Design Guarantees for Resilient Robot Formations on Lattices

Luis Guerrero-Bonilla, David Saldaña, and Vijay Kumar

***Abstract*—This paper presents guarantees to satisfy resilience on the communication network of robot formations. In these resilient networks, cooperative robots can achieve consensus in the presence of faulty or malicious robots. We propose a design framework on triangular and square lattices, providing an underlying structure for proximity-based robot networks. We present sufficient conditions on the robot communication range to guarantee resilient consensus. Our results can be used to design robot formations considering obstacles, number of robots, and energy usage. Additionally, robot networks with homogeneous and heterogeneous communication range are studied. We support our theoretical analysis with simulations on selected scenarios.**

***Index Terms*—Distributed Robot Systems, Networked Robots, Multi-Robot Systems.**

I. Introduction

One of the most relevant techniques in distributed robotic systems is consensus, where each robot only needs to communicate with its nearest neighbors to agree on a global variable for coordination [1]–[8]. Most of the approaches in distributed robotic systems assume that all the robots are cooperative and their sensors do not fail. However, real applications are very susceptible to failures or external attackers who want to control the whole robot network [9]–[11].

The work in [9], [12], [13] presents the *Weighted Mean-Subsequence-Reduced (W-MSR)* algorithm, which provides an update rule for a networked system to achieve asymptotic convergence to a value in the convex hull of the initial values of non-misbehaving nodes. The algorithm is effective if the communication graph satisfies a property known as r-*robustness*, which provides sufficient conditions for the W-MSR algorithm to work. Although there are algorithms to check and determine the robustness of a graph, they are computationally inefficient [14], as the work in [13] provides an analysis of the complexity of determining the extent of r-robustness of any given network, concluding that it is NP-hard.

In order to provide the sufficient conditions for the W-MSR algorithm to provide resilience against misbehaving agents in a robot formation, it is required to ensure that the communication network of such formation satisfies the desired r-robustness. Our previous work [15] provides a systematic method to construct r-robust graphs. However, the work in [16] shows that not all graphs are realizable on the plane under euclidean distance constraints, and therefore a well designed robust communication graph might not correspond to a feasible planar arrangement of robots, making the construction of robust robot formations a challenge.

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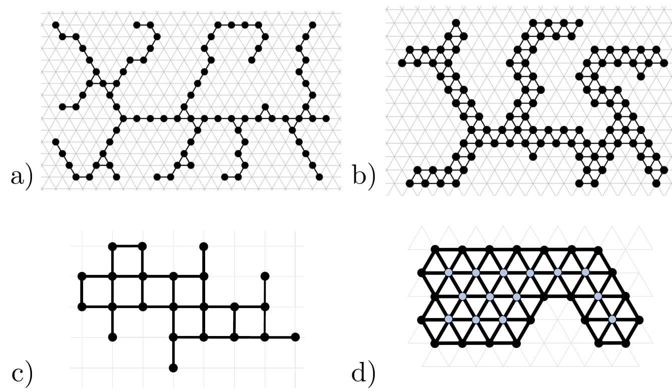

Fig. 1. Examples of robot formations: a) connected formation, b) triangular formation, c) square formation, d) formation with heterogeneous communication radii. The robots with large and small communication range are depicted in black and light blue, respectively.

Some approaches in the literature aim to increase the robustness of the robot network by increasing the network connectivity. The use of the algebraic connectivity of the communication network can directly increase the network robustness [11], [17], but it also conglomerates the robots when the communication radius is fixed. Taking advantage of their motion capabilities, robots can coordinate to generate periodic connections [18]. However, this method requires a previous coordination of the robots to identify the way the robots can periodically meet up. Our recent work [15] suggests an underlying lattice structure in the robot formation to design and extend robust formations. These properties in well-structured formations can generate modular configurations that can be extended to large networks. In [19], we presented a triangular formation for robot networks that can achieve resilient consensus in the presence of a single non-cooperative robot. The work in [20] introduces the use of a lattice structure for hexagonal robot formations in relation to r-robustness. Such structure allows to calculate a communication range for all robots that guarantees the maximum r-robustness possible given the number of robots. However, it requires a large number of robots to be distributed in a very specific pattern, which may hinder the use or viability of such formations in the presence of obstacles, narrow areas, or limited number of robots.

In this paper, we address the problem of designing the communication network of a group of robots distributed in the plane on triangular and square lattices, in order to ensure r-robustness. We look for spare formations that allow to have guarantees on robustness, but do not demand a highly structured pattern as in [20], allowing flexibility to arrange robots in the plane and deal with obstacles or limited number of robots. In Figure 1, we illustrate some of the configurations that can be analyzed using our methodology. In contrast to the methods in the literature, we can have complex environments or small number of robots to generate robust networks. The main contribution of this paper is offering a systematic method to determine the sufficient communication range to ensure resilient consensus in a robot formation. We propose a sufficient conditions to design complex formations that can be deployed in environments with obstacles, or narrow regions. Additionally, we propose a method to use of heterogeneous robots in their communication range, allowing for some optimization in energy usage.

II. FUNDAMENTALS OF r-ROBUSTNESS

Based on the nomenclature in [12] and [13], let an undirected graph be described by the pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes, and $\mathcal{E}$ is the set of edges of the graph, so that an edge $(i, j) \in \mathcal{E}$ indicates that nodes $i, j \in \mathcal{V}$ are connected. The set of neighbors of node i is denoted by $\mathcal{V}_i = \{j \in \mathcal{V} | \, (i, j) \in \mathcal{E}\}$, and the degree of a node i is denoted by $|\mathcal{V}_i|$, where the operator $|\cdot|$ denotes the cardinality of a set. Suppose the ith node shares a value η_i with its neighbors in the network, and updates its value over time according to a nominal rule of the form

$$\eta_i\left[t+1\right] = w_{ii}\left[t\right]\eta_i\left[t\right] + \sum_{j\in\mathcal{V}_i} w_{ij}\left[t\right]\eta_j\left[t\right], \qquad (1)$$

where $\eta_j\left[t\right]$ is the shared value from the neighbor j to i at time t, $w_{ij} > 0$, $\sum_j w_{ij}\left[t\right] = 1$.

Definition 1 (Malicious node). *A node $i \in \mathcal{V}$ is said to be malicious if it sends $\eta_i\left[t\right]$ to all of its neighbors at each time-step, but does not follow the nominal rule* (1) *at some time-step.*

Note that the definition of a malicious agent can be extended to include intentionally non-cooperative or manipulative robots, as well as defective or unintentionally non-cooperative robots. The W-MSR algorithm [9], [12], [13] involves three steps: First, node i creates a sorted list of the received values. Secondly, the list is compared to $\eta_i[t]$, and the F larger and smaller values are removed. The remaining values in the list are denoted by $\mathcal{R}_i\left[t\right]$. Third, node i updates its value with the following rule:

$$\eta_i\left[t+1\right] = w_{ii}\left[t\right]\eta_i\left[t\right] + \sum_{j\in\mathcal{R}_i[t]} w_{ij}\left[t\right]\eta_j\left[t\right], \qquad (2)$$

Sufficient conditions for the W-MSR to ensure asymptotic convergence of the consensus are given by the graph property known as r-robustness, stated in the following terms.

Definition 2 (F-local set). *A set $\mathcal{S} \subset \mathcal{V}$ is F-local if it contains at most F nodes in the neighborhood of all other nodes for all t, i.e., $|\mathcal{V}_i\left[t\right] \bigcap \mathcal{S}| \leq F, \forall i \in \mathcal{V} \setminus S$, $\forall t \in \mathbb{Z}_{\geq 0}, F \in \mathbb{Z}_{\geq 0}$.*

Definition 3 (r-reachable subset). *The subset $\mathcal{S} \subset \mathcal{V}$ is said to be r-reachable if there exists $i \in \mathcal{S}$ such that $|\mathcal{V}_i \setminus \mathcal{S}| \geq r$, where $r \in \mathbb{Z}_{\geq 0}$, that is, if it contains a node that has at least r neighbors outside that set.*

Definition 4 (r-robust graph). *A graph $\mathcal{G}$ is said to be r-robust if for every pair of nonempty disjoint subsets of $\mathcal{V}$, at least one of the subsets is r-reachable.*

The work in [12] relates the asymptotic convergence of the consensus to the W-MSR algorithm and r-robustness:

Theorem 1 ([12]). *Consider a time-invariant network modeled by a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where each normal node updates its value according to the W-MSR algorithm with parameter F. Under the F-local malicious model, resilient asymptotic consensus is achieved if the topology of the network is $(2F+1)$-robust. Furthermore, a necessary condition is for the topology of the network to be $(F+1)$-robust.*

The work in [13] provides an analysis of the complexity of determining the extent of r-robustness of any given network, concluding that it is coNP-complete. The work in [9] presents a method to increase the number of nodes in a r-robust graph by continually adding nodes with incoming edges from at least r nodes in the existing graph:

Theorem 2 ([9]). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an r-robust graph. Then the graph $\mathcal{G}' = (\mathcal{V} \cup \{v'\}, \mathcal{E} \cup \mathcal{E}')$ where v' is a new vertex added to $\mathcal{G}$ and $\mathcal{E}'$ is the edge set related to v', is r-robust if $|\mathcal{V}_{v'}| \geq r$.*

The work in [15] introduced the concept of F-elemental graphs and proposed a method to build them as follows:

Definition 5 (F-elemental graph). *An F-elemental graph is a graph with $n = 4F+1$ nodes that is r-robust with $r = 2F+1$ for some positive integer value of F.*

Theorem 3 ([15]). *A graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with $|\mathcal{V}| = 4F+1$ is $2F+1$-robust if:*

1) *There is a set $\mathcal{V}' \subset \mathcal{V}$ of $2F$ nodes that are connected to all nodes in the graph.*
2) *The set of nodes $\mathcal{V} \backslash \mathcal{V}'$ forms a connected subgraph.*

The link between the r-robustness theory and formations of robots is given by the communication network among the robots, which gives rise to a graph. Let us denote $\mathbf{x}_i \in \mathbb{R}^2$ as the position of robot i on the plane. In the following, we use a very simplistic disk model to describe the communication network among robots, where the operator $\|\cdot\|$ denotes the 2-norm.

Definition 6 (Communication graph). *Given a set of robots $\mathcal{V}$ with communication radius R, a graph $\mathcal{G}_R\left(\mathcal{V}, \mathcal{E}_R\right)$ with node set $\mathcal{V}$ and edge set defined by*

$$\mathcal{E}_R = \{(i, j) \, | \|\mathbf{x}_i - \mathbf{x}_j\| \leq R\}, \qquad (3)$$

is called the communication graph of the set $\mathcal{V}$.

III. r-ROBUST FORMATIONS ON LATTICES

In this section, we explore the design of the communication network of a formation of robots with an underlying lattice structure. A lattice is a set of linear combinations with integer coefficients of the elements of a basis of $\mathbb{R}^2$. The elements of the set are *lattice points*. Let $\mathbf{v}_1$ and $\mathbf{v}_2$, be such basis, and let $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \ell$, where we call ℓ the *lattice length*. A lattice in the plane is given by

$$\mathbb{L} = \{a_i\mathbf{v}_1 + b_i\mathbf{v}_2 : \mathit{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \mathbb{R}^2; a_i, b_i \in \mathbb{Z}\}. \quad (4)$$

The work in this paper is mainly focused on two types of lattices. The first one is the *triangular lattice* $\mathbb{L}_\triangle$ with basis

$$B_\triangle = \left\{\mathbf{v}_{1\triangle} = \frac{\ell}{2}\begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}, \mathbf{v}_{2\triangle} = \ell\begin{bmatrix}1\\0\end{bmatrix}\right\}. \quad (5)$$

The second one is the *square lattice* $\mathbb{L}_\square$, with basis

$$B_\square = \left\{\mathbf{v}_{1\square} = \ell\begin{bmatrix}1\\0\end{bmatrix}, \mathbf{v}_{2\square} = \ell\begin{bmatrix}0\\1\end{bmatrix}\right\}. \quad (6)$$

A robot is part of the lattice $\mathbb{L}$ if its position is equal to a lattice point, such that $\mathbf{x}_i \in \mathbb{L}$. A lattice point that is not occupied by a robot is referred to as a *available*. Given a lattice length ℓ, we now define a graph capturing the proximity of the robots in a set.

Definition 7 (Proximity graph). *Given a set of robots $\mathcal{V}$ and a distance ℓ, the graph $\mathcal{G}_\ell(\mathcal{V}, \mathcal{E}_\ell)$ with node set $\mathcal{V}$ and edge set defined by*

$$\mathcal{E}_\ell = \{(i,j) \,|\, \|\mathbf{x}_i - \mathbf{x}_j\| \le \ell\}. \quad (7)$$

is called the proximity graph *of the set $\mathcal{V}$.*

While the lattice provides some underlying structure to the robot formation by predefining the allowed positions of the robots, it allows for different formations to be realized. The particular structure of each formation can be exploited to ensure the desired robustness. We build our mathematical framework based on the sufficient communication ranges to satisfy the requirements for the creation of F-elemental graphs in the communication network of a robot formation, as well as to preserve the robustness by appending new robots to the formation. We describe the communication range by a function $R_* : \mathbb{Z}_{\ge 1} \to \mathbb{R}$ that maps the number of robots m to a distance where m robots are ensured to be reached. In particular, we study three main communication ranges R_-, $R_\triangle$, $R_\square$, corresponding to formations of robots satisfying the constraints of a *connected formation*, *triangular formation* and *square formation* respectively. Examples of these formations are shown in Figure 1.

A. Connected formations

We begin our study by only imposing the connectivity constraint in the associated proximity graph of a formation of robots, with no additional constraints regarding the placement of the robots in the plane.

Definition 8 (Connected formation). *A formation of n robots is said to be connected if its associated proximity graph $\mathcal{G}_\ell$ is connected.*

A connected formation on a lattice allows to easily calculate the minimum number of robots within a given distance around a single robot in the formation.

Lemma 1. *In a connected formation of n robots, every robot has at least $1 \le m \le n-1$ robots within a distance*

$$R_-(m) = m\ell. \quad (8)$$

Proof. Consider robot 0 at position $\mathbf{x}_0$. Since the formation is connected, there is at least one more robot within a distance ℓ. Let us denote the distance between robots i and j by $d_{i,j} = \|\mathbf{x}_j - \mathbf{x}_i\|$, and consider the sparsest case where there is only one robot, denoted by 1, at the maximum distance such that $d_{0,1} = \ell$.

Considering again the sparsest case, the next robot in the connected formation, identified as 2, will be located at the maximum distance from $\mathbf{x}_0$. Then, $d_{0,2}$ can be expressed using the cosine law $d_{0,2}^2 = d_{0,1}^2 + d_{1,2}^2 - 2d_{0,1}d_{1,2}\cos\theta = 2\ell^2(1-\cos\theta)$, where θ is the angle between the line segments $\overline{\mathbf{x}_0\mathbf{x}_1}$ and $\overline{\mathbf{x}_1\mathbf{x}_2}$. The angle $\theta = \pi$ maximizes $d_{0,2}$, arranging the robots in a collinear fashion. Assuming the sparsest case, the maximum distance from robot 0 to the mth robot can be expressed as

$$\begin{aligned} d_{0,m}^2 &= d_{0,m-1}^2 + d_{m-1,m}^2 - 2d_{0,m-1}d_{m-1,m}\cos\theta \\ &= d_{0,m-1}^2 + \ell^2 - 2d_{0,m-1}\ell\cos\theta, \end{aligned} \quad (9)$$

which is maximized at $\theta = \pi$, where all robots are arranged in a collinear formation. This simplifies the equation to

$$d_{0,m} = d_{0,m-1} + \ell. \quad (10)$$

Therefore, the maximum distance from robot 0 to the mth robot can be expressed as a recursive function $R_-(m) = R_-(m-1) + \ell$ with initial condition $R_-(0) = 0$. Its solution is given by $R_-(m) = (1)^n R_-(0) + \sum_{k=0}^{m} \ell = m\ell$. □

Considering the sparsest arrangement of robots in a connected formation, we now compute a minimum communication range for the robots to guarantee resilience in the communication network.

Lemma 2. *Given a set $\mathcal{V}$ of $4F+1$ robots in a connected formation, if the communication range of every robot is $R \ge R_-(3F)$, the associated communication graph of the formation is $(2F+1)$-robust.*

Proof. Let $\mathcal{C}_0 \subset \mathcal{V}$ be a connected subset with $3F+1$ robots. Since each robot has a communication radius of at least $R = R_-(3F)$, each of them has at least $3F$ neighbors (by Lemma 1). Thus the associated communication graph of $\mathcal{C}_0$ is complete. Let us define the set $\mathcal{N}_0 = \mathcal{C}_0$, which contains the robots that are communicated with every other robot in $\mathcal{C}_0$.

Consider the subsets $\mathcal{C}_i = \mathcal{C}_{i-1} \cup \{i\}$, where $i \in \{1, \ldots, F\}$ represents one of the remaining robots in $\mathcal{V}/\mathcal{C}_{i-1}$ such that the associated proximity graph of $\mathcal{C}_i$ is connected. This ensures that i can communicate with at least $3F$ robots in $\mathcal{C}_i$. Let the set of robots that can communicate with every robot in $\mathcal{C}_i$ be denoted by $\mathcal{N}_i$.

Suppose that i is adjacent to a robot in $\mathcal{N}_0$. Then, at most one robot k_i may not be communicated with i. On the other hand, suppose i is adjacent to a robot $j \in \mathcal{V} \backslash \mathcal{C}_0$. Then, i can communicate with at least $3F - 1$ robots in $\mathcal{N}_j$, so that at most one robot k_i may not communicate with i. In either case, suppose the one robot k_i belongs to the set $\bigcap_{j=0}^{i-1} \mathcal{N}_j$. Then, $\mathcal{N}_i = \bigcap_{j=0}^{i-1} \mathcal{N}_j \backslash \{k_i\}$, removing at most one robot from the set that can communicate with every other robot.

Once robot $i = F$ is considered, the set $\mathcal{N}_F$ contains the robots that can communicate with every robot in $\mathcal{C}_F = \mathcal{V}$. At most F robots are removed from $\mathcal{N}_0$, so that $|\mathcal{N}_F| = |\mathcal{N}_0| - F = 2F + 1$. Thus, the conditions to create an F-elemental graph are satisfied: *i)* there is a subset of $2F$ robots in $\mathcal{N}_F$ that are connect to every robot in $\mathcal{V}$, and *ii)* the additional robot that is connected to every robot in $\mathcal{V}$ generates a star-subgraph connecting the rest of the robots. Therefore, the communication network of $\mathcal{V}$ is $(2F+1)$-robust by Theorem 3. □

We can guarantee r-robustness of a connected formation by selecting the proper communication radius for the robots as stated in the following theorem.

Theorem 4. *A group of $n \geq 4F+1$ robots in a connected formation is ensured to be $2F+1$-robust, if the communication radius of each robot satisfies $R \geq R_{-}(3F)$.*

Proof. The communication graph of a set of $4F+1$ robots in a connected formation with $R \geq R_{-}(3F)$ is $(2F+1)$-robust by Theorem 2. From Lemma 1, we know that we can connect a new robot to the connected formation and it will be within the communication range of at least $3F$ robots. Since $3F \geq 2F+1$, the communication graph of the initial $4F+1$ robots and the new robot preserves the $(2F+1)$-robustness by Theorem 2. We can add the rest of the robots one by one, using a communication rage of $R \geq R_{-}(3F)$ for each robot, satisfying the minimum number of neighbors to preserve the robustness (by Lemma 1). Following this procedure, it is possible to construct any connected formation with the desired number of robots. □

B. Triangular formations

The connectivity constraint on the proximity graph of the formations is simple and allows a great flexibility to distribute the robots on the plane, but it does not take advantage of the particular structure of the lattice. We now define a formation that exploits the triangular lattice, and calculate the corresponding required communication range to have m reachable robots.

Definition 9 (Triangular formation). *A connected formation of $n \geq 3$ robots with positions $\mathbf{x}_i \in \mathbb{L}_\triangle$ is said to be triangular if:*

(i) Every robot in the formation is the vertex of an equilateral triangle with edge length equal to ℓ.

(ii) Adjacent triangles share two vertexes and an edge.

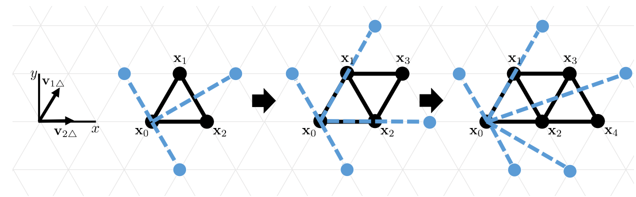

Fig. 2. The construction of a triangular formation that maximizes the distance from the first allocated robot. This shows how every subsequent robot in the formation is selected as to maximize the distance from robot 0 to the next robot. The blue lines denote the lattice locations where the next robot can be placed.

An example of a triangular formation is shown in Figure 1 c). These constraint is stronger than just maintaining connectivity, and straight line formations are no longer allowed. Every time a new robot is added to the formation, a new triangle must be created.

Lemma 3. *If a group of $n \geq 3$ robots are in a triangular formation, every robot has at least $1 \leq m \leq n-1$ robots within a distance*

$$R_\triangle(m) = \frac{1}{2}\ell\sqrt{m^2 + 3\left(\frac{1+(-1)^{m+1}}{2}\right)}. \quad (11)$$

Proof. Similarly to Lemma 1, the proof consists on constructing the triangular formation in which each subsequent robot is as far away as possible from the initial one, so that the maximum distance to the mth robots is considered. Satisfying the constraints of the triangular formation, let robots 0, 1 and 2 be located at $\mathbf{x}_0$, $\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{v}_{1\triangle}$, $\mathbf{x}_2 = \mathbf{x}_0 + \mathbf{v}_{2\triangle}$, all of them lattice points in $\mathbb{L}_\triangle$ (refer to Figure 2). Keeping up with the triangular formation constraints, the position $\mathbf{x}_3$ of robot 3 can be chosen among three lattice locations, given by $\mathbf{x}_0 + \mathbf{v}_{1\triangle} - \mathbf{v}_{2\triangle}$, $\mathbf{x}_0 - \mathbf{v}_{1\triangle} + \mathbf{v}_{2\triangle}$, and $\mathbf{x}_0 + \mathbf{v}_{1\triangle} + \mathbf{v}_{2\triangle}$. We can directly verify that the third option maximizes $\|\mathbf{x}_3 - \mathbf{x}_0\|$. Robot 4 can be located on four lattice locations, but two of them have already been verified to be closer to $\mathbf{x}_0$ than $\mathbf{x}_3$, so only two locations need to be evaluated, namely $\mathbf{x}_0 + 2\mathbf{v}_{1\triangle}$ and $\mathbf{x}_0 + 2\mathbf{v}_{2\triangle}$. Since both are at the same distance from robot 0, without loss of generality, let robot 4 be at $\mathbf{x}_4 = \mathbf{x}_0 + 2\mathbf{v}_{2\triangle}$. This pattern repeats, having to check two new lattice positions for the one that maximizes the distance to robot 0. For the mth robot in the formation, these two positions are given by $\{\mathbf{x}_0 + \mathbf{v}_{1\triangle} + \frac{m-1}{2}\mathbf{v}_{2\triangle}, \mathbf{x}_0 - \mathbf{v}_{1\triangle} + \frac{m-1}{2}\mathbf{v}_{2\triangle}\}$ for $m \geq 3$ odd, and $\{\mathbf{x}_0 + \frac{m}{2}\mathbf{v}_{2\triangle}, \mathbf{x}_0 + 2\mathbf{v}_{1\triangle} + \frac{m-4}{2}\mathbf{v}_{2\triangle}\}$ for $m \geq 4$ even. Evaluating $\|\mathbf{x}_m - \mathbf{x}_0\|$, it is straight forward to verify that the position for the mth robot that maximizes the distance from $\mathbf{x}_0$ is given by

$$\mathbf{x}_m = \begin{cases} \mathbf{x}_0 + \mathbf{v}_{1\triangle} + \frac{m-1}{2}\mathbf{v}_{2\triangle} & \text{if } m \text{ is odd} \\ \mathbf{x}_0 + \frac{m}{2}\mathbf{v}_{2\triangle} & \text{otherwise.} \end{cases} \quad (12)$$

Then, we can compute the maximum distance between the initial robot 0 and the mth robot as

$$\|\mathbf{x}_m - \mathbf{x}_0\| = \begin{cases} \frac{1}{2}\ell\sqrt{m^2+3} & \text{if } m \text{ is odd} \\ \frac{1}{2}m\ell & \text{otherwise,} \end{cases} \quad (13)$$

and can be rewritten as in (11). □

Lemma 4. *Given a set $\mathcal{V}$ of $4F+1$ robots in a triangular formation, if the communication range of every robot is $R \geq R_{\triangle}(3F)$, the associated communication graph of the formation is $(2F+1)$-robust.*

Proof. The proof follows the same steps of Lemma 2 using $R = R_{\triangle}(m)$. □

Theorem 5. *A group of $n \geq 4F+1$ robots in a triangular formation is ensured to be $(2F+1)$-robust, if the communication radius of each robot satisfies $R \geq R_{\triangle}(3F)$.*

Proof. The proof follows the steps of Theorem 4 using $R = R_{\triangle}(m)$. □

C. r-robust formations on a square lattice

The work in [21] suggests that studying formations on square lattices is highly relevant for practical applications. Consider the square lattice with basis given by (6). We define a *squared formation* and describe our framework for robust networks as follows.

Definition 10 (Square formation). *A connected formation of $n \geq 4$ robots with positions $\mathbf{x}_i \in \mathbb{L}_{\square}$ is said to be square, if every robot in the formation is the vertex of square of edge length equal to the lattice length ℓ, or adjacent to a robot which is a vertex of a square of edge length ℓ.*

An example of a square formation is shown in Figure 1.d).

Lemma 5. *In a locally square formation of $n \geq 5$ robots, every robot has at least $1 \leq m \leq n-1$ robots within a distance $R_{\square}(m)$ given by*

$$R_{\square}(m) = \begin{cases} 2\ell & \text{if } m = 3 \\ \ell\sqrt{\lfloor \frac{m}{2} \rfloor^2 + 1} & \text{otherwise.} \end{cases} \quad (14)$$

Proof. Satisfying the constraints of the square formation, let robot 0 be located at $\mathbf{x}_0 \in \mathbb{L}_{\square}$, and the robots 1 to 4 be located according to

$$\mathbf{x}_m = \begin{cases} \mathbf{x}_0 + \left(\frac{m+1}{2}\right)\mathbf{v}_{1\square} + \mathbf{v}_{2\square} & m \geq 1 \quad \text{odd,} \\ \mathbf{x}_0 + \left(\frac{m}{2}\right)\mathbf{v}_{1\square} & m \geq 2 \quad \text{even,} \end{cases} \quad (15)$$

for $1 \leq m \leq 4$ (refer to Figure 3). There are seven lattice positions suitable for robots 5 and 6. By direct evaluation, it is straight forward to verify that the next two locations, the farthest from $\mathbf{x}_0$, are the ones adjacent to the opposite edge of the square, $\mathbf{x}_0 + 3\mathbf{v}_{1\square} + \mathbf{v}_{2\square}$ and $\mathbf{x}_0 + 3\mathbf{v}_{1\square}$. Let robots five and six be placed in those locations. The next two robots can be located in any of the ten available locations, but six have already been verified to be closer to $\mathbf{x}_0$ than the robots 5 and 6, therefore only four locations need to be evaluated (refer to Figure 3 for an illustration of the pattern). In general, the position of the mth odd robot that maximizes the distance to $\mathbf{x}_0$ is then one of the four positions given by

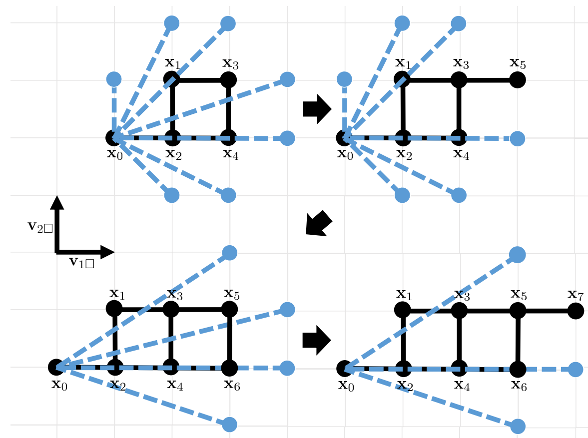

Fig. 3. Sequential addition of robots in a square lattice. Each subsequent robot in the formation is placed maximizing the distance from robot 0. The pattern repeats for all the successive robots.

$\mathbf{x}_0 + (m-1)/2\mathbf{v}_{1\square} + 2\mathbf{v}_{2\square}$, $\mathbf{x}_0 + (m-1)/2\mathbf{v}_{1\square} - \mathbf{v}_{2\square}$, and the two positions given by (15). Evaluating these expressions, we can verify that the positions given by (15) maximize $\|\mathbf{x}_m - \mathbf{x}_0\|$. Thus, a robot has at least m robots around it within a distance given by

$$\|\mathbf{x}_m - \mathbf{x}_0\| = \begin{cases} \ell\sqrt{\left(\frac{m+1}{2}\right)^2 + 1} & \text{if } m \text{ is odd,} \\ \ell\sqrt{\left(\frac{m}{2}\right)^2 + 1} & \text{otherwise,} \end{cases} \quad (16)$$

for $m \geq 5$. Adjusting for $m = 0$ to 4, the result can be stated as in (14). □

Figure 3 shows the construction of a square formation that maximizes the distance from the first allocated robot.

Lemma 6. *Given a set $\mathcal{V}$ of $4F+1$ robots in a square formation, if the communication range of every robot is $R \geq R_{\square}(3F)$, the associated communication graph of the formation is $(2F+1)$-robust.*

Proof. The proof follows the proof of Lemma 2 using $R = R_{\square}(m)$. □

Theorem 6. *A group of $n \geq 4F+1$ robots in a square formation is ensured to be $(2F+1)$-robust, if the communication range of each robot satisfies $R \geq R_{\square}(3F)$*

Proof. The proof follows the proof of Theorem 4 using $R = R_{\square}(m)$. □

IV. Formations around Obstacles, and Heterogeneous Communication Capabilities

In the previous section, we showed sufficient conditions to build robot formations that ensure r-robust communication networks (Theorems 5 and 6). Such formations can be designed to cover an area around obstacles as shown in Figure 4. Adjusting both the radius R and the lattice length ℓ, the formation can be tailored to the desired coverage as long as there are enough robots to satisfy the r-robustness conditions. Figure 5 shows a scenario where two sets of robots can be connected through a narrow passage while maintaining

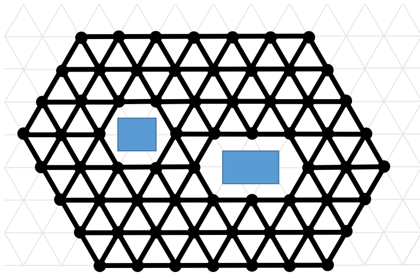

Fig. 4. Robots on a triangular lattice surrounding obstacles. The robots and the obstacles are depicted in black and blue respectively.

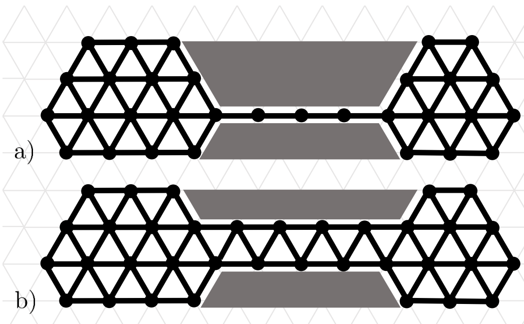

Fig. 5. Two sets of robots are connected through a narrow passage between obstacles. a) A connected formation with $R = R_{-}(3F)$ is used. b) A locally triangular formation with $R = R_{\triangle}(3F + 1)$ is used, requiring more robots, but smaller communication range.

the robustness. Depending on the number of robots and the obstacle-free space available, a connected formation or a triangular formation can be implemented. The latter uses more robots, but requires a smaller communication range.

The scenarios above show the use of the same communication radius for each robot in the formation, however, a heterogeneous set of communication radii can also be used. Using the results of the previous section, it is possible to combine robots of different communication range to optimize energy usage as in the following application example. Consider the scenario of Figure 5.a, where two sets of robots are connected through a narrow passage using the same communication range, e.g. we can use $R_{-}(3)$ to ensure 3-robustness.

An alternative solution can be obtained by setting an initial formation to the left of the passage made of robots with a range of $R_{\triangle}(3)$, as shown at the top of Figure 6. Then, it is possible to increase the range of the robots 1, 2 and 3 to $R_{-}(3)$ and maintain the robustness, since we are only adding more edges to the graph. However, thanks to the increased communication range of those three robots, more robots can be deployed through the narrow passage with the same range $R_{-}(3)$, so that each new node has at least three incoming edges from other three nodes with enlarged communication range. Finally, once the passage has been cleared, a triangular configuration can be continued with robots with the smaller communication range of $R_{\triangle}(3)$. Using Lemmas 1 and 3, it can be verified that there is a way to construct the formation ensuring three incoming edges for every new node, thus ensuring the desired r-robustness is maintained. It is worth mentioning that there

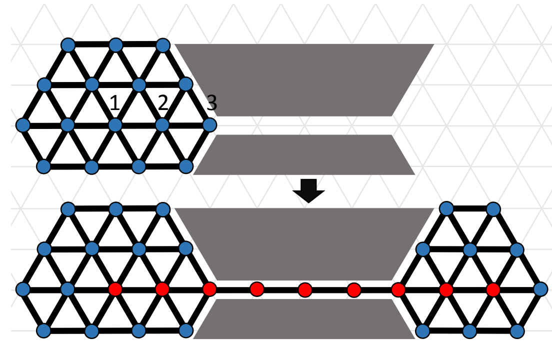

Fig. 6. Robots of different communication range can be used to deal with obstacles in the environment and optimize energy usage in the communications. The dots in red represent robots with a communication radius $R_{-}(3)$, while the ones in blue have $R_{\triangle}(3)$. The formation is 3-robust.

can be multiple solutions. For example, changing three robots initially is necessary if the obstacles block the communication. However, if the robots can communicate through the obstacles, then fewer robots would need to be changed initially, since some blue robots can contribute to the required incoming edges for new robots. The development of a rigorous algorithm for particular cases is left for future research.

We end this section by showing how a bounded region in a triangular or square lattice can be filled with robots of smaller communication range compared to the ones at the boundary.

Theorem 7. *Suppose there is a region in a triangular lattice, bounded by a connected formation of $n \geq 4F + 1$ robots satisfying $(2F + 1)$-robustness, and every robot has a communication range $R \geq F\ell$. If the lattice locations inside the region are occupied by robots of communication range*

$$R_{\triangle int} = F\ell, \quad F \geq 1, \tag{17}$$

then the communication graph of the extended formation is $(2F + 1)$-robust.

Proof. Consider the set $\mathcal{B}_0$ of robots at the boundary of the empty region as the vertices of a polygon. Every polygon has at least 3 inner corners with an interior angle between 0 and π radians. Consider the robot at an inner corner of the polygon, as well as the two robots adjacent to it. Adjacent to these two robots, there is an unoccupied lattice point. A robot placed on such an unoccupied lattice point will have three robots at a distance of ℓ, and at least two more robots at every distance increase of ℓ from it. Therefore, a robot placed in a lattice point by an inner corner inside the empty region will have $3 + 2(m - 1) = 2m + 1$ neighbors within a distance of $m\ell$. For such a robot to have a degree of at least $2F + 1$ to preserve the robustness according to Theorem 2, the other robots must have a communication range R satisfying $2R/\ell + 1 \geq 2F + 1$, leading to $R_{\triangle int} \geq F\ell$. Assigning $R_{\triangle int}$ to the robots inside the empty region, occupy the lattice points by the inner corners of the polygon corresponding to the set $\mathcal{B}_0$. Once there are no more inner corners of $\mathcal{B}_0$, consider the new set of robots bounding the remaining empty space, $\mathcal{B}_1$, and allocate robots in the corners of the corresponding polygon. This process can be repeated until there are no more

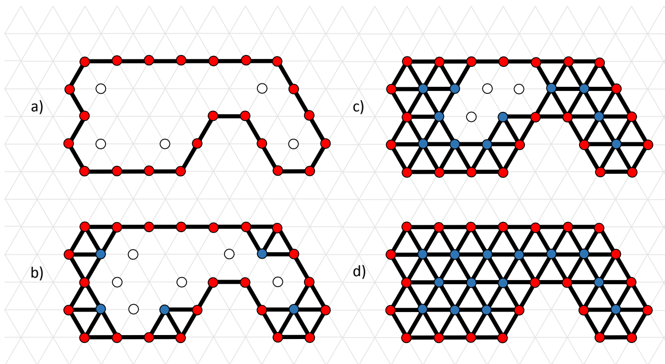

Fig. 7. A 3-robust formation of robots, depicted as black circles, surrounds an empty region on a triangular lattice. New robots can be added while preserving the robustness at the locations with blue circles. The sequence of images from a) to d) show how the region can filled.

unoccupied internal lattice points. Hence, using Theorem 2, we can construct a formation that combines the boundary robots with communication range $R \geq F\ell$, and the internal robots with $R = F\ell$, so that the resulting formation has an associated $(2F+1)$-robust communication graph. □

Figure 7 shows an example of the sequential filling of an empty region to illustrate the proof of the theorem above. This example allows to use either $R_{-}(3F)$ or $R_{\triangle}(3F)$ to satisfy the robustness conditions of the boundary, since the minimum of both radii is greater than $F\ell$ for all values of F. This logic procedure can also applied to squared lattices.

Corollary 1. *Suppose there is a region in a square lattice, bounded by a connected formation of* $n \geq 4F+1$ *robots satisfying* $(2F+1)$*-robustness, and every robot has a communication range* $R \geq \max\{\sqrt{2}\ell, F\ell\}$*. If the lattice locations inside the region are occupied by robots of communication range* $R_{\square int} = \max\{\sqrt{2}\ell, F\ell\}$*, then the communication graph of the extended formation is* $(2F+1)$*-robust.*

Proof. The proof follows the steps of the proof of Theorem 7, but requires adjusting the distance to the first three robots at the corners, since one of them is at a distance $\sqrt{2}\ell$. Hence, every robot is required to have a range $R \geq \sqrt{2}\ell$. □

V. Simulations and Results

In order to support our theoretical analyses, we simulate two critical scenarios for robot formations and resilient consensus. The first scenario is presented in Figure 1.a). It shows a formation that only satisfies the conditions of connected formation. Figure 8 shows the consensus convergence in the presence of four malicious agents, located in disjoint neighborhoods, using a $R_{-}(3)$ for all robots, leading to a 3-robust formation resilient against 1 malicious agent in the vicinity of every robot.

In our second scenario, we have a formation of 398 robots in a complicated configuration due to a bottleneck, illustrated in Figure 9. This scenario can satisfy the conditions of Theorems 4, 5 or 7. So, we can ensure r-robustness using different communication ranges. Considering the formation as just connected, taking advantage of its triangular formation properties,

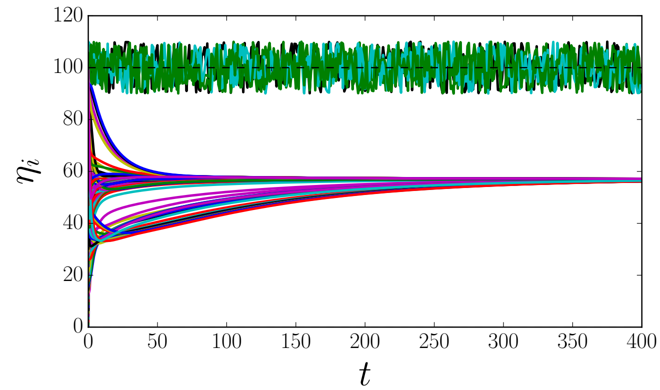

Fig. 8. Achieving consensus in a homogeneous network of 89 robots and the 4 malicious robots, at most 1 in the vicinity of every robot, for the formation in Figure 1 a). The malicious robots share random values, with mean 100, inside and outside of the convex hull of the initial values of all robots.

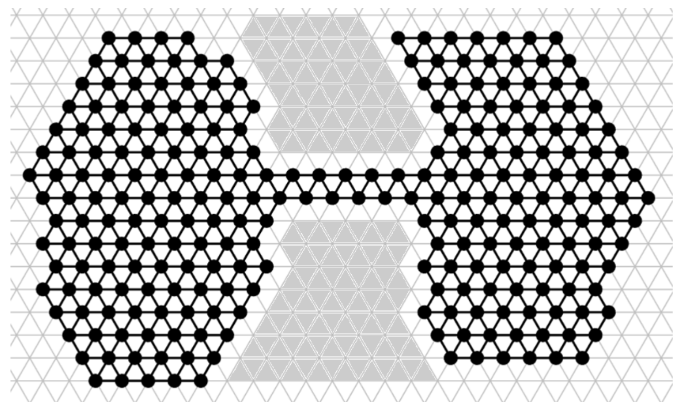

Fig. 9. A robot formation in a triangular lattice. This special configuration can take advantage of the boundary properties to reduce the required communication radii. The robustness of the formation shown can be guaranteed by using the appropriate communication range of: a connected formation, a triangular formation, or a heterogeneous communication radii. The lattice length is $\ell = 1$.

or using different communication ranges for different robots. Figure 10 shows the sum of squares of the communication range of each robot using the different strategies as a function of F, in order to provide a measure of the power required to ensure the robustness of the network. Since $R_{-} > R_{\triangle}$, the power required using R_{-} is higher than with $R_{\triangle}$. Using a heterogeneous strategy by having the outer robots be triangular formation boundary with $R_{\triangle}(3F)$ and assigning the inner robots a range of $R_{\triangle int}$ helps decrease the required power while guaranteeing the same robustness. Figure 11 shows the consensus convergence in the presence of malicious agents with the heterogeneous communication range, which is the case that requires less power.

VI. Conclusions

We present sufficient conditions on the robot communication range to ensure r-robust communication networks in formations over triangular and square lattices. The main idea throughout the results of this paper is the exploitation of the underlying structure in the formation that is provided by the

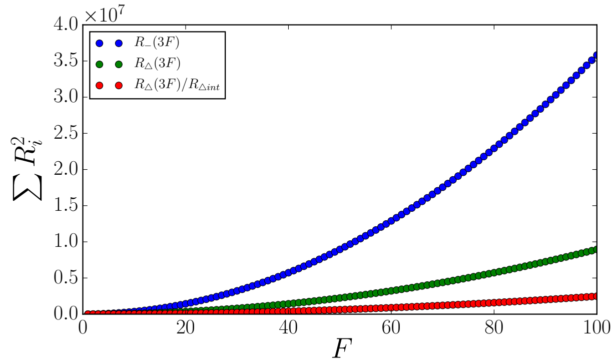

Fig. 10. The sum of the squares of the communication ranges of the robots in a formation is presented as a measure proportional to the power required to satisfy the desired robustness. For the scenario of Figure 9, the shown graphs correspond to the strategies with all the robots having a communication range of $R_-(3F)$, $R_\triangle(3F)$, and the heterogeneous strategy. For the last one, out of the 398 robots, the 150 robots making the triangular boundary are assigned a range of $R_\triangle(3F)$, while the 248 robots inside the boundary are assigned a range of F.

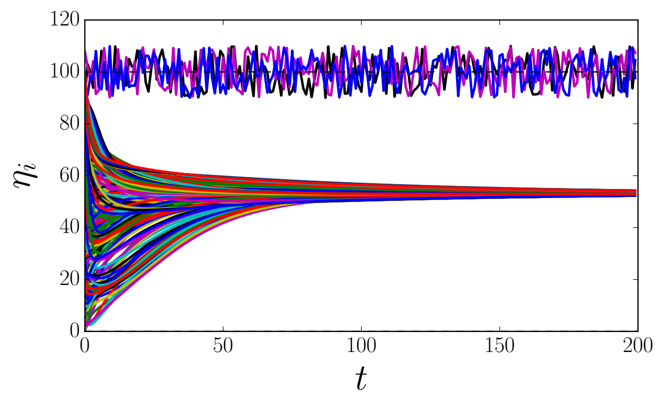

Fig. 11. Achieving consensus in a heterogeneous network of 398 robots and 4 malicious robots for the formation in Figure 9. The malicious robots share random values with mean 100, both inside and outside of the convex hull of the initial values.

lattices. Compared to the results in the literature, our analysis allows to select formations with great flexibility, enabling the adjustment of the formation and its communication network based on challenging environmental obstacles, number of robots and energy constraints.

It is important to emphasize that the results in this paper are only sufficient. They are based on the sparsest-case scenarios in the formations, where the maximum distance between a robot and its mth neighbor is considered. Making such a conservative assumption provides the guarantees of robustness for any formation satisfying the constraints, in exchange of an increase in the communication range of each robot. However, by exploiting the lattice constraints and using different communication ranges in the formation, we are able to balance the increased cost in the communication range size with the great flexibility to select the formation as needed or desired. Studying the optimization of of the communication range assignment to the robots in a particular formation is an area left for future research.

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