MINIMAL SURFACES IN CIRCLE BUNDLES OVER RIEMANN SURFACES

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Abstract. For a compact 3-manifold M which is a circle bundle over a Riemann surface Σ with even Euler number $e(M)$, and with a Riemannian metric compatible with the bundle projection, there exists a compact minimal surface S in M. S is embedded and is a section of the restriction of the bundle to the complement of a finite number of points in Σ. If the Euler number is zero, a smooth minimal section S exists, and for any nonzero Euler number a smooth minimal surface exists which is a double-section over all but finitely many points of Σ.

1. Introduction

Let M be a 3-manifold which is a circle bundle (to be specific, a principal $U(1)$-bundle) over a compact Riemann surface Σ, with projection $\pi : M \to \Sigma$. Assume that the metric on M is compatible with the bundle projection, that is, π is a Riemannian submersion and the fibers are geodesics. Assume that the Euler class $e(M)$ of the associated rank-2 vector bundle E over Σ is even. The goal of this paper is to show the existence of a smooth minimal surface in M, which is a section of the bundle except over a finite set of points, and is topologically the Riemann surface Σ with a finite number of cross-caps.

An example of such a minimal surface is described in [1]. Consider $M = T_1(S^2)$, the unit tangent bundle of the standard 2-sphere. For any choice of a unit tangent vector v at $p \in S^2$, the Pontryagin cycle P, the section defined by parallel translation of v along each longitude line from p, will be a smooth minimal surface in M which is a smooth section except over $-p$. P is in this case a totally-geodesic $\mathbb{RP}^2$ embedded in M.

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2. Minimal graphs

Let X be a compact, n-dimensional manifold, and let $\pi : B \to X$ be a fiber bundle over X with compact fiber F of dimension k. Any such bundle admits a class of Riemannian metrics, due to Sasaki [11], for which the projection π is a Riemannian submersion with totally-geodesic fibers isometric to F under inclusion, determined by a choice of connection on the associated principal bundle.

Definition 2.1. A *rectifiable section* T in B is a countably-rectifiable, integer-multiplicity n-current in B so that,

(1) $\left\langle \overrightarrow{T}(q), \mathbf{e}(q) \right\rangle \geq 0$, $\|T\|$-almost everywhere; where $\mathbf{e}(q)$ is the unique horizontal (orthogonal to the fibers) n-plane at q which maps onto $T_*(X, \pi(q))$ under π_* (preserving orientation), and $\overrightarrow{T}$ is the unit oriented n-vector tangent plane of T at q.

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(2) The image current $\pi_{\#}(T)$ is the fundamental class $1[X]$ as an n-dimensional current on X with integer coefficients.
(3) If $\partial X = \emptyset$, $\partial T \equiv 0 (mod\, 2)$ as flat chains modulo 2 (If $\partial X \neq \emptyset$, ∂T must have support contained in $\pi^{-1}(\partial X) = \partial B$).

The space of all such *rectifiable sections* of the bundle B over X will be denoted $\widetilde{\Gamma}(B)$.

Remark 2.2. This definition differs slightly from that in [6], in that here the currents are only required to be relative cycles mod-2. This is necessary because the currents constructed will not be cycles as integral chains. As rectifiable currents they are by definition oriented, but currents corresponding to smooth submanifolds may be non-orientable as manifolds, or, equivalently, may have interior boundaries as rectifiable currents. Compact non-orientable manifolds without boundary (as manifolds) are mod-2 cycles as flat chains modulo 2.

In [6] it is shown that any homology class of rectifiable sections has a minimal-mass representative, which is a continuous section over an open, dense subset of X. As remarked in [8, 11.1] or [3, 4.2.26], the extension from integer coefficients to $\mathbb{Z}/2\mathbb{Z}$ coefficients, and considering the currents as flat chains modulo 2 for the boundary condition will not alter the arguments of [6].

If σ is a C^1 section, then the mass of the image (usually called the *volume* of the section in this case) is given by

$$\mathcal{V}(\sigma) := \int_X \sqrt{1 + \|\nabla\sigma\|^2 + \cdots + \left\|\nabla\sigma \wedge \overset{\min\{n,k\}}{\cdots} \wedge \nabla\sigma\right\|^2}\, dV_X.$$

Theorem 2.3. [6] *Let X be a compact manifold, and let B be a fiber bundle over X with compact smooth fiber F and with an associated Sasaki metric. In any nonempty mod-2 homology class of rectifiable sections $\sigma : X \to B$, there is a mass-minimizing, rectifiable section which is continuous except over a set S of measure 0 in X.*

Remark 2.4. It should be noted that the original result shows the section to be C^1 on an open dense set; continuity may hold on a slightly larger set. Also, the theorem does not say there will not be other mass-minimizers that may have worse regularity, only that there is one which is this nicely-behaved. Finally, Proposition (2.5) below will imply that there is such a nonempty homology class of rectifiable sections in the cases we need.

Now, let M be a 3-manifold which is a circle bundle $\pi : M \to \Sigma$ over a compact Riemann surface Σ, with a Sasaki metric. The Euler class $e(M)$ of M is the Euler class of the associated orientable rank-2 vector bundle $E \to \Sigma$. In order to show the existence of the claimed minimal surface in M, we first have to show that $\widetilde{\Gamma}(M) \neq \emptyset$ when $e(M)$ is even.

Proposition 2.5. *If $M \to \Sigma$ is a circle bundle over a compact Riemann surface, with even Euler class $e(M)$, then $\widetilde{\Gamma}(M) \neq \emptyset$.*

Proof. If $k := |e(M)|/2$, choose k points $\{p_1, \ldots, p_k\} \subset \Sigma$. Essentially by the Poincaré-Bendixon theorem, there is a smooth section of E with zeros only at the points p_j, of index ± 2 at all points, where the sign is that of $e(M)$. Equivalently, given $\epsilon > 0$ sufficiently small, there is a smooth section τ of $M|_{\Sigma\setminus\{B_\epsilon(p_1),\ldots,B_\epsilon(p_k)\}}$ with the following boundary conditions: for each j, $M|_{\partial B_\epsilon(p_j)} \cong S^1 \times S^1$, so the map $z \mapsto z^n$ defines a section on the boundary component $M|_{\partial B_\epsilon(p_j)}$ with index $n = \pm 2$ for each $j \leq k$. A smooth section τ exists with these boundary conditions, where we choose the sign of n to match the sign of $e(M)$. This section can be constructed to extend, for any $0 < \delta < \epsilon$, to $M|_{\Sigma\setminus\{B_\delta(p_1),\ldots,B_\delta(p_k)\}}$ with similar boundary conditions. The limit of these extensions, as $\delta \to 0$, has closure which is a mod-2 cycle in M, as in [1]. This limit is a rectifiable section, so the space $\widetilde{\Gamma}(M)$ of rectifiable sections is nonempty. ☐

Following [6], with the slight modification to the boundary conditions, there is a mass-minimizing rectifiable section σ in the mod-2 homology class of any such τ above, which is a continuous section over an open dense subset. We show below that the exceptional set is a finite collection of fibers over points $\{x_1, \ldots, x_n\} \subset \Sigma$, and that the mod-2 cycle which is the closure of this section is a smooth minimal surface in M. We emphasize that the exceptional points of the minimizer need not be the points used in Proposition (2.5); in particular, the number n of points may be larger than k by an even number (since the indices must cancel).

The mod-2 homology class $[\tau] \in H_2(M, \mathbb{Z}/2\mathbb{Z})$ will project to $\pi_*([\tau]) = [\Sigma] \in H_2(\Sigma, \mathbb{Z}/2\mathbb{Z})$. Moreover, such a rectifiable section τ exists for each class $\alpha \in H_2(M, \mathbb{Z}/2\mathbb{Z})$ which projects to the generator, any element of $(\pi_*)^{-1}([\Sigma]) \subset H_2(M, \mathbb{Z}/2\mathbb{Z})$. This is the case since, for any map $\alpha : \Sigma \to S^1$, the product $\tau \cdot \alpha$, thinking of M as an S^1-principal bundle over Σ, will be defined by the right action of S^1 on M, and $\tau \cdot \alpha$ will be another rectifiable section of M. These sections will be homologous whenever α is homotopic to 1, and the homotopy classes of $\tau \cdot \alpha$ will correspond to the homotopy classes of α in $\pi^1(\Sigma) \cong H^1(\Sigma, \mathbb{Z})$. Since $H_2(M, \mathbb{Z}/2\mathbb{Z}) \cong H_2(\Sigma, \mathbb{Z}/2\mathbb{Z}) \oplus H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ by the Gysin sequence (with the mod-2 reduction of the Euler class being 0), any element of $(\pi_*)^{-1}([\Sigma]) \subset H_2(M, \mathbb{Z}/2\mathbb{Z})$ has a representative of this form. Thus, the results of [6] will imply that there will be a mass-minimizing representative rectifiable section in each such homology class.

In the case of Euler class 0, this description gives a better picture of the various components of the space of rectifiable sections, and the nature of the existence of a mass-minimizer within a homology class. In that case, arguing as above, there are rectifiable sections in each class of $(\pi_*)^{-1}([\Sigma]) \subset H_2(M, \mathbb{Z})$, which is bijective with $H_1(\Sigma, \mathbb{Z})$, and those lying within different classes are not homologous, so there would be at least a separate mass-minimizer for each such homology class of sections. We make no claim that these various homology classes will have distinct minimum volumes, but it may be possible to show such a result, arguing as in [2].

3. Singularities

Consider now a mass-minimizing rectifiable section T of a circle bundle M over a compact Riemann surface Σ, with the metric as described earlier. An *exceptional point,* or a *singular point* $x \in \Sigma$ is a point over which T is not a continuous section. Since $\pi(Supp(T)) = \Sigma$, this implies that there are two points, at least, in $\pi^{-1}(x) \cap Supp(T)$ for an exceptional point x. Our first goal will be to show that the entire fiber is enclosed in T over any exceptional point. This step uses a basic construction which will be needed elsewhere as well, a horizontal sequence of stretches of the current.

3.1. **H-cones.** If $S \in \widetilde{\Gamma}(M)$ is a rectifiable section with finite mass, and if $x_0 \in \Sigma$ is an arbitrary point, then for sufficiently small $r > 0$, and for all $\lambda > 1$, S defines a rectifiable section $S_{\lambda,R}$ in $B(0,R) \times S^1$ by $S_{\lambda,R} = \left[(\phi_\lambda)_\# \left(S \llcorner \pi^{-1}(B(0,r))\right)\right] \llcorner B(0,R) \times S^1$, if $\lambda r \geq R$, where x_0 corresponds with the center 0 of the coordinate system, $\phi_\lambda(x,y) = (\lambda x, y)$, and $\pi^{-1}(B(0,R))$ is identified with $B(0,R) \times S^1$, having the Riemannian metric induced from M and the dilation ϕ_λ. $B(0,\lambda r) \times S^1$ also has a specific Riemannian metric, the metric from M stretched horizontally by ϕ_λ. Clearly, for an arbitrary $R > 0$, if $\lambda > 1$ is sufficiently large, $S_{\lambda,R}$ will be well-defined in $\widetilde{\Gamma}(B(0,R) \times S^1)$.

An *h-cone* H of S at $x_0 \in \Sigma$, for a given sequence $\lambda_i \to \infty$, is the limit, for each $R > 0$, of the sequence of restricted stretches $S_{\lambda_i,R}$, if that limit exists as a rectifiable section (thinking of $S_{\lambda,R}$ as rectifiable sections of $B(0,R) \times S^1$ in order to define the limit), *and* if $H_{\lambda,R} = H \llcorner B(0,R) \times S^1$ for all $\lambda > 1$. The limit will be a rectifiable section of $B(0,R) \times S^1$ with the flat Euclidean metric.

For a given point, current, and sequence of stretches, an h-cone may or may not exist, just as tangent cones for rectifiable currents may or may not exist at a given point. In addition, we make no claim for uniqueness of such h-cones (the h-cone may depend upon the sequence of stretches) even when they do exist. However, if S is a mass-minimizing rectifiable section, then an h-cone will exist over each base point. Over a regular point, h-cones are simply horizontal planes, but over singular points they reveal some of the singular structure.

Theorem 3.1. *Let $S \in \widetilde{\Gamma}(M)$ be mass-minimizing and continuous over an open dense subset, as in Theorem 2.3. At each point $x_0 \in \Sigma$, there is an h-cone.*

Proof. Certainly there is nothing to prove unless x_0 is a singular point. In that case, the stretches satisfy, for $\lambda > 1$,

$$\begin{aligned} \mathcal{V}(S_{\lambda,R}) &= \int_{B(0,R)} \sqrt{1 + \frac{1}{\lambda^2} \left\| \nabla u|_{x/\lambda} \right\|^2} dA \\ &\leq \frac{1}{\lambda} \int_{B(0,R)} \sqrt{1 + \left\| \nabla u|_{x/\lambda} \right\|^2} dA \\ &\leq \lambda \int_{B(0,R/\lambda)} \sqrt{1 + \|\nabla u\|^2} r dr d\theta \\ &= \lambda f(R/\lambda) \\ &= R \frac{f(R/\lambda)}{R/\lambda}, \end{aligned}$$

where $f(t) = \mathcal{V}(S \llcorner B(0,t) \times S^1)$, and S is the graph of u *a.e.* Now, f is increasing, thus is almost-everywhere differentiable. We have that

$$\begin{aligned} \frac{d}{dt}\left(\frac{f(t)}{t}\right) &= \frac{tf'(t) - f(t)}{t^2} \\ &\geq 0. \end{aligned}$$

To see this, let C_t be the *horizontal cone* over $S \llcorner \partial B(0,t) \times S^1$ defined by extending rays inward horizontally to the fiber over 0, that is, if $S = graph(u)$, then C_t would be the graph of $v(x) := u(tx/\|x\|)$. Then, since S is volume-minimizing, and $f'(t)$ is the mass of $S \llcorner \partial B(0,t) \times S^1$

$$\begin{aligned} tf'(t) &\geq \mathcal{V}(C_t) \\ &\geq \mathcal{V}(S \llcorner B(0,t) \times S^1) \\ &= f(t). \end{aligned}$$

Thus, $\frac{f(t)}{t}$ is increasing, and so, for $t < R$, $f(t)/t \leq A$, where $A = f(R)$, or

$$R\left(\frac{f(R/\lambda)}{R/\lambda}\right) \leq RA,$$

and the mass of $S_{\lambda,R}$ is uniformly bounded (in λ) for all $\lambda > 1$ sufficiently large. Thus any sequence $\{\lambda_n\}$ of stretches, as $\lambda \to \infty$, is uniformly bounded in mass over a fixed R. In order to apply the compactness theorem, we need to also show that the mod-2 boundaries $\partial_2 S_{\lambda_n} = S_{\lambda_n} \llcorner \partial B(0,R) \times S^1$ have bounded mass. But, by slicing, for any $\lambda > 0$,

$$\int_{s/2}^{s} \mathcal{V}\left(\partial\left(S_\lambda \llcorner B(0,r)\right)\right) dr \leq \mathcal{V}\left(S_\lambda \llcorner B(0,s)\right) \leq \lambda f(s/\lambda) \leq sA,$$

following [8, Theorem 9.8] and [3, 5.4.3(6)], and so for some r, $R/2 < r < R$, $\mathcal{V}\left(\partial\left(S_\lambda \llcorner B(0,r)\right)\right) \leq \frac{RA}{R/2} = 2A$. Also, by slicing, almost-all such choices of r have slices that are rectifiable. Then, the further stretch $S_{(\lambda R/r),R}$ has rectifiable mod-2 boundary $\partial\left(S_{\lambda R/r} \llcorner B(0,R)\right)$, with boundary volume $\mathcal{V}\left(\partial\left(S_{\lambda R/r} \llcorner B(0,R)\right)\right) \leq \left(\frac{R}{r}\right) 2A \leq 4A$, So, any sequence $\lambda_n \to \infty$ can be modified to one with a convergent subsequence. Set $S_{0,R}$ to be the limit of this subsequence, $S_{0,R} := \lim_n S_{\lambda_n} \llcorner B(0,R) \times S^1$. Taking a further subsequence, since the boundaries $\partial_2 S_{\lambda_n,R}$ are rectifiable and have no boundary themselves, it can be assumed that $\partial_2 S_{\lambda_n,R}$ also converges, to a rectifiable section $B \in \widetilde{\Gamma}(S^1_R \times S^1)$.

To see that S_0 is an h-cone, we use the fact that at each point of S there is an oriented tangent cone [6, Prop. 4.1] in the usual sense. Any non-vertical ray in the tangent cone at a point $p \in \pi^{-1}(x_0)$, under the sequence of horizontal stretches λ_n, will converge to a horizontal ray in $S_{0,R}$, and any point of $S_{0,R}$ is on such a horizontal ray, so $S_{0,R}$ is an h-cone. $\square$

Each element of a sequence S_{λ_i} of horizontal stretches of a mass-minimizing rectifiable section S in turn minimizes a modified functional, $\mathcal{V}_{\lambda_i}$ defined by

$$\mathcal{V}_{\lambda_i}(T) := \mathcal{V}\left(\left(\phi_{\frac{1}{\lambda_i}}\right)_{\#}(T)\right)\lambda_i,$$

where $T \in \widetilde{\Gamma}(B(0,R)\times S^1)$ with the metric induced from M by the stretch as before, and $\left(\phi_{\frac{1}{\lambda_i}}\right)_{\#}(T) \in \widetilde{\Gamma}(B(0,\frac{R}{\lambda_i})\times S^1)$ has the original metric from M. S_{λ_i} will minimize $\mathcal{V}_{\lambda_i}$ among all rectifiable sections with the same mod-2 boundary as $\partial_2 S_{\lambda_i} = S_{\lambda_i}\llcorner\partial B(0,R)\times S^1$.

The functionals $\mathcal{V}_{\lambda_i}$ will converge to a limiting functional $\mathcal{V}_0$. If T is a graph of some smooth function $u : B(0,R)\to S^1$, then $\mathcal{V}(u) = \int_{B(0,R)}\sqrt{1+\|\nabla u\|^2}dA$, and

$$\begin{aligned}\mathcal{V}_{\lambda_i}(T) &:= \int_{B(0,R/\lambda_i)}\sqrt{1+\lambda_i^2\left\|\nabla u|_{\lambda_i x}\right\|^2}dA\lambda_i\\ &= \int_{B(0,R)}\sqrt{1+\lambda_i^2\|\nabla u\|^2\frac{1}{\lambda_i^2}}dA\lambda_i\\ &= \int_{B(0,R)}\sqrt{\frac{1}{\lambda_i^2}+\|\nabla u\|^2}dA,\end{aligned}$$

where the second line is just change of variables. On such a current, clearly

$$\mathcal{V}_0(T) = \int_{B(0,R)}\|\nabla u\|\,dA.$$

This functional is called the "twisting" of the current in [1], at least in the case of the unit tangent bundle. The main property of the limiting functional is that it will be minimized by the h-cone of the volume-minimizer (the minimizer of the limit is the limit of the minimizers), among all rectifiable sections in $B(0,R)\times S^1$ with the same boundary as the h-cone. This property is of course not the general situation for arbitrary sequences of functionals, but will hold in this case.

Proposition 3.2. *If, for some sequence $\lambda_i\to\infty$, the stretches S_{λ_i} converge in $\widetilde{\Gamma}(B(0,R)\times S^1)$ to S_0, where S minimizes $\mathcal{V}$ (and so S_{λ_i} minimize $\mathcal{V}_{\lambda_i}$) among all such currents with the same boundary in $\partial B(0,R)\times S^1$, then S_0 minimizes $\mathcal{V}_0$ among all elements of $\widetilde{\Gamma}(B(0,R)\times S^1)$ with the same boundary as S_0.*

Proof. Recall that, taking an appropriate subsequence, the sequence of mod-2 boundaries

$$\partial_2 S_{\lambda_i} = S_{\lambda_i}\llcorner\partial(B(0,R)\times S^1)$$

converge to $C := \partial_2 S_0 = S_0\llcorner\partial(B(0,R)\times S^1)$ in $\widetilde{\Gamma}(\partial B(0,R)\times S^1)$. Then, for any $\epsilon>0$, there is an I sufficiently large so that, if $i>I$, then there is a rectifiable current T_i in $\partial B(0,R)\times S^1$ so that $\partial_2 T_i = (\partial_2 S_{\lambda_i} - \partial_2 S_0) = (S_{\lambda_i} - S_0)\llcorner\partial(B(0,R)\times S^1)$ with mass $\mathcal{M}(T_i)<\epsilon$.

Assume that S_0 does not minimize $\mathcal{V}_0$ among all rectifiable sections with the same boundary. Then, there is some $\epsilon>0$, and a rectifiable section $T\in\widetilde{\Gamma}(B(0,R)\times S^1)$ with $\partial_2 T = \partial_2 S_0$ so that $\mathcal{V}_0(T) < \mathcal{V}_0(S_0) - 4\epsilon$. Choose I above for this ϵ. Since the functionals also converge, choose $i>I$ sufficiently large so that $\mathcal{V}_{\lambda_i}(T) < \mathcal{V}_0(T)+\epsilon$, $\mathcal{V}_0(S_0) < \mathcal{V}_{\lambda_i}(S_0)+\epsilon$, and $\mathcal{V}_{\lambda_i}(S_0) < \mathcal{V}_{\lambda_i}(S_{\lambda_i})+\epsilon$ Then, $\partial_2(T+T_i) = \partial_2 S_{\lambda_i}$, $T+T_i\in\widetilde{\Gamma}(B(0,R)\times S^1)$, and

$$\begin{aligned}\mathcal{V}_{\lambda_i}(T+T_i) &\le \mathcal{V}_{\lambda_i}(T)+\mathcal{V}_{\lambda_i}(T_i)\\ &< \mathcal{V}_0(T)+\epsilon+\mathcal{M}(T_i)\\ &< \mathcal{V}_0(T)+2\epsilon\\ &< \mathcal{V}_0(S_0)-2\epsilon\\ &< \mathcal{V}_{\lambda_i}(S_0)-\epsilon\\ &< \mathcal{V}_{\lambda_i}(S_{\lambda_i}),\end{aligned}$$

which contradicts the fact that S_{λ_i} minimizes $\mathcal{V}_{\lambda_i}$. $\square$

With this result. we can identify the kind of singular behavior that can occur.

3.2. **Index of singularities.** A singular point $x \in \Sigma$ of a mass-minimizing $S \in \widetilde{\Gamma}(M)$ as above determines an index, an integer k_x generalizing the index of vector fields.

Let $x \in \Sigma$, and let $S \in \widetilde{\Gamma}(M)$ be mass-minimizing. For almost-all $\epsilon > 0$ sufficiently small, the restriction $S_\epsilon := S \llcorner \pi^{-1}(\partial B(x,\epsilon))$ determines a rectifiable section $S_\epsilon \in \widetilde{\Gamma}(S^1 \times S^1)$. If $P : S^1 \times S^1 \to S^1$ is the projection onto the second factor (the fiber), then k_ϵ, defined by $P_\#(S_\epsilon) = k_\epsilon \left[S^1\right]$, is just the degree of the map. Choose a sequence $\lambda_i \to \infty$ so that, on $B(0,1) \times S^1$, S_{λ_i} converges to an h-cone H. Define the *index* of S at x, k_x, to be

$$k_x := \lim_{i \to \infty} k_{\lambda_i} = k_H,$$

which can be viewed as either a limiting index, or, equivalently, the index of the h-cone. Of course, the index is also defined for non-minimizing sections, but it does seem to require something like the existence of an h-cone to guarantee existence and boundedness of the limit.

Proposition 3.3. *If x has index 0, then x is a regular point.*

Proof. Assume that x is a singular point with index 0. Then, the h-cone H of S at x, in $\widetilde{\Gamma}(B(0,R) \times S^1)$, also has index 0, so that $H \llcorner \partial B(0,R) \times S^1$ is a rectifiable section of degree 0, represented as a map $u : S^1 \to S^1$ of degree 0, possibly with singularities or vertical portions. Thus $u = e^{if}$ for some real-valued map f (in general, rectifiable section of the trivial line bundle), and H is the graph of $u(r,\theta) = e^{if(\theta)}$. The h-cone minimizes the functional

$$\begin{aligned} \mathcal{V}_0(u) &= \int\int_{B(0,R)} \|\nabla f\|\, r dr d\theta \\ &= \int\int_{B(0,R)} \left\| \frac{\partial f}{\partial \theta} \right\| dr d\theta. \end{aligned}$$

For any function $h(r,\theta)$ with support in the interior of $B(0,1)$,

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\bigg|_0 \int\int_{B(0,R)} \|\nabla(f+th)\|\, r dr d\theta \\ &= \int\int_{B(0,R)} \frac{\frac{1}{r}\frac{\partial f}{\partial \theta}\frac{\partial h}{\partial \theta}}{\left\|\frac{\partial f}{\partial \theta}\right\|} r dr d\theta \\ &= \int\int_{B(0,R)} \frac{\frac{\partial f}{\partial \theta}\frac{\partial h}{\partial \theta}}{\left\|\frac{\partial f}{\partial \theta}\right\|} dr d\theta. \end{aligned}$$

For some small $\epsilon > 0$, take h to be $h(r,\theta) := (M - \epsilon(R-r)/R - f(\theta))_-$, where by $()_-$ we mean the nonpositive part of the function, $(g)_-(x) := \inf\{g(x), 0\}$, and $M = \sup\{f(\theta)\}$. Since, in $Supp(h)$, which has positive measure, $\partial h/\partial\theta = -\partial f/\partial \theta$, $\frac{\frac{\partial f}{\partial \theta}\frac{\partial h}{\partial \theta}}{\left\|\frac{\partial f}{\partial \theta}\right\|} = -\left\|\frac{\partial f}{\partial \theta}\right\|$. Unless f is *a.e.* constant, the integral will be negative, which would contradict minimality of S. Thus f must be constant, and so the graph of S is continuous at x. $\square$

As a corollary, we can now show that at any singular point, the entire fiber over the point is contained in the support of S.

Corollary 3.4. *If $x \in \Sigma$ is a singular point for a mass-minimizing $S \in \widetilde{\Gamma}(M)$, then $\pi^{-1}(x) \subset Supp(S)$.*

Proof. The singularity has to have nonzero index, which implies that the entire fiber is in the support. $\square$

Corollary 3.5. *The singular points of a mass-minimizing $S \in \widetilde{\Gamma}(M)$ are isolated.*

Proof. If a singular point x_0 is the limit of other singular points and is of index k, consider the current defined in $B(0,R)\times S^1$ as Sv, where $v(r,\theta)=e^{-ik\theta}$, which has support

$$Supp(Sv)=\left\{ (x,yv(x))\in B(0,R)\times S^1 \middle| (x,y)\in S\right\}$$

and has the obvious tangent planes and multiplicities inherited from S. Clearly Sv has index 0 at x_0. However, since there is a sequence of singular points of S approaching x_0, and for each such point x_i the entire fiber is contained in the support of S, this will also be true for Sv since v has only x_0 as a singular point, so is regular at each x_i, and thus Sv is still singular at x_i as claimed. Since $Supp(Sv)$ is closed, it must contain the entire fiber over x_0, so x_0 is a singular point of Sv, and is of index 0 at x_0.

Sv does not minimize the volume, but it does minimize a twisted volume $\mathcal{V}_v$ defined for rectifiable sections of $B(0,R)\times S^1$ by $\mathcal{V}_v(T)=\mathcal{V}(Tv^{-1})$. Stretching as before, the h-cone Hv (where H is the h-cone of S for a specific sequence of stretches) will minimize

$$\mathcal{V}_{v,0}(w)=\int\int_{B(0,R)\times S^1}\left\|\nabla(wv^{-1})\right\| dA.$$

Since Hv, the minimizer of this functional (with the boundary conditions inherited from S), is an h-cone, we have the variational condition, as before, if $w=e^{if}$,

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_0 \int_0^R\int_0^{2\pi}\|\nabla(f+th(r,\theta)+k\theta)\|\,rd\theta dr \\ &= \int_0^R\int_0^{2\pi}\frac{\left(\frac{\partial f}{\partial\theta}+k\right)\frac{\partial h}{\partial\theta}}{\left\|\frac{\partial f}{\partial\theta}+k\right\|}d\theta dr. \end{aligned}$$

As with Proposition (3.3), taking h to be $h(r,\theta):=(M-\epsilon(R-r)/R-f(\theta)-k\theta)_{-}$, where M is the maximum of f, would provide a contradiction unless $\frac{\partial f}{\partial\theta}+k\equiv 0$.

Thus, the h-cone of S at x_0 is that of v^{-1} itself, which has only the singularity at 0. If S had a sequence of singularities approaching x_0, the h-cone would also. Thus the singularities of S are isolated. □

This result also shows:

Corollary 3.6. *Each singularity is of index* ± 2.

Proof. In [1], it is shown that only an isolated singularity of index ±2 can be a rectifiable section. □

We finally are in a position to prove the main result, which is

Theorem 3.7. *Let M be a circle bundle over a compact Riemann surface Σ, with the Sasaki metric, so that the Euler number $e(M)$ of the circle bundle is even. Then, there is a mass-minimizing rectifiable section S which is moreover a smooth, embedded minimal surface in M. Topologically, S is Σ with an finite number of cross-caps.*

Proof. For a volume-minimizing section $S\in\widetilde{\Gamma}(M)$ as shown to exist by Theorem 2.3, we have shown that there are a finite number of pole points $x\in\Sigma$, over each of which S contains the entire fiber, and the index is ±2. On the complement of those singular fibers, S is a continuous graph and is a minimal surface, so it is smooth (since it is codimension 1 in a 3-manifold.).

In an ϵ-neighborhood of the singular fibers, the graph is asymptotically that of $u=e^{\pm 2i\theta}$, and so the current is C^1 at these points. Since it is of class C^1 and minimal (weak mean curvature vanishing), it is a smooth minimal surface. That the structure of the surface in a neighborhood of a singularity is a cross-cap can be found in [1]. The topological statement then follows. □

Remark 3.8. While it seems clear that a volume-minimizing rectifiable section should have a minimal number of pole points (cross-caps), since singularities add to volume [2], we do not make that claim here. It may be the case that the smallest number of cross-caps will depend on the homology class of the section, since for "large" elements of $\pi^1(\Sigma)$, the section constructed by this theorem may manage to have less mass with additional cross-caps. If, however, the Euler class of the bundle is 0, there will be a smooth minimizer which is a global section, since in that case we can work with the class of currents which are limits of smooth sections. All singularities would then be of index 0, and so could not exist.

A similar statement holds for bundles with odd Euler class, and in fact for any nonzero Euler class, except that the minimizer will not be a section, but will generically be the double of a section with a finite number of vertical fibers. The Hopf fibration $S^3 \to S^2$ provides a model for such double-sections, with the equatorial S^2 in S^3 being the double-section, meeting each fiber at two points except for one fiber contained in the surface. These surfaces will then be 2-sided minimal surfaces in M which are covers of the base space Σ except over finitely many points.

Corollary 3.9. *Let M be a circle bundle over a compact Riemann surface Σ, with the Sasaki metric, so that the Euler number $e(M)$ of the circle bundle is nonzero. Then, there is a smooth, embedded 2-sided minimal surface S in M which intersects all but finitely many fibers of $\pi : M \to \Sigma$ twice. Topologically, S is a two-fold cover of Σ except over a finite number of ramification points.*

Proof. The construction of such a section is to take the projection of $M \to \Sigma$ to $\overline{M} := M/x \sim x \cdot (-1)$ identifying "antipodal" points on each fiber under the right $U(1)$-action. $\overline{M}$ will again be a circle bundle over Σ, with double the Euler class. The main theorem then shows the existence of a mass-minimizing rectifiable section σ of $\overline{M}$ which is a smooth minimal submanifold with finitely many cross-caps. Pulling that back to M will no longer be a rectifiable section, but will be a minimal submanifold S of M so that all but finitely many fibers intersect the submanifold at 2 points (antipodal points). The preimages of the cross caps (connected sums with a projective plane) become a trivial connected sum as in the example of $M = S^3$ above, and the minimal surface S will be two-sided. □

Remark 3.10. The "exceptional" points $\{x_1, \ldots, x_n\}$ of Σ over which S is not a two-fold cover satisfy $S \cap \pi^{-1}(x_i) = \pi^{-1}(x_i) \cong S^1$.

4. Applications to geometric structures on 3-manifolds

Thurston's geometrization conjecture, famously proved by G. Perelman, is that any compact oriented 3-manifold can be decomposed into components, each of which have a specific geometric structure from a list of 8 possible types (cf. [13, 12]). In this context a geometric structure is a complete, locally-homogeneous metric. Of these 8 types of geometries, 4 occur as Sasaki metrics on circle bundles over compact Riemann surfaces with even Euler class and constant curvature, specifically structures of the types S^3, $\mathbb{H}^2 \times \mathbb{R}$, $S^1 \times S^2$, and $\mathbb{R}^3$. In each such class, the main theorem of this paper asserts the existence of a smooth, minimal submanifold transverse to the fibers of the bundle except for finitely many fibers contained within the submanifold. The existence of such surfaces is known in several of these cases, but this theorem provides an explicit realization of the submanifold.

One caution is that the Seifert fibrations considered in these geometries often have exceptional fibers, so are not circle bundles over a Riemann surface (instead, over an orbifold). For example, the lens spaces $L(p, q)$ as Seifert fibered spaces fibered by the image of the Hopf fibration have exceptional fibers, unless $q = 1$ [4, p. 87]. Similarly, spaces with the geometry of $PSL(2, \mathbb{R})$ will have an exceptional fiber. The results of this paper do not apply in that situation, although it should be possible to extend the result to this case with some modification.

In the case $e(M) = 0$, zero Euler class, there will be smooth mass-minimizing sections of the bundle $M \to \Sigma$. It was proved in [7] that such a minimal surface exists; any incompressible horizontal 2-sided surface is isotopic to a unique minimal surface. In the present case, for any homology class of sections, there will be at least one smooth mass-minimizer by [6]. Moreover, applying an argument of [5], used there only for the unit tangent bundle, each such minimizing section σ defines a flat connection ω on $M \to \Sigma$ as a principal S^1-bundle, simply by taking the horizontal distribution to be the tangents to the section and translates of it. The unit normal field to this foliation will be divergence free precisely when the section is minimal, and so the form $*\omega / |\omega|$ will be closed, hence a calibration, only when the section is minimal. In that case, the minimal section is calibrated by this form, and so, any other section in the same homology class will have more volume than σ or its translates, showing uniqueness up to translation. The one-dimensional family of translations reflects the one-dimensional nullity of the second variation.

If the Euler class $e(M)$ is nonzero and even, then the rectifiable section σ produced by this theorem will be one-sided, because of the existence of cross-caps. In [9], such surfaces are shown to exist, the present Theorem 3.7 provides an alternate construction of them.

For the case of the lens spaces $L(p,1)$, since the Euler characteristic of $L(p,1)$ as a circle bundle over S^2 is just p, whenever p is even then $\widetilde{\Gamma}(BL(p,1)) \neq \emptyset$, so there will be a mass-minimizing rectifiable section with a finite number of cross-caps (at least $p/2$, of course, but not necessarily exactly $p/2$). This section is neither horizontal nor vertical in the sense of Seifert fibrations (neither transverse to all fibers, nor a union of fibers). They also are one-sided Heegaard splittings of the manifold [10]. When p is odd we can apply Corollary 3.9 to show the existence of a minimal 2-sided minimal surface. Again, these minimal surfaces are neither horizontal nor vertical. As a result, according to [9], the surface cannot be stable. The example of the equatorial S^2 in S^3 shows all of these properties.

Another interesting consequence of this construction on the lens spaces $L(2p,1)$ is that these minimal surfaces will then lift to be minimal surfaces in S^3, containing a finite number of fibers of the Hopf fibration, but otherwise transverse to the fibers. However, neither the number of cross-caps, nor the number of exceptional points, is determined by the conditions we have, so it is not clear which minimal surfaces in S^3 are represented in this way.

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