ricci curvature, needed to guarantee at one point in the argument that ∂R is connected, is not necessary for small volumes by Theorem 2.2. The restriction for $K_0 > 0$ to $V \leq (\mathrm{vol}\, M_0)/2$ is needed because Bishop's theorem no longer guarantees vol $M \leq \mathrm{vol}\,(M_0)$ (but see Bray [Br, Section 3.3]). Such generalizations probably fail for general n, probably in $\mathbb{S}^3 \times (\mathbb{S}^1)^3$, where small geodesic spheres have relatively large perimeter (see Section 3.1), though isoperimetric surfaces would be some unknown amount smaller; the isoperimetric profile is not known exactly for nonsymmetric examples.

Proof. First suppose $K_0 > 0$, so that $P(V)$ is concave by (3.7), and of course $P(\mathrm{vol}\,(M) - V) = P(V)$; similarly for P_0 and $\mathrm{vol}\,(M_0)$. Since vol $M \leq \mathrm{vol}\,(M_0)$ by Bishop's theorem, it suffices to consider $0 < V < (\mathrm{vol}\,(M))/2$, where $P'(V)$ and $P_0'(V)$ are positive. For almost all V,

$$\frac{d(P^{2/n}P'^2)}{dP} = 2P^{-1+2/n}\left(\frac{P'^2}{n} + PP''\right) \leq -2nK_0P^{-1+2/n}$$

by (3.7), (3.6). In particular, $\lim_{P\to 0} P^{2/n}P'^2$ exists. By comparison with geodesic balls (3.2),

$$\lim_{P\to 0} P^{2/n}P'^2 \le n^2|\mathbb{S}^n|^{2/n}.$$

Because P' and P_0' are positive, it follows by (3.6) that

$$P^{2/n}P_2' \le -n^2K_0P^{2/n} + n^2|\mathbb{S}^n|^{2/n},$$

with equality for geodesic balls in the space form (sphere). Therefore $P(V) \le P_0(V)$.

Second suppose $K_0 \le 0$, so that $P_0'(V)$ is positive and decreasing. Initially $P'(V)$ is positive (for example by Theorem 2.2 or the Heinze-Karcher inequality (see (2.3)), and the argument is as before. Suppose $P'(V)$ goes negative and $P \not\le P_0$. If V_0 is the last time $P(V_0) \le P_0(V_0)$, then $P_R'(V_0) > P_0'(V_0)$. Decrease V until $P_R'(V_1) = P_0'(V_2)$, where V_2 is determined so that $P(V_1) = P_0(V_2)$; $V_2 < V_1$. Now by comparison of P with a translate of P_0, $P(V_0) < P_0(V_2 + (V_0 - V_1)) \le P_0(V_0)$, the desired contradiction.

Finally suppose that for some $V_0 > 0$, $P(V_0) = P_0(V_0)$. We claim that for all $V \le V_0$, $P(V) = P_0(V)$. As before, the only interesting case is $K_0 \le 0$, $P' \not> 0$. We may consider an interval (V_3, V_0) such that $P_R'(V_3) = 0$, $P(V) < P_0(V)$ on (V_3, V_0), and $P(V_0) = P_0(V_0)$. Somewhere on that interval $P_R'(V) > P_0'(P_0^{-1}(P(V)))$; hence somewhere on that interval $P_R'(V_1) = P_0'(P_0^{-1}(P(V_1)))$. Now a comparison as before of P with a translate of P_0 shows that $P(V_0) < P_0(V_0)$, the desired contradiction. Now by Proposition 3.3, small minimizers are totally umbilic.

By section 3.1, the scalar curvature of M satisfies $R \le R_0$. Since by hypothesis $R \ge (n+1)(nK_0) = R_0$, equality must hold and the Ricci curvature must be nK_0 everywhere. ☐

Theorem 3.5 improves Theorem 3.4 by showing that under a Ricci curvature bound, all metric balls are isoperimetrically superior to the model's. The proof uses a strong form of Bishop's theorem, which says that the ratio of perimeter in the surface to perimeter in the model is a nonincreasing function of radius [Cha, Prop. 3.3].

Theorem 3.5. *Let M be a smooth, compact, connected $(n+1)$-dimensional Riemannian manifold with Ricci curvature* Ric $\ge nK_0$. *Then any metric ball has no more perimeter than a ball of the same volume in the model space form of constant curvature K_0. If equality holds, the ball is isometric to the model's.*

Proof. Fix a point in M. By the strong form of Bishop's theorem [Cha, Prop. 3.3], the ratio of the perimeter P of a metric ball of radius r about the fixed point in M to the perimeter P_0 of a ball of radius r in the model is a nonincreasing function of r. In particular, for $r \leq r_0$,

$$\delta_0 \leq \frac{P(r)}{P_0(r)} \leq 1,$$

where

$$\delta_0 = \frac{P(r_0)}{P_0(r_0)}.$$

Hence volume satisfies

$$\delta_0 \leq \delta_1 = \frac{V(r_0)}{V_0(r_0)} \leq 1. \tag{3.8}$$

Since as a function of volume, P_0 is concave,

$$P_0(\delta_1 V_0(r_0)) \geq \delta_1 P_0(V_0(r_0)).$$

Therefore for volume $V(r_0) = \delta_1 V_0(r_0)$,

$$P_0 \geq \delta_1 P_0(r_0) \geq \delta_0 P_0(r_0) = P(r_0), \tag{3.9}$$

as desired. If equality holds, then by (3.9) $\delta_1 = \delta_0$. That equality in (3.8) implies that $\delta_1 = 1$. Now by equality for a more standard form of Bishop's theorem [Cha, Theorem 3.9], that the volume of a geodesic ball is no greater than in the model, the ball is isometric to the model's. □

4. SHARP LOWER BOUNDS ON PERIMETER

Theorem 4.4 derives sharp lower bounds on perimeter from an upper bound on the sectional curvature for regions of small volume in Riemannian manifolds. The proof depends on the Gauss-Bonnet-Chern formula and two algebraic lemmas.

4.1 Gauss-Bonnet-Chern formula. (Chern [Che], Allendoerfer and Weil [AW]; cf. Spivak [Sp, Vol. V, p. 573], [M3, Section 8.5], and the excellent expository undergraduate thesis of Hutchings [Hut]).

For a smooth, $(n+1)$-dimensional Riemannian manifold R with boundary,

$$\frac{2}{|\mathbb{S}^{n+1}|}\int_R G + \frac{1}{|\mathbb{S}^n|}\int_{\partial R}\Phi = \chi(R), \tag{4.1}$$

where $|\mathbb{S}^n|$ denotes the volume of the unit, Euclidean n-sphere, χ denotes the Euler characteristic, G is the Gauss-Bonnet-Chern integrand

$$G = \begin{cases} \text{ave}\,\{\pm R_{i_1 i_2 j_1 j_2}\cdots R_{i_n i_{n+1} j_n j_{n+1}}\} & n \text{ odd,} \\ 0 & n \text{ even,} \end{cases} \tag{4.2}$$

and Φ is the boundary integrand

$$\Phi = \sum C_m \text{ave}\Big\{ \pm\, \kappa_{i_1}\cdots\kappa_{i_m} R_{i_{m+1} i_{m+2} j_{m+1} j_{m+2}}\cdots R_{i_{n-1} i_n j_{n-1} j_n} : \\ n-m \text{ is even and } j_k = i_k \text{ for } k \le m\Big\} \tag{4.3}$$

In (4.2) and (4.3), $R_{ijk\ell}$ are the components of the Riemannian curvature tensor of R in an orthonormal basis and the $\pm$ sign depends on whether the two permutations i_k and j_k have the same parity. In (4.3), we assume an orthonormal basis in the directions of the inward principal curvatures κ_i. The $C_m(n)$ are absolute positive constants, with $C_n = 1$, so that for κ_i large, the leading term is simply $\kappa_1\cdots\kappa_n$ (the so-called Gauss-Kronecker curvature).

Chern writes these formulas in terms of connection forms $\omega_{i,n+1} = -\kappa_i\, dx_i$ and curvature forms $\Omega_{ij} = -R_{ijk\ell}\, dx_k\, dx_\ell$. (Actually Chern uses n where we have used $n+1$. In our formula (4.3), the constants are not so bad:

$$C_m = \binom{n/2}{m/2} = \frac{\frac{n}{2}\left(\frac{n}{2}-1\right)\cdots\left(\frac{m}{2}+1\right)}{\frac{n-m}{2}!},$$

which makes sense because $n-m$ is even.)

Remark. Formula (4.3) for the Gauss-Bonnet-Chern boundary integrand Φ may be rewritten

$$\Phi - \sum C_m \text{ave}\{\kappa_{i_1} \cdots \kappa_{i_m} G_{i_{m+1}\dots i_n}\}, \tag{4.4}$$

where $G_{i_{m+1}\dots i_n}$ is the $(n-m)$-dimensional Gauss-Bonnet-Chern integrand on the $e_{i_{m+1}} \dots e_{i_n}$ section of R. In a space form of constant curvature K_0, for n even, Φ takes the simple form

$$\Phi = \text{ave}\{\bar{K}_{i_1 i_2} \cdots \bar{K} i_{n-1} i_n\}, \tag{4.5}$$

where $\bar{K}_{ij}$ is the intrinsic sectional curvature of ∂R. For a round sphere of constant mean curvature H, this becomes

$$\Phi = (H^2 + K_0)^{n/2}$$

by the Gauss equations. For example, for a round sphere of radius r in hyperbolic space $\mathbb{H}^{n+1}$,

$$\Phi = (\coth^2 r - 1)^{n/2} = \operatorname{csch}^n r.$$

The Gauss-Bonnet formula, $(1/|\mathbb{S}^n|)\int_{\partial R}\Phi = 1$, implies that the area is $|\mathbb{S}^n|\sinh^n r$, which is correct.

For n odd, the Gauss-Bonnet-Chern formula is more complicated, involving the Gauss-Bonnet-Chern integrand G. For example, for a round sphere in $\mathbb{H}^4$ of radius r, area $|\mathbb{S}^3|\sinh^3 r$, and volume $V = \int |\mathbb{S}^3|\sinh^3 r\,dr$,

$$\Phi = H^3 - \frac{3}{2}H = \coth^3 r - \frac{3}{2}\coth r, \quad G = 1,$$

and the Gauss-Bonnet-Chern formula,

$$\frac{2}{|\mathbb{S}^4|}\int_R G + \frac{1}{|\mathbb{S}^3|}\int_{\partial R}\Phi = 1,$$

says that

$$\frac{3}{2}\int_0^r \sinh^3 t = 1 - \left(\cosh^3 r - \frac{3}{2}\cosh r \sinh^2 r\right),$$

a standard integration formula. (The first coefficient is given by $2|\mathbb{S}^3|/|\mathbb{S}^4| = 2(2\pi^2)/\frac{8}{3}\pi^2 = \frac{3}{2}$.)

Lemma 4.2. *Let $R_{ijk\ell}$ be the Riemannian curvature at a point in an orthonormal basis $e_1, \dots, e_n$ (cf. [M3, Section 5.3]). Suppose that the sectional curvature satisfies $K \le K_0$. Then the $R_{ijk\ell}$ other than $R_{ijij} = K_{ij}$ (and $R_{ijji} = -R_{ijij}$) satisfy*

$$|R_{ijk\ell}| \le 4\sup\{K_0 - K_{ij}\}.$$

Proof. Let $\epsilon = \sup\{K_0 - K_{ij}\} \geq 0$. By symmetries it suffices to show that $|R_{1213}| \leq \epsilon \leq 4\epsilon$ and $|R_{1234}| \leq 4\epsilon$. Note that (cf. [M3, (5.3)])

$$K_0 \geq K\left(e_1 \wedge \frac{e_2 \pm e_3}{\sqrt{2}}\right) = \frac{R_{1212}}{2} + \frac{R_{1313}}{2} \pm R_{1213} \geq \frac{K_0 - \epsilon}{2} + \frac{K_0 - \epsilon}{2} \pm R_{1213},$$

so that $|R_{1213}| \leq \epsilon$, as asserted. Similarly note that

$$\begin{aligned} K_0 &\geq K\left(\frac{e_1 + e_2}{\sqrt{2}} \wedge \frac{e_3 \pm e_4}{\sqrt{2}}\right) \\ &= \frac{R_{1313} + R_{1414} + R_{2323} + R_{2424}}{4} \\ &\quad + \frac{\pm R_{1314} \pm R_{1413} \pm R_{2324} \pm R_{2423} + R_{1323} + R_{2313} + R_{1424} + R_{2414}}{4} \\ &\quad \pm \frac{R_{1324} + R_{2314} + R_{1423} + R_{2413}}{4} \\ &\geq (K_0 - \epsilon) - 2\epsilon \pm \frac{R_{1324} + R_{1423}}{2}, \end{aligned}$$

so that

$$|R_{1234} - R_{1423}| = |R_{1324} + R_{1423}| \leq 6\epsilon.$$

Similarly $|R_{1342} - R_{1234}| \leq 6\epsilon$. These two inequalities, together with the first Bianchi identity [M3, (5.7)]

$$R_{1234} + R_{1342} + R_{1423} = 0$$

imply that $|R_{1234}| \leq 4\epsilon$, as desired. □

Lemma 4.3. *Consider a symmetric polynomial of the form*

$$f(\kappa_1, \ldots, \kappa_n) = \kappa_1 \cdots \kappa_n + C_1 \sum \kappa_{i_1} \cdots \kappa_{i_{n-1}} + C_2 \sum \kappa_{i_1} \cdots \kappa_{i_{n-2}} + \cdots .$$

Let $H = (\kappa_1 + \cdots + \kappa_n)/n$. *Then for some* κ_0, *for all* $\kappa_i \geq \kappa_0$,

$$f(\kappa_1, \ldots, \kappa_n) \leq f(H, \ldots, H),$$

with equality only if $\kappa_1 = \cdots = \kappa_n = H$.

Proof. The analogous statement for $\sum \kappa_{i_1} \cdots \kappa_{i_m}$, with $\kappa_0 = 0$, is standard, with an easy calculus proof (the analysis of equality requires $m > 1$).

Now choose κ_0, depending only on n and the C_i, such that

$$g(\kappa_1, \ldots, \kappa_n) = f(\kappa_1 + \kappa_0, \ldots, \kappa_n + \kappa_0)$$

has nonnegative coefficients. Then for $\kappa_i \geq \kappa_0$,

$$f(\kappa_1, \ldots, \kappa_n) = g(\kappa_1 - \kappa_0, \ldots, \kappa_n - \kappa_0) \leq g(H - \kappa_0, \ldots, H - \kappa_0) = f(H, \ldots, H),$$

with equality only as asserted. □

Our main Theorem 4.4 derives sharp lower bounds on perimeter from an upper bound on the sectional curvature. The proof uses the Gauss-Bonnet-Chern formula. The analysis of equality is due to Victor Bangert. The argument of Kleiner [K] does not yield the analysis of equality when $K_0 > 0$, because the rigidity results he uses [K, p. 42] do not hold for $K_0 > 0$.

For a two-dimensional ambient, the result holds for any disc by the Bol-Fiala inequality (1.1); for analysis of equality see [Os1, Cor. p. 9] or [BuZ, Section 2.2] or [MHH, Remarks after 4.3].

Theorem 4.4. *Let M be a smooth, compact $(n + 1)$-dimensional Riemannian manifold with sectional curvature K and Gauss-Bonnet-Chern integrand G. Suppose that*

(a) $K < K_0$, or
(b) $K \leq K_0$ and $G \leq G_0$,

where G_0 is the Gauss-Bonnet-Chern integrand of the model space form of constant curvature K_0. Then for small prescribed volume, the perimeter of a region R is at least as great as the perimeter $P_0(V)$ of a round ball B_V in the model, with equality only if R is isometric to B_V.

Of course for n even, $G = G_0 = 0$ and (b) reduces to $K \leq K_0$. We do not know whether the hypothesis $G \leq G_0$ is necessary for $n \geq 3$ odd.

Remarks. We do not know whether the strict inequality $K < K_0$ on the sectional curvature may be relaxed to a strict inequality $R < n(n + 1)K_0$ on the scalar curvature. The weaker hypothesis $R < n(n+1)K_0$ suffices for geodesic balls (see section 3.1), but minimizers are generally better than geodesic balls.

Although a nonstrict inequality $K \leq K_0$ probably suffices in all dimensions, a nonstrict inequality $\mathrm{Ric} \leq nK_0$ on the Ricci curvature does not suffice, even for $n = 3$. Indeed, $\mathbb{S}^2 \times \mathbb{S}^2$ has constant Ricci curvature 1, as in the model $\mathbb{S}^4(\sqrt{3})$, but for given small volume, even geodesic balls in $\mathbb{S}^2 \times \mathbb{S}^2$ have less perimeter than round balls in $\mathbb{S}^4(\sqrt{3})$, as can be deduced from the asymptotic expression for the perimeter $P(r)$ of a geodesic ball of radius r in $\mathbb{S}^2 \times \mathbb{S}^2$,

$$P(r) = 2\pi^2 r^3 \left(1 - \frac{1}{6}r^2 + \frac{11}{1080}r^4 - \cdots\right),$$

in comparison with the analog in $\mathbb{S}^4(\sqrt{3})$,

$$P_0(r) = 2\pi^2 r^3 \left(1 - \frac{1}{6}r^2 + \frac{13}{1080}r^4 - \cdots\right)$$

(cf. section 3.1). An alternative argument, avoiding computation, uses Theorem 3.4 to deduce that otherwise a small minimizer in $\mathbb{S}^2 \times \mathbb{S}^2$ is totally umbilic (as well as constant-mean-curvature), so that all the principal curvatures everywhere are equal. By Alexandrov's reflection argument, it is invariant under an action of $\mathbb{S}^1 \times \mathbb{S}^1$ about a point (p_1, p_2) in $\mathbb{S}^2 \times \mathbb{S}^2$. The generating curve must be a circular arc in the Euclidean plane, while the circular orbit in $\mathbb{S}^2 \times \{p_2\}$ must have the same radius and curvature, a contradiction.

Proof of Theorem 4.4. Fix n. By the comment preceding Theorem 4.4, we may assume that $n \geq 2$. We may assume that R is a region of least perimeter $P(V)$. By Theorem 2.1, for small V, R is a nearly round small ball. The principal curvatures $\kappa_1, \ldots, \kappa_n$ and mean curvature $H = (\kappa_1 + \cdots + \kappa_n)/n$ are large. At any point p in ∂R, consider an orthonormal basis $\{e_i\}$ of principal directions. Let $R_{ijk\ell}$ denote components of the Riemannian curvature and $K_{ij} = R_{ijij}$ the sectional curvature of M (cf. [M3, Section 5.3]). By Lemma 4.2, for $\kappa_i \geq 0$, the Gauss-Bonnet-Chern boundary integrand Φ (4.3) satisfies

$$\begin{aligned}(4.6)\qquad \Phi \leq\ & \kappa_1 \cdots \kappa_n + C_{n-2}\mathrm{ave}\{\kappa_{i_1} \cdots \kappa_{i_{n-2}} K_{i_{n-1}i_n}\} \\ & + \sum_{m \leq n-4} C_m \mathrm{ave}\{\kappa_{i_1} \cdots \kappa_{i_m}(K_{i_{m+1}i_{m+2}} \cdots K_{i_{n-1}i_n} + A_1\epsilon)\},\end{aligned}$$

where $\epsilon = \sup\{K_0 - K_{ij}\} \geq 0$ and A_1 is a positive constant depending on upper and lower bounds on the curvature of M. For V small (and H large and $H/2 \leq \kappa_i \leq 2H$), for some positive constant $A_2(n)$,

$$\begin{aligned}(4.7)\qquad \Phi \leq\ & \kappa_1 \cdots \kappa_n + C_{n-2}\mathrm{ave}\{\kappa_{i_1} \cdots \kappa_{i_{n-2}} K_0\} - A_2\epsilon H^{n-2} \\ & + \sum_{m \leq n-4} C_m \mathrm{ave}\{\kappa_{i_1} \cdots \kappa_{i_m}(K_{i_{m+1}i_{m+2}} \cdots K_{i_{n-1}i_n} + A_1\epsilon)\} \\ \leq\ & \sum C_m \mathrm{ave}\{\kappa_{i_1} \cdots \kappa_{i_m}\} K_0^{(n-m)/2} \leq \sum C_m H^m K_0^{(n-m)/2}\end{aligned}$$

by Lemma 4.3, with equality only if $\kappa_1 = \cdots = \kappa_n = H$ and ∂R is umbilic. By the Gauss-Bonnet-Chern formula (4.1),

$$\frac{1}{|\mathbb{S}^n|}\int_{\partial R}\Phi = \chi(R) - \frac{2}{|\mathbb{S}^{n+1}|}\int_R G \geq 1 - \frac{2G_0}{|\mathbb{S}^{n+1}|}V,$$

and therefore,

$$\sum C_m H^m K_0^{(n-m)/2} \geq |\mathbb{S}^n|\left(1 - \frac{2G_0}{|\mathbb{S}^{n+1}|}V\right)\frac{1}{P}.$$

Meanwhile for round balls in the model,

$$\sum C_m H_0^m K_0^{(n-m)/2} \geq |\mathbb{S}^n|\left(1 - \frac{2G_0}{|\mathbb{S}^{n+1}|}V\right)\frac{1}{P_0}.$$

Since $C_n = 1$ and the H_0^n term dominates, H_0 is a function $H_0 = f(V, P_0)$. Therefore for V small (and H and H_0 large),

$$H \geq f(V,P), \quad H_0 = f(V,P_0).$$

Recall that one geometric interpretation of nH is the rate of change of area with respect to volume under perturbations of the given surface (cf. [M3, Theorem 5.1], where the definition of mean curvature differs by a factor of $-n$). For $\Delta V < 0$, the new minimizers must do at least as well as perturbations of the given one, and the left derivative (which exists everywhere by Proposition 3.3) satisfies $P_L'(V) \geq f(V,P)$ everywhere. Since locally $P(V) - CV^2$ is concave by Proposition 3.3 and $P_0'(V)$ is continuous, actually

$$P_L'(V) \geq P_R'(V) \geq f(V,P), \quad \text{while} \quad P_0'(V) = f(V,P_0).$$

It follows that

$$P(V) \geq P_0(V),$$

as desired.

If equality holds at V_0, then $P(V) = P_0(V)$ for all $V \leq V_0$ and $P_L'(V0) = P_0'(V_0)$. Since even beyond V_0, $P(V) \geq P_0(V)$, actually $P'(V_0) = P_0'(V_0)$ and $H = H_0$. Since ∂R is totally umbilic, ∂R has the same second fundamental form as ∂B_{V_0}. By equality in the derivation of (4.7), each $K_{ij} = K_0$. Hence by the Gauss equations ∂R has the same constant sectional curvature $H_0^2 + K_0$ as ∂B_{V_0}, and hence is isometric to ∂B_{V_0}. For V small and H large with respect to a lower bound on the ambient sectional curvature K, interior equidistants to ∂R stay convex (cf. [Pet, Theorem 2.3.6]) and hence embedded until the focal distance r, which is at least as great as the radius r_0 of ∂B_{V_0}, because $K \leq K_0$. Also because $K \leq$

K_0, the equidistants have at least as much area as those in B_{V_0}. Since $P(V_0) = P_0(V_0)$, all comparable areas and volumes must be equal, all equidistants to ∂R are isoperimetric and isometric to geodesic spheres in B_{V_0}, and R is isometric to B_{V_0}, completing the proof under hypothesis (b). Under the alternative hypothesis (a), the volume term may be absorbed into the H^{n-2} term because $n \geq 2$, and of course equality never holds. $\square$

4.5 Gauss-Kronecker curvature. Our argument shows that in a smooth Riemannian manifold with sectional curvature $K_1 \leq K \leq K_0 < 0$, there exists a $\kappa_0 > 0$ such that for any smooth ball B such that ∂B has principal curvatures $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n \geq \kappa_0$, the Gauss-Kronecker curvature $\kappa_1 \kappa_2 \cdots \kappa_n$ satisfies

$$\int_{\partial B} \kappa_1 \kappa_2 \cdots \kappa_n > |\mathbb{S}^n|,$$

where $|\mathbb{S}^n|$ denotes the area of the unit Euclidean n-sphere.

4.6 Noncompact ambients. Theorem 2.2, Proposition 3.3, Theorem 3.4, and Theorem 4.4 hold as well for noncompact ambient M with compact quotient M/Γ by the isometry group Γ, which suffices to guarantee the existence of minimizers [M2, Chapter 13], [M1, 4.5].

5. Isoperimetric inequalities in surfaces

Proposition 5.2 and Theorem 5.4 give in a surface S an isoperimetric estimate for area in terms of perimeter and generalize the Bol-Fiala inequality (1.1) to regions of unrestricted topological type. The proofs use results of Grayson's curve shortening [Gr], as applied to isoperimetric estimates by Benjamini and Cao [BeC]. This section does not depend on the previous sections.

Lemma 5.1. *Let S be a smooth Riemannian surface with Gauss curvature $K \leq K_0$. Consider a smooth Jordan curve C of length P which flows under curve shortening to a point and encloses area A. If $K_0 P^2 \leq 4\pi^2$, then*

$$A \leq \begin{cases} \dfrac{2\pi - \sqrt{4\pi^2 - K_0 P^2}}{K_0} & (K_0 \neq 0), \\[2ex] \dfrac{P^2}{4\pi} & (K_0 = 0). \end{cases} \tag{5.1}$$

Proof. Run the flow backwards from 0 to A. As in [BeC, (1.14)],

$$\frac{dP^2}{dA} \geq 4\pi - 2K_0 A. \tag{5.2}$$

Integration yields the desired inequality (5.1). (We do not need to know that A increases monotonically, because P^2 is increasing, so we can ignore contributions from decreases in A.) ☐

Proposition 5.2. *Let M be a smooth complete Riemannian surface with convex boundary and convex at infinity, with Gauss curvature $K \leq K_0$. (M may be compact and the boundary may be empty.) Let L_0 be the infimum of the lengths of simple closed geodesics. Consider a smooth curve C of length P. If every component of C has length less than L_0 and $K_0 P^2 \leq 4\pi^2$, then C bounds a region of area A satisfying*

$$A \leq \begin{cases} \dfrac{2\pi - \sqrt{4\pi^2 - K_0 P^2}}{K_0} & (K_0 \neq 0), \\[2ex] \dfrac{P^2}{4\pi} & (K_0 = 0). \end{cases} \tag{5.3}$$

If C has one component, it bounds such a disc.

Consequently the perimeter of any region of area $A \leq$ (area (M))/2 satisfies

$$P^2 \geq \min\{L_0^2,\ 4\pi A - K_0 A^2\}. \tag{5.4}$$

Remarks. Inequalities (5.3) are sharp in surfaces of constant curvature K_0. If $K_0 \leq 0$ and C bounds a disc, then (5.3) follows from the Bol-Fiala inequality (1.1). The need for the restriction to curves shorter than any closed geodesic is illustrated by a dumbbell surface as in Figure 1. Of course when $K_0 > 0$, $K_0 P^2 \leq 4\pi^2$ is necessary for the square root to be defined; note also that a polar cap on a sphere of curvature slightly less than K_0 with perimeter slightly greater than $4\pi^2/K_0$ can have area greater than $2\pi/K_0$. The need to assume M convex at infinity is illustrated by a sphere with a cusp as in Figure 2.

Inequality (5.4) is sharp as illustrated in Figure 3 by the hyperbolic ($K = K_0 = -1$) two-holed torus with a narrow neck of circumference L_0. Small discs have perimeter P satisfying $P^2 = 4\pi A + A^2$, while half the surface has perimeter L_0.

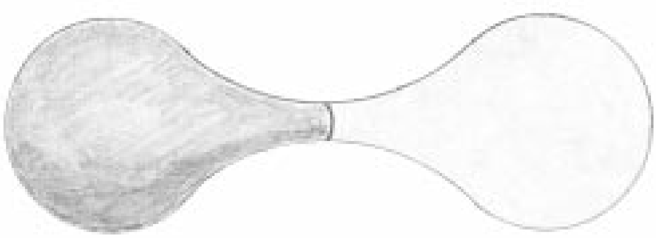

FIGURE 1. The isoperimetric inequality (5.3) can fail for curves as long as the shortest geodesic, which bounds lots of area.

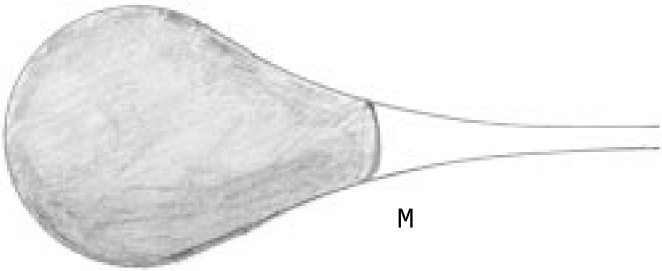

FIGURE 2. If M is not convex at infinity, a short curve may bound lots of area.

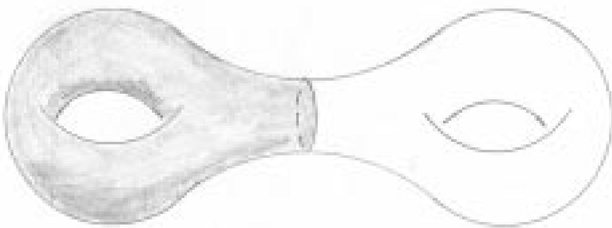

FIGURE 3. A short geodesic can enclose large area in a hyperbolic surface.

Proof of Proposition 5.2. According to Grayson [Gr], each component of C flows under curve shortening to a point or a simple closed geodesic (actually he does not rule out a family of limiting geodesics). Since by hypothesis such geodesics are longer, it must flow to a point. By Lemma 5.1, it bounds a disc of area A_i and perimeter P_i satisfying (5.3).

The set of points lying in an odd number of discs has the right boundary and area $A \leq \sum A_i$. Inequality (5.3) follows because the right-hand side is convex, because $\sqrt{4\pi^2 - K_0 x^2}$ is concave or convex, according to whether K_0 is positive or nonpositive.

Inequality (5.4) follows from (5.3). The condition $K_0 P^2 \leq 4\pi^2$, vacuous when $K_0 \leq 0$, is unnecessary when $K_0 > 0$, because $4\pi^2/K_0$ is the maximum value of $4\pi A - K_0 A^2$. □

Theorem 5.3 generalizes the Bol-Fiala inequality (1.1) from discs to arbitrary regions.

Theorem 5.3. *Let S be a smooth Riemannian sphere, or complete plane convex at infinity, or compact convex disc, with Gauss curvature $K \leq K_0$. Let L_0 be the infimum of the lengths of simple closed geodesics. Then the perimeter P of any smooth region R of area A satisfies*

$$P^2 \geq \min\{(2L_0)^2,\ 4\pi A - K_0 A^2\}. \tag{5.5}$$

Remarks. The theorem is sharp, as illustrated in Figure 1 by two unit spheres connected by a thin cylinder. Small discs have perimeter P satisfying $P^2 = 4\pi A - A^2$, while sections of the cylinder have perimeter $2L_0$.

For the relatively easy case of a plane with $K \leq K_0 \leq 0$ (which is automatically convex at infinity), the Gauss-Bonnet formula implies that there are no simple closed geodesics, and therefore $P^2 \geq 4\pi A - K_0 A^2$, as already follows easily from the Bol-Fiala inequality (1.1).

Note that the theorem can fail for the complement of a disc in an $\mathbb{R}P^2$, torus, or compact hyperbolic surface.

We do not know whether the theorem generalizes to higher dimensions. Even in dimension three, mean curvature flow may develop singularities, and worse the possibility of perimeter of higher topological type spoils the Gauss-Bonnet estimate on dP/dV as in (4.6). In dimensions above three, we know no simple inequality on dP/dV for the flow.

5.1 Proof of Theorem 5.3. For convenience we may assume $K_0 \neq 0$, since the case $K_0 = 0$ follows from the case $K_0 > 0$. By scaling, we may assume $K_0 =$

± 1. We begin with the more interesting case $K_0 = 1$. Let $C_1, \ldots, C_k$ denote the boundary components. We may assume that $P^2 < 4\pi^2$ (the maximum of $4\pi A - A^2$). Similarly we may assume that $P < 2L_0$ and hence that each C_i, with the possible exception of C_1, flows under curve shortening to a point rather than to a geodesic. Each such component bounds an associated disc D_i, with area A_i and perimeter P_i satisfying

$$A_i < 2\pi - \sqrt{4\pi^2 - P_i^2} \tag{5.6}$$

by Lemma 5.1. (For an exceptional C_1 in the sphere, choose D_1 on the same side of C_1 as the region R.) Any two D_i are either disjoint or nested, since otherwise by the maximum principle they could not flow to points.

We focus now on the cases of S a plane or disc. By the Bol-Fiala inequality (1.1),

$$P_1^2 \geq 4\pi A_1 - A_1^2.$$

Hence one of the following inequalities holds:

$$A_1 \leq 2\pi - \sqrt{4\pi^2 - P_1^2}, \tag{5.7a}$$

$$A_1 \geq 2\pi + \sqrt{4\pi^2 - P_1^2}. \tag{5.7b}$$

If (5.7a) holds, then

$$A \leq \sum_{i=1}^{k} A_i \leq 2\pi - \sqrt{4\pi^2 - P^2}$$

because the function $\sqrt{4\pi^2 - x^2}$ is concave, and hence $P^2 \geq 4\pi A - A^2$. If S is a sphere, R could be the complementary region, and

$$A \geq \text{area}\,(S) - \left(2\pi - \sqrt{4\pi^2 - P^2}\right) \geq 2\pi + \sqrt{4\pi^2 - P^2}$$

because area $(S) \geq 4\pi$ by the Gauss-Bonnet theorem, and hence $P_2 \geq 4\pi A - A^2$ still.

Suppose (5.7b) holds. Then C_1 does not satisfy (5.5) and does not flow to a point. By the maximum principle, D_1 cannot be contained in any other D_i. Now if $Q = \sum_{i=1}^{k} A_i$, by (5.5) and (5.7b),

$$\begin{aligned} A \geq A_1 - \sum_{i=2}^{k} A_i &\geq \left(2\pi + \sqrt{4\pi^2 - P_1^2}\right) - \left(2\pi - \sqrt{4\pi^2 - Q^2}\right) \\ &\geq \sqrt{4\pi^2 - 0^2} + \sqrt{4\pi^2 - (P_1 + Q)^2} = 2\pi + \sqrt{4\pi^2 - P^2}, \end{aligned}$$

by two more applications of the concavity of $\sqrt{4\pi^2 - x^2}$. Therefore $P^2 \geq 4\pi A - A^2$, as desired. (The sphere requires no special treatment, because we chose D_1 on the correct side of C_1.)

Finally if $K_0 = -1$, the right-hand side of (5.5) and similarly (5.7a) become $\sqrt{4\pi^2 + P_i^2} - 2\pi$, inequality (5.7b) does not occur, S cannot be a sphere, $\sqrt{4\pi^2 + x^2}$ is convex, and the argument is just much simpler. $\square$

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