ABSTRACT. Let $M$ be a compact Riemannian manifold of dimension $n$, and let $\mathcal{F}$ be a smooth foliation on $M$. A topological obstruction is obtained, similar to results of R. Bott and J. Pasternack, to the existence of a metric on $M$ for which $\mathcal{F}$ is totally geodesic. In this case, necessarily that portion of the Pontryagin algebra of the subbundle $\mathcal{F}$ must vanish in degree $n$ if $\mathcal{F}$ is odd-dimensional. Using the same methods simple proofs of the theorems of Bott and Pasternack are given.


Key words and phrases: Foliations, totally geodesic foliations, integrability obstructions, geodesibility.

0. INTRODUCTION

If $\mathcal{F}$ is a codimension-$k$ distribution on a compact smooth manifold $M$, there is a well-known topological obstruction, due to R. Bott, to the integrability of $\mathcal{F}$; the Pontryagin algebra of $T(M)/\mathcal{F}$ must vanish in degrees greater than $2k$ [1]. This result was greatly improved by J. Pasternack in his thesis under the additional assumption that the metric on $M$ is fiberlike with respect to the foliation $\mathcal{F}$ [7]. In that case, the characteristic algebra of $T(M)/\mathcal{F}$ must vanish in degrees greater than $k$. This article gives a simple proof of these facts, using tensors similar to those introduced by B. O'Neill [6] (cf., [5]). Also, there is a similar obstruction theorem derived in the case where $\mathcal{F}$ is totally geodesic and of odd dimension. However, in this case the obstruction is in the characteristic algebra of the subbundle $\mathcal{F}$ itself; if $M$ is $n$-dimensional, the characteristic algebra of $\mathcal{F}$ must vanish in degree $n$.

1. PRELIMINARIES

Let $M$ be, as above, a smooth, compact Riemannian $n$-manifold. Let $\mathcal{F}$ be a foliation on $M$ of codimension $k$. Denote also by $\mathcal{F}$ the associated distribution and the orthogonal projection onto this distribution. Similarly, if $\mathcal{H} = \mathcal{F}^\perp$ is the orthogonal distribution, denote by $\mathcal{H}$ the orthogonal projection, and, if $\mathcal{H}$ is integrable, denote the resulting foliation also by $\mathcal{H}$. Vectors in $\mathcal{H}$ (resp., $\mathcal{H}$) will be called horizontal (resp., vertical). As in [5] and [6], define tensors $T$ and $A$ on $M$ by, for all vector fields $E, F \in \mathcal{X}(M)$,

$$T_E F = \mathcal{H} \nabla_{\mathcal{F}E} \mathcal{F} F + \mathcal{F} \nabla_{\mathcal{F}E} \mathcal{H} F,$$

$$A_E F = \mathcal{H} \nabla_{\mathcal{F}E} \mathcal{F} F + \mathcal{F} \nabla_{\mathcal{F}E} \mathcal{H} F.$$

As in [5], it is easily seen that $\mathcal{F}$ is totally geodesic if and only if $T = 0$, 


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and that the metric is fiberlike (i.e., locally there are Riemannian submersions defining the foliation) if and only if $A_X Y = - A_Y X$ for all $X, Y \in \Gamma(H)$.

The properties of the tensors $A$ and $T$ may equivalently be given in terms of a single tensor $\mathcal{P}$, where $\mathcal{P}$ is the automorphism $\mathcal{P} : T_v(M) \to T_v(M)$ given by $\mathcal{P} = \mathcal{F} - H$. In a forthcoming article the second author will classify the various geometric almost-product and foliated structures defined naturally in terms of this automorphism, analogously to the work of A. Gray and L. M. Hervella on almost-complex structures [4]. At present there is the following partial classification.

**Proposition (1.1)**

(a) $\mathcal{P}$ is parallel if and only if $M$ is locally isometric to a Riemannian product.

(b) $\nabla_Y(\mathcal{P}) = 0$ for $Y \in \Gamma(H)$ if and only if $\mathcal{F}$ is totally geodesic.

(c) For $X, Y \in \Gamma(H)$, $\nabla(\mathcal{P})_X Y + \nabla(\mathcal{P})_Y X = 0$ if and only if the metric is fiberlike.

(d) For $X, Y \in \Gamma(H)$, $\nabla(\mathcal{P})_X Y - \nabla(\mathcal{P})_Y X = 0$ if and only if $H$ is integrable.

**Proof.** A calculation verifies that

$$\nabla(\mathcal{P})_E F = - 2 \mathcal{F} \nabla_{\mathcal{F}E} \mathcal{H} F + 2 \mathcal{H} \nabla_{\mathcal{F}E} \mathcal{F} F - 2 \mathcal{F} \nabla_{\mathcal{H}E} \mathcal{H} F + 2 \mathcal{H} \nabla_{\mathcal{H}E} \mathcal{F} F.$$

By taking the various cases of $E$ and $F$ either vertical or horizontal it is clear that $\mathcal{P}$ is parallel if and only if both $A$ and $T$ vanish. In that case [5] shows that $M$ is locally isometric to a Riemannian product, verifying part (a). The remaining portions of the Proposition follow from this formula for $\nabla(\mathcal{P})$ and [5].

$X \in \Gamma(H)$ is basic if, for some local submersion $f_U : U \to \mathbb{R}^k$ defining $\mathcal{F}_U | U$, $X$ is $f_U$-related to a vector field $\tilde{X}$ on $\mathbb{R}^k$; that is, $f_U(X) = \tilde{X}$.

**Proposition (1.2)**

(a) If $X$ is basic, and if $V$ is vertical, then $[X, V]$ is vertical.

(b) If the metric is fiberlike, it is possible to choose $X$ and $Y$ basic (with arbitrary horizontal values at a given point) so that $\nabla_X Y$ is also vertical.

**Proof.** The first statement is trivial; since $X$ is $f_U$-related to $\tilde{X}$, and $V$ is $f_U$-related to zero, $[X, V]$ is $f_U$-related to $[\tilde{X}, 0] = 0$. For the second, the definition of a fiberlike metric [5] implies the existence on $\mathbb{R}^k$ of a metric for which $f_U : U \to \mathbb{R}^k$ is a Riemannian submersion. In this case, if $X$ and $Y$ are basic, $f_U$-related to $\tilde{X}$ and $\tilde{Y}$, respectively, then, for $\mathcal{V}$ the Riemannian covariant derivative on $\mathbb{R}^k$, $\mathcal{H} \nabla_{X} Y$ is $f_U$-related to $\mathcal{V}_{\tilde{X}} \tilde{Y}$ [6]. Choosing vector fields $\tilde{X}, \tilde{Y}$ so that $\mathcal{V}_{\tilde{X}} \tilde{Y} = 0$ completes the proof. Note that, in the general case $X$ and $Y$ may be chosen with $[X, Y]$ vertical by a similar argument.

2. Theorems of Bott and Pasternack

On $\mathcal{H}$, define a connection $\tilde{\nabla}$ by $\tilde{\nabla}_E X = \mathcal{H} \nabla_{E} X - A_X \mathcal{F} E$, for $E \in \mathcal{H}(M)$ and $X \in \Gamma(H)$. It is evident that $\tilde{\nabla}$ is a connection; $\tilde{\nabla}$ is a geometrically natural
choice of Bott’s connection on $T^*(M)/\mathcal{F} \simeq \mathcal{H}$. Unfortunately $\tilde{\nabla}$ is not, in general, symmetric.

**Theorem (2.1).** If $\tilde{\Omega}$ is the curvature of $\tilde{\nabla}$, $\tilde{\Omega}(V, W) = 0$ if both $V$ and $W$ are vertical.

**Proof.** For $X$ basic,

$$\tilde{\nabla}_V X = \mathcal{H}(\nabla_V X - \nabla_X V) = \mathcal{H}[V, X] = 0,$$

by Proposition (1.2). Then,

$$\tilde{\Omega}(V, W) X = \tilde{\nabla}_V \tilde{\nabla}_W X - \tilde{\nabla}_W \tilde{\nabla}_V X - \tilde{\nabla}_{[V, W]} X = 0$$

due to the integrability of $\mathcal{F}$.

**Corollary (2.2) [Bott].** $\text{Char}^p(\mathcal{H}) = 0$ for $p > 2k$, where $\text{Char}^p(\mathcal{H})$ is that part of the real characteristic algebra (Pontryagin or Chern) of $\mathcal{H}$ in degree $p$.

**Remark.** In general, this is the Pontryagin algebra of $\mathcal{H}$, and specifically does not include terms involving the Euler class, since the connection is not symmetric. In the case where $\mathcal{H}$ is complex, all appropriate Chern classes must vanish.

**Proof.** If $\mathcal{F}^{p/2}$ is the space of all $\Omega(k, \mathbb{R})$-invariant polynomials of degree $p/2$ (resp., $\Omega(k/2, \mathbb{C})$-invariant polynomials), it is well-known [2] that $\text{Char}^p(\mathcal{H})$ is generated by all $P(\tilde{\Omega})$, for $P \in \mathcal{F}^{p/2}$. As $P(\tilde{\Omega})$ is tensorial, it suffices to compute $P(\tilde{\Omega})(A_1, \ldots, A_p)$ where each $A_j$ is chosen to be either vertical or horizontal. However, if $p > 2k$ each monomial must possess a component of $\tilde{\Omega}(A_i, A_j)$ with both $A_i$ and $A_j$ vertical. \[ \square \]

In the case where the metric is fiberlike the connection $\tilde{\nabla}$ will be symmetric; an exactly analogous argument yields Pasternack’s theorem.

**Proposition (2.3).** If the metric is fiberlike, $\tilde{\nabla}$ is symmetric.

**Proof.** The condition that

$$\langle \tilde{\nabla}_E X, Y \rangle + \langle X, \tilde{\nabla}_E Y \rangle = E\langle X, Y \rangle$$

is clearly tensorial, thus it suffices to consider only the case where $X$ and $Y$ are basic. If $E$ is vertical, Proposition (1.2) implies that the left-hand side vanishes. That the right-hand side is also zero may be found in [5]. If $E$ is horizontal, taking $E$ to be basic yields

$$\langle \tilde{\nabla}_E X, Y \rangle + \langle X, \tilde{\nabla}_E Y \rangle = \langle \tilde{\nabla}_E X, \tilde{\nabla}_E Y \rangle + \langle \tilde{\nabla}_E X, \tilde{\nabla}_E Y \rangle$$

by [6]. As $\tilde{\nabla}$ is symmetric the proposition is verified, since $E\langle X, Y \rangle = E\langle \tilde{\nabla}_E X, \tilde{\nabla}_E Y \rangle$ at corresponding points.

**Theorem (2.4).** If the metric is fiberlike, $\tilde{\Omega}(X, V) = 0$ for $X$ horizontal, $V$ vertical.

**Proof.** Let $Y$ be chosen to be basic and so that $\nabla_X Y \in \Gamma(\mathcal{F})$ by Proposition (1.2). $X$, as usual, will be assumed to be basic. Then, $\tilde{\nabla}_X Y = \mathcal{H} \nabla_X Y = 0$ as well as $\tilde{\nabla}_V Y = 0$. As $[X, V]$ is vertical, evidently $\tilde{\Omega}(X, V) Y = 0$. \[ \square \]
COROLLARY (2.5) [Pasternack]. If the metric on $M$ is fiberlike, $\text{Char}^p(\mathcal{H}) = 0$ for $p > k$.

Remark. Here the appropriate terms involving the Euler class may be included; that is, if $\mathcal{H}$ is orientable, consider all $so(k)$-invariant polynomials of degree $p/2$.

Proof. In this case it is necessary that each monomial in $P(\bar{Q})(A_1, \ldots, A_p)$ has a component of $\bar{Q}(A_i, A_j)$ where at least one of $A_i$ and $A_j$ is vertical. □

3. Totally Geodesic Foliations

A foliation $\mathcal{F}$ is totally geodesic if each leaf is a totally geodesic submanifold of $M$. In [5] the first author and L. Whitt found a strong obstruction to the existence of a totally geodesic foliation $\mathcal{F}$ of codimension one under the assumption that $\mathcal{F}$ has at least one closed leaf; in that case $M$ must fiber over a circle. In contrast, H. Gluck has shown that there is no obstruction to the existence of a totally geodesic foliation of dimension one on a simply-connected manifold of odd dimension. It thus seems reasonable to suspect that the topological obstructions to geodesibility of a foliation $\mathcal{F}$, above the integrability obstructions, should lie in the bundle $\mathcal{F}$ rather than the normal bundle.

Define a connection $\hat{\nabla}$ on $\mathcal{F}$ by $\hat{\nabla}_E V = \mathcal{F}\nabla_E V$. $\hat{\nabla}$ is clearly a symmetric connection. Note that, since $\mathcal{F}$ is totally geodesic, $T = 0$. More generally it would be desirable, analogously to Bott's connection, to consider $\mathcal{F}\nabla_E V - T_v \mathcal{H}E$; however, the nonintegrability of $\mathcal{H}$ prevents any transparent consequences in general.

PROPOSITION (3.1). If $X_m \in T_m(M, m)$ is horizontal, and $Y_m \in T_m(M, m)$ is vertical, there are extensions $X \in \Gamma(\mathcal{H})$ and $Y \in \Gamma(\mathcal{F})$ so that $\hat{\nabla}_X V = 0$.

Proof. Choose $X$ to be basic. Let $\bar{\gamma}$ be any integral curve of $\bar{X}$ on $\mathbb{R}^k$, where $f^\circ : U \mapsto \mathbb{R}^k$ is a chosen local submersion with $f^\circ(X) = \bar{X}$. Let $\Sigma = f^{-1}(\bar{\gamma})$.

LEMMA (3.2). If $\Sigma$ is given the induced metric, the restriction $\mathcal{F}^\Sigma$ of $\mathcal{F}$ to $\Sigma$ is totally geodesic. Also, note that the orthogonal distribution $\mathcal{H}^\Sigma$ is integrable. The metric on $\Sigma$ is fiberlike with respect to the foliation $\mathcal{F}^\Sigma$.

Proof. Since the Riemannian covariant derivative $\nabla^\Sigma$ on $\Sigma$ is given by the orthogonal projection $\Pi_{T_\Sigma \mathcal{V}} \nabla$, the first statement is trivial. That the metric on $\Sigma$ is fiberlike with respect to $\mathcal{F}$ follows from the duality between fiberlike metrics and totally geodesic foliations described in [5]. □

Now let $g_\Sigma : \Sigma \to \mathbb{R}$ be a local submersion defining $\mathcal{H}$. As the induced metric on $\Sigma$ is fiberlike, there is a metric on $\mathcal{R}$ so that $g_\Sigma$ is a Riemannian submersion. Choose $V$ to be basic with respect to $g_\Sigma$. Proposition (1.2) then implies that $[V, X] \in \Gamma(\mathcal{F}^\Sigma)$. However, as $V$ is vertical and $\mathcal{F}$ is totally geodesic, $\nabla_V X \in \Gamma(\mathcal{H}^\Sigma)$ as well, so that $\nabla_X V \in \Gamma(\mathcal{H}^\Sigma) \subseteq \Gamma(\mathcal{H})$ ($V$ may be
extended to a vector field on $U$ using a smooth family of $g_\lambda$’s for the various integral curves of $\tilde{X}$). Thus, $\tilde{\nabla}_V X = 0$.

\textbf{THEOREM (3.3).} If $\mathcal{F}$ is totally geodesic, and if $\tilde{\Omega}$ is the curvature of $\tilde{\nabla}$, then $\tilde{\Omega}(V, X) = 0$ if $V$ is vertical and $X$ is horizontal.

\textit{Proof.} Extend $X$ to be basic, and, as in Proposition (3.1), choose $V$ to be $\mathcal{H}^2$-basic and so that $\tilde{\nabla}_X V = 0$. Let $W$ be another $\mathcal{H}^2$-basic vector field, for which, using Proposition (1.2), $\tilde{\nabla}_V W$ is in $\mathcal{T}(\mathcal{H})$. As $\mathcal{F}$ is totally geodesic, $\tilde{\nabla}_V W = 0$, thus $\tilde{\nabla}_V W = 0$. The proof of Proposition (3.1) implies that $\tilde{\nabla}_X W = 0$ as well, since $W$ is $\mathcal{H}^2$-basic. Also, $[X, V] = 0$ as $[X, V]$ must be both horizontal and vertical, applying Proposition (1.2) twice. Thus $\tilde{\Omega}(X, V)W = 0$.

\textbf{COROLLARY (3.4).} If $\mathcal{F}$ is totally geodesic and if $\dim(\mathcal{F})$ is odd, $\text{Char}^n(\mathcal{F}) = 0$, where $n = \dim(M)$.

\textit{Proof.} If $P$ is any $o(n-k)$-invariant polynomial of degree $n/2$, consider $P(\tilde{\Omega})(A_1, \ldots, A_n)$ where $A_i$ is either vertical or horizontal. Each monomial must possess a component of $\tilde{\Omega}(A_i, A_j)$ where one is vertical and the other is horizontal, as $\dim(\mathcal{F})$ is odd, thus each monomial must vanish.

\section{4. An example}

Let $M$ be a compact 8-dimensional orientable manifold with $\chi(M) = 0$ but Hirzebruch signature nonzero. Thurston [8] has shown that there is a foliation $\mathcal{F}$ on $M$ of codimension one. However,

\textbf{PROPOSITION (4.1).} No codimension-one foliation $\mathcal{F}$ on $M$ is geodesible.

\textit{Proof.} Let $\mathcal{H} = \mathcal{F}^\perp$. As $T_*(M) \cong \mathcal{H} \oplus \mathcal{F}$, the total Pontryagin class $p_*(M)$ is given by $p_*(M) = p_*(\mathcal{H})p_*(\mathcal{F})$. But $\mathcal{H}$ is one-dimensional, so that $p_1(\mathcal{H}) = 1$, hence $p_4(M) = 0$ and $p_1(M) = p_1(\mathcal{F})$. By the Hirzebruch signature theorem, the signature $\sigma(M)$ of $M$ satisfies $\sigma(M) = -\frac{1}{48}(7p_2(M) - p_1(M)^2) = -\frac{1}{48}p_1(\mathcal{F})^2$, which is nonzero by assumption. Corollary (3.4) then implies that $\mathcal{F}$ cannot be totally geodesic.

\textbf{REFERENCES}


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