Dense r-robust formations on lattices

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***Abstract*— Robot networks are susceptible to fail under the presence of malicious or defective robots. Resilient networks in the literature require high connectivity and large communication ranges, leading to high energy consumption in the communication network. This paper presents robot formations with guaranteed resiliency that use smaller communication ranges than previous results in the literature. The formations can be built on triangular and square lattices in the plane, and cubic lattices in the three-dimensional space. We support our theoretical framework with simulations.**

I. INTRODUCTION

Sharing information among robots in a group is an essential mechanism for distributed decision making through coordination. Nearest neighbor rules have been developed to achieve consensus among the robots [1]–[3], but those rules rely on all the robots being cooperative. As a consequence, large networks are susceptible to failure when at least one robot is non-cooperative, sharing incorrect information [4]. The algorithm known as the Weighted Mean-Subsequence-Reduced (W-MSR) algorithm [5]–[8] provides an update rule that can achieve asymptotic consensus in the presence of malicious agents, mitigating the effect of these agents on the final value of the consensus. The W-MSR algorithm requires the communication graph of the robot network to satisfy a property known as r-robustness. While there are approaches on how to determine the r-robustness of graphs [9] and how to create graphs that satisfy a desired r-robustness [10], algorithms to drive a group of robots into a formation that satisfies such property are limited. It has been shown that r-robustness can be met by ensuring a minimum algebraic connectivity [11], [12], allowing for the use of known control laws for formation control [13], [14]. However, this approach conglomerates the robots, making it challenging for the group of robots to cover a desired area. A body of work to achieve robust formations of robots has been developed based on the W-MSR algorithm [10], [15]–[17].

Resilient networks in the literature require high connectivity and large communication ranges, leading to high energy consumption in the communication network. The main contribution of this paper is the design of resilient robot formations in the plane and the three-dimensional space with guaranteed r-robustness, using smaller communication ranges than previous results in the literature. We extend the methods in [17] to create dense formations, sacrificing the sparsity and flexibility of the formation studied in [16] in exchange for structural conditions on the formations. We use the word *dense* as an antonym of *sparse* to emphasize the difference between the highly structured formations studied in this paper, and the less constrained sparse formations studied previously in the literature, which require a larger communication range.

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II. FUNDAMENTALS

In this section, we summarize the main concepts of resilient networks and lattices.

A. Resilient consensus and r-robustness

We model the communication network among robots as a graph. Let an undirected graph be described by the pair $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, ..., n\}$ is the set of n nodes, and $\mathcal{E}$ is the set of edges of the graph, so that an edge $(i, j) \in \mathcal{E}$ indicates that nodes $i, j \in \mathcal{V}$ are connected. The set of neighbors of node i is denoted by $\mathcal{V}_i = \{j \in \mathcal{V} | \, (i, j) \in \mathcal{E}\}$, and the degree of a node i is denoted by $|\mathcal{V}_i|$. At every time-step, each node $i \in \mathcal{V}$ shares a value η_i with its neighbors in the network, and it updates its value over time according to some nominal rule of the form

$$\eta_i \left[t + 1\right] = w_{ii} \left[t\right] \eta_i \left[t\right] + \sum_{j \in \mathcal{V}_i} w_{ij} \left[t\right] \eta_{ji} \left[t\right], \qquad (1)$$

where $\eta_{ji} \left[t\right]$ is the shared value from the neighbor j to i at time t. Conditions on the weights w_{ij} and the graph properties to ensure consensus have been thoroughly studied in the literature [18], [19].

Definition 1 (Malicious node). *A node $i \in \mathcal{V}$ is said to be malicious if it sends $\eta_i \left[t\right]$ to all of its neighbors at each time-step, but does not follow the nominal rule* (1) *at all time-steps.*

Note that the definition of a malicious agent covers intentionally non-cooperative robots, such as robots hacked by an outsider attempting to manipulate the network, as well as defective and unintentionally non-cooperative robots, such as robot with a malfunctioning location sensor. The W-MSR algorithm provides an update rule which ensures asymptotic consensus and mitigates the effect of the malicious agents. We refer the reader to [5], [7], [8] for details on the algorithm. A required condition for the algorithm to ensure asymptotic consensus is that the graph satisfies a property known as *r-robustness*.

Definition 2 (**r-robust graph**). *A graph $\mathcal{G}$ is said to be r-robust if for every pair of nonempty disjoint subsets of $\mathcal{V}$, at least one of the subsets contains a node that has at least r neighbors outside that subset.*

A set $\mathcal{S} \subset \mathcal{V}$ is *F-local* if it contains at most F nodes in the neighborhood of the other nodes for every time step t, i.e., $|\mathcal{V}_i[t] \bigcap \mathcal{S}| \leq F, \forall i \in \mathcal{V} \setminus S$, $\forall t \in \mathbb{Z}_{\geq 0}, F \in \mathbb{Z}_{\geq 0}$. The following theorem establishes r-robustness as a sufficient and necessary condition for the W-MSR algorithm to ensure asymptotic convergence of the consensus:

Theorem 1 ([5]). *Consider a time-invariant network modeled by a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where each normal node updates its value according to the W-MSR algorithm with parameter F. Under the F-local malicious model, resilient asymptotic consensus is achieved if the topology of the network is $(2F+1)$-robust. Furthermore, a necessary condition is for the topology of the network to be $(F+1)$-robust.*

The work in [7] presents a method to increase the number of nodes in a r-robust graph:

Theorem 2 ([7]). *Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be an r-robust graph. Then the graph $\mathcal{G}' = \{\{\mathcal{V}, v_{new}\}, \{\mathcal{E}, \mathcal{E}_{new}\}\}$ where v_{new} is a new vertex added to $\mathcal{G}$ and $\mathcal{E}_{new}$ is the edge set related to v_{new}, is r-robust if $|\mathcal{V}_{new}| \geq r$.*

The work in [10] introduced the concept of F-elemental graphs and proposed a method to build them as follows:

Definition 3 (**F-elemental graph**). *An F-elemental graph is a graph with $n = 4F+1$ nodes that is r-robust with $r = 2F+1$ for some positive integer value of F.*

Theorem 3 ([10]). *A graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with $|\mathcal{V}| = 4F+1$ is $(2F+1)$-robust if:*

1) *There is a set $\mathcal{V}' \subset \mathcal{V}$ of $2F$ nodes that are connected to all nodes in the graph.*
2) *The set of nodes $\mathcal{V} \backslash \mathcal{V}'$ forms a connected subgraph.*

If the subset of nodes with full connectivity has at least $2F+1$ elements, then the robustness can be immediately ensured, as shown in the following corollary.

Corollary 1. *A graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with $|\mathcal{V}| = 4F+1$ is $(2F+1)$-robust if there is a subset $\mathcal{V}' \subset \mathcal{V}$ of at least $2F+1$ nodes that are connected to all nodes in the graph.*

Proof. Since $\mathcal{V}'$ has at least $2F+1$ nodes connected to every other node in $\mathcal{V}$, there is a subset $\mathcal{V}'' \subset \mathcal{V}'$ of $2F$ nodes connected to every other node, and at least one more node ensuring that the rest of the nodes in $\mathcal{V} \backslash \mathcal{V}''$ form a connected subgraph. By Theorem 3, the graph is $(2F+1)$-robust. □

B. Robot Networks and Formations on Lattices

Given a set of n robots, let $\mathbf{x}_i \in \mathbb{R}^2$ be the position of robot $i \in \{1, ..., n\}$ on the plane. We use the disk model to describe the communication network.

Definition 4 (**Communication graph**). *Given a set of robots, $\mathcal{V}$, with communication range R, the graph $\mathcal{G}_R = (\mathcal{V}, \mathcal{E}_R)$ with edge set defined by*

$$\mathcal{E}_R = \{(i,j) \,|\, \|\mathbf{x}_i - \mathbf{x}_j\| \leq R\}, \tag{2}$$

is called the communication graph of the set $\mathcal{V}$.

To build a communication network, we will use an underlying lattice structure for the formation of robots. A lattice on the plane is a set of linear combinations with integer coefficients of the elements of a basis of a $\mathbb{R}^2$. The elements of the set are *lattice points*. Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be such a basis, and let $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \ell$, where ℓ is the *lattice length*. A lattice on the plane is given by

$$\mathbb{L} = \{a\mathbf{v}_1 + b\mathbf{v}_2 : \text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \mathbb{R}^2; a, b \in \mathbb{Z}\}. \tag{3}$$

The work in this paper studies formations on two types of lattices on the plane. First, the *triangular lattice* $\mathbb{L}_\triangle$ with basis

$$B_\triangle = \{\mathbf{v}_{1\triangle} = \frac{\ell}{2}\begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}, \mathbf{v}_{2\triangle} = \ell\begin{bmatrix}1\\0\end{bmatrix}\}. \tag{4}$$

Second, the *square lattice* $\mathbb{L}_\square$ with basis

$$B_\square = \{\mathbf{v}_{1\square} = \ell\begin{bmatrix}1\\0\end{bmatrix}, \mathbf{v}_{2\square} = \ell\begin{bmatrix}0\\1\end{bmatrix}\}. \tag{5}$$

Formations on a lattice in the space are also studied. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be a basis for $\mathbb{R}^3$, and let $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \|\mathbf{v}_3\| = \ell$. A lattice in the three-dimensional space is given by

$$\mathbb{L}_{\cube} = \{a_i\mathbf{v}_1 + b_i\mathbf{v}_2 + c_i\mathbf{v}_3 : \\ \textit{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3; a_i, b_i, c_i \in \mathbb{Z}\}. \tag{6}$$

We will consider formations on a *cubic lattice*, which has a basis given by

$$B_{\cube} = \left\{\mathbf{v}_{1\cube} = \ell\begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{v}_{2\cube} = \ell\begin{bmatrix}0\\1\\0\end{bmatrix}, \mathbf{v}_{3\cube} = \ell\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}. \tag{7}$$

In the robot formations discussed in this paper, each robot is located at a lattice point, i.e., $\mathbf{x}_i \in \mathbb{L}$. Given a lattice length ℓ, we now define a graph that describes the proximity of the robots on the lattice.

Definition 5 (Proximity graph). *Given a set of robots $\mathcal{V}$ and a distance ℓ, the graph $\mathcal{G}_\ell = (\mathcal{V}, \mathcal{E}_\ell)$ with edge set defined by*

$$\mathcal{E}_\ell = \{(i,j) \,|\, \|\mathbf{x}_i - \mathbf{x}_j\| \leq \ell\}, \tag{8}$$

is called the proximity graph of the set $\mathcal{V}$.

III. General results for robot formations with r-robust communication networks

Given a set of robots $\mathcal{V}$ distributed in a finite region, we look into selecting a sufficient communication range to ensure r-robustness. Let $\rho \in \mathbb{R}_{\geq 0}$ and $\mathbf{x}_c \in \mathbb{R}^m$ be the radius and center of a ball in the corresponding space, with $m = 2$ for robots on a plane, or $m = 3$ for three-dimensional space. The set of robots within such a ball is given by

$$\mathcal{B} = \{i \in \mathcal{V} : \|\mathbf{x}_i - \mathbf{x}_c\| \leq \rho\}. \tag{9}$$

The next lemma allows us to communicate between robots in concentric balls.

Lemma 1. *Let $\mathcal{B}_1$ be the set of robots within a ball of radius ρ_1, and $\mathcal{B}_2 \subseteq \mathcal{B}_1$ be the subset of robots in a concentric ball of radius $\rho_2 \leq \rho_1$. If the communication range of every robot in $\mathcal{B}_1$ satisfies*

$$R \geq \rho_1 + \rho_2, \tag{10}$$

then every robot in $\mathcal{B}_2$ is communicated with every robot in $\mathcal{B}_1$.

Proof. Let $\mathbf{x}_c$ denote the center of the ball. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the positions of robots in $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, so that they satisfy $0 \leq \|\mathbf{x}_1 - \mathbf{x}_c\| \leq \rho_1$ and $0 \leq \|\mathbf{x}_2 - \mathbf{x}_c\| \leq \rho_2$. Then,

$$\begin{aligned}\|\mathbf{x}_1 - \mathbf{x}_2\| &= \|\left(\mathbf{x}_1 - \mathbf{x}_c\right) - \left(\mathbf{x}_2 - \mathbf{x}_c\right)\| \\ &\leq \|\mathbf{x}_1 - \mathbf{x}_c\| + \|\mathbf{x}_2 - \mathbf{x}_c\| \leq \rho_1 + \rho_2. \end{aligned} \tag{11}$$

Since $\rho_1+\rho_2 \geq \|\mathbf{x}_1-\mathbf{x}_2\|$, a communication range $R \geq \rho_1+\rho_2$ ensures that each robot in $\mathcal{B}_2$ is within the communication range of each robot in $\mathcal{B}_1$. □

With enough robots, the r-robustness of the robot communication network can be ensured as follows.

Lemma 2. *Let $\mathcal{B}_1$ be the set of robots within a ball of radius ρ_1 with $|\mathcal{B}_1| \geq 4F+1$, and $\mathcal{B}_2 \subseteq \mathcal{B}_1$ be the subset of robots within a concentric ball of radius $\rho_2 \leq \rho_1$ with $|\mathcal{B}_2| \geq 2F+1$. If the communication range of every robot is*

$$R \geq \rho_1 + \rho_2, \tag{12}$$

then the communication graph of the robots in $\mathcal{B}_1$ is $(2F+1)$-robust.

Proof. By Lemma 1, each robot in $\mathcal{B}_1$ is within the communication range of each robot in $\mathcal{B}_2$. Let $\mathcal{V}$ be a subset of $4F+1$ robots such that $\mathcal{B}_2 \subset \mathcal{V} \subseteq \mathcal{B}_1$. If $\mathcal{V} = \mathcal{B}_1$, then by Corollary 1 the communication graph of $\mathcal{B}_1$ is $(2F+1)$-robust. Otherwise, if $\mathcal{V} \subset \mathcal{B}_1$, then by Corollary 1, the communication graph corresponding to the robots in $\mathcal{V}$ is $(2F+1)$-robust. However, each robot in $\mathcal{B}_1 \setminus \mathcal{V}$ has at least $2F+1$ neighbors in $\mathcal{V}$, and by Theorem 2, each of these robots can be added to the $(2F+1)$-robust network of $4F+1$ robots preserving the robustness, and therefore, the communication graph of $\mathcal{B}_1$ is $(2F+1)$-robust. Then, for $|\mathcal{B}_1| \geq 4F+1$, the communication graph of $\mathcal{B}_1$ is $(2F+1)$-robust. □

We can extend a formation and preserve the r-robustness by placing new robots in the vicinity of a group of robots that already belong to an r-robust formation, as shown in the following lemma.

Lemma 3. *Consider a set of robots $\mathcal{V}$ with an r-robust communication graph, each robot with a communication range R. Let $\mathbf{x}_c$ and ρ be the center and radius of a ball such that the subset of robots $\mathcal{B} = \{i \in \mathcal{V} : \|\mathbf{x}_i - \mathbf{x}_c\| \leq \rho\} \subseteq \mathcal{V}$ satisfies $|\mathcal{B}| \geq r$. If $R \geq 2\rho$, then a robot that does not belong to $\mathcal{V}$ located within the ball of center $\mathbf{x}_c$ and radius $R-\rho$ can be added to the robot network, preserving the r-robustness.*

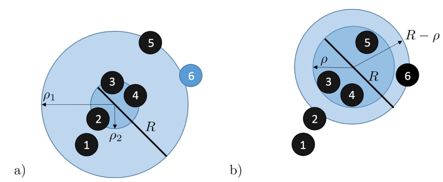

Fig. 1. a) The subset of robots $\{1, 2, 3, 4, 5\}$, all of which have a communication range equal to $R = \rho_1 + \rho_2$ denoted with a thick black line, form a $2F+1$-robust communication graph with $F = 1$, according to Lemma 2. b) Robot 6 can be added to the formation using Lemma 3 preserving the 3-robustness, since it is within the communication range of the Robots 3,4, and 5.

Proof. By the lemma's premise, $R \geq 2\rho$, and therefore $R-\rho \geq \rho$. Let $\mathcal{B}_1$ be the set of robots within a ball of radius $\rho_1 = R-\rho$ centered at $\mathbf{x}_c$, and $\mathcal{B}_2 = \mathcal{B} \subseteq \mathcal{B}_1$ the subset of robots in a concentric ball of radius $\rho_2 = \rho \leq R-\rho = \rho_1$. Since $\rho_1 + \rho_2 = R - \rho + \rho = R$, then by Lemma 1, every robot in $\mathcal{B}$ is communicated with every robot in $\mathcal{B}_1$. Since $\mathcal{B}$ contains at least r robots belonging to $\mathcal{V}$, then each of the robots in $\mathcal{B}_1$ will have at least r neighbors in $\mathcal{V}$. Applying Theorem 2 to each of the robots in $\mathcal{B}_1 \setminus \mathcal{V}$, we conclude that the addition of these robots to the network, which do not belong to $\mathcal{V}$ and are within a distance $R-\rho$ from $\mathbf{x}_c$, preserves the robustness. □

Lemmas 2 and 3 are general for arbitrary locations on the two-dimensional and three-dimensional space, and can be applied to a wide range of robot formations. Figure 1 shows an example.

IV. Densely packed r-robust formations on lattices

In this section, we define and describe the construction of a class of densely packed and highly structured formations, for which we specialize Lemmas 2 and 3, and calculate lower bounds on the communication range of each robot to ensure a desired r-robustness.

A. p-formations

Given a lattice $\mathbb{L}$ and a distance d, the formations are built in *layers* $L_k^{\mathbb{L},d}$ of lattice points, starting from layer $L_0^{\mathbb{L},d} = \{\mathbf{x}_0\}$, which contains the lattice point at the center of the formation with position $\mathbf{x}_0$. The layer $L_k^{\mathbb{L},d}$ for $k \geq 1$ is constructed with the lattice points at a distance d of the lattice points in the layer $L_{k-1}^{\mathbb{L},d}$, such that

$$\begin{aligned} L_k^{\mathbb{L},d} = \{\mathbf{x}_j \in \mathbb{L} \setminus \bigcup_{l=0}^{k-1} L_l^{\mathbb{L},d} : \\ \|\mathbf{x}_j - \mathbf{x}_i\| \leq d \,\forall\, \mathbf{x}_i \in L_{k-1}^{\mathbb{L},d}\}. \end{aligned} \tag{13}$$

Using either $d = \ell$ or $d = \sqrt{2}\ell$, we can define the following formations.

Definition 6 (**p-hexagonal formation**)**.** *A p-hexagonal formation on a triangular lattice is the set of robots $\mathcal{H}_p$ located*

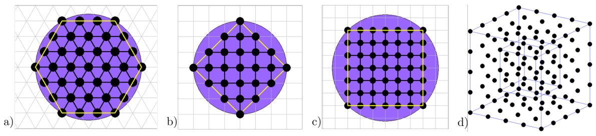

Fig. 2. a) A p-hexagonal formation. b) A p-small square formation. c) A p-large square formation. In all cases, $p=3$. The corresponding circle in purple and the hexagonal and square footprints in yellow are shown for reference. d) A p-cubic formation with $p=2$.

on the set of lattice points with center at $\mathbf{x}_0$ *given by* $\mathcal{F}_{\mathcal{H},p}=\bigcup_{k=0}^{p} L_k^{\mathbb{L}_\triangle,\ell}$.

Definition 7 (p-small square formation). *A p-small square formation on a square lattice is the set of robots s_p located on the set of lattice points with center at $\mathbf{x}_0$ given by* $\mathcal{F}_{s,p}=\bigcup_{k=0}^{p} L_k^{\mathbb{L}_\square,\ell}$.

Definition 8 (p-large square formation). *A p-large square formation on a square lattice is the set of robots $\mathcal{S}_p$ located on the set of lattice points with center at $\mathbf{x}_0$ given by* $\mathcal{F}_{\mathcal{S},p}=\bigcup_{k=0}^{p} L_k^{\mathbb{L}_\square,\sqrt{2}\ell}$.

Definition 9 (p-cubic formation). *A p-cubic formation on a cubic lattice is the set of robots $\mathcal{C}_p$ located in the set of lattice points with center at $\mathbf{x}_0$ given by* $\mathcal{F}_{\mathcal{C},p}=\bigcup_{k=0}^{p} L_k^{\mathbb{L}_{\boxdot},\sqrt{2}\ell}$.

Figure 2 shows examples of the p-formations. We refer to the sets $\mathcal{F}_{\mathcal{H},p},\mathcal{F}_{s,p},\mathcal{F}_{\mathcal{S},p}$ and $\mathcal{F}_{\mathcal{C},p}$ in Definitions 6-9 as the *footprint* of the formation, and each $L_k^{\mathbb{L},d}$ for $0\le k\le p$ is a *layer* of the formation footprint. We refer to these four formations as *p-formations*. These formations satisfy the following condition.

Lemma 4. *If a set of robots $\mathcal{V}$ is organized in a p-formation, then all the robots in $\mathcal{V}$ are within a ball of radius $\rho_\beta(p)$, where*

$$\rho_\beta(p)=\begin{cases}\rho_{\mathcal{H}}(p)=p\ell & \text{for } p\text{-hexagonal},\\ \rho_s(p)=p\ell & \text{for } p\text{-small square},\\ \rho_{\mathcal{S}}(p)=p\sqrt{2}\ell & \text{for } p\text{-large square},\\ \rho_{\mathcal{C}}(p)=p\sqrt{3}\ell & \text{for } p\text{-cubic}.\end{cases} \tag{14}$$

Proof. The robots in each layer can be at most at a distance ℓ from some robot in the previous layer in the hexagonal and small square formations, $\sqrt{2}\ell$ in the large square formation, and $\sqrt{3}\ell$ in the cubic formations. Since layer 0 is only one robot at some position $\mathbf{x}_c$, then all the robots from layers 0 to layer p are within a radial distance $p\ell$, $p\sqrt{2}$ or $\sqrt{3}\ell$ from $\mathbf{x}_c$. $\square$

Note that $\rho_\beta(p)$ is a linear function of p. Let $n_\beta(p)$ the number of lattice points as a function of the number of layers p. It is straight-forward to show that $n_\beta(p)$ can be calculated through the closed-form equations

$$n_\beta(p)=\begin{cases}n_{\mathcal{H}}(p)=3p^2+3p+1 & \text{for } p\text{-hexagonal},\\ n_s(p)=2p^2+2p+1 & \text{for } p\text{-small square},\\ n_{\mathcal{S}}(p)=4p^2+4p+1 & \text{for } p\text{-large square},\\ n_{\mathcal{C}}(p)=(2p+1)^3 & \text{for } p\text{-cubic}.\end{cases} \tag{15}$$

The smallest integer size p_{min} of a p-formation to have at least n_d robots is given by

$$p_{min}(n_d)=\min\left\{p\in\mathbb{Z}_{>0}:n_\beta(p)-n_d\ge 0\right\}. \tag{16}$$

Choosing $n_d=4F+1$ gives the smallest number of layers with enough robots for an F-elemental graph. Closed form solutions to $p_{min}(n_d)$ can be obtained by solving the quadratic or cubic polynomial corresponding to the p-formations discussed in this paper. We can also compute the maximum resiliency of a p-formation as a function of the number of layers p. The maximum number F is given by

$$\begin{aligned}F_{max}(p)&=\max\left\{F\in\mathbb{Z}_{>1}:n_\beta(p)-(4F+1)\ge 0\right\}\\ &=\left\lfloor\frac{n_\beta(p)-1}{4}\right\rfloor,\end{aligned} \tag{17}$$

where $\lfloor\cdot\rfloor$ is the floor function.

B. *p-formations and r-robustness*

We use Lemmas 2 and 3 to calculate a sufficient communication range that ensures r-robustness in p-formations.

Theorem 4. *If a p-formation satisfies that $p=p_{min}(4F+1)$ and every robot has a communication range*

$$R\ge\rho_\beta\left(p_{min}(4F+1)+p_{min}(2F+1)\right), \tag{18}$$

then the communication graph of the p-formation is at least $(2F+1)$-robust.

Proof. Given that $p=p_{min}(4F+1)$, there are at least $4F+1$ robots within circle of radius $\rho_\beta(p_{min}(4F+1))$. Since the function p_{min} is non decreasing, there is a concentric p-formation with $p=p_{min}(2F+1)$ with at least $2F+1$ robots. Since $\rho_\beta(p)$ is proportional to p, $\rho_\beta(p_{min}(2F+1))\le\rho_\beta(p_{min}(4F+1))$. By Lemma 2, if

$$\begin{aligned}R&\ge\rho_\beta(p_{min}(4F+1))+\rho_\beta(p_{min}(2F+1))\\ &=\rho_\beta\left(p_{min}(4F+1)+p_{min}(2F+1)\right),\end{aligned} \tag{19}$$

where the linearity of $\rho_\beta(p)$ was used, then the communication graph of the p-formation is at least $(2F+1)$-robust. $\square$

Theorem 4 allows for the construction of at least $(2F+1)$-robust communication graphs in p-formations. However, the resulting formation may have a higher robustness, since the number of robots in each layer increases quadratically or cubically while the demand of robots to satisfy a desired resiliency F increases linearly. Given a p-formation, equation (17) allows us to calculate the maximum resiliency $F_{max}(p)$ feasible with $n_\beta(p)$ robots in the formation. We now reformulate our result to satisfy the maximum resiliency $F_{max}(p)$, given p.

Corollary 2. *If the communication range of every robot in a p-formation is given by*

$$R\ge\rho_\beta\left(p+p_{min}(2F_{max}(p)+1)\right), \tag{20}$$

then the associated communication graph is $(2F_{max}(p)+1)$-robust.

Proof. Since $F_{max}(p)$ satisfies (17), then $n_\beta(\rho) \geq 4F_{max}(p)+1$, and by (16), we conclude that $p \geq p_{min}(4F_{max}(p)+1)$. Therefore,

$$\begin{aligned} R &\geq \rho_\beta\left(p + p_{min}\left(2F_{max}(p)+1\right)\right) \qquad (21)\\ &\geq \rho_\beta\left(p_{min}\left(4F_{max}(p)+1\right) + p_{min}\left(2F_{max}(p)+1\right)\right), \end{aligned}$$

and, by Theorem 4, the communication graph of the p-formation is at least $(2F_{max}(p)+1)$-robust. □

C. Extending p-formations

If $p = p_{min}(4F+1) > p_{min}(2F+1)$, then there are enough robots in the internal layers of the p-formation to communicate with every other robot, ensuring the $(2F+1)$-robustness following Theorem 4. Moreover, relying exclusively on robots within the inner layers to communicate with every other robot allows to increase the size and extend the shape of the formations while preserving the robustness, shown as follows.

Theorem 5. *Let $\mathcal{V}$ be a set of robots arranged in a p-formation with $p = p_{min}(4F+1)$, such that it has $(2F+1)$-robust communication network, $p_{min}(4F+1) > p_{min}(2F+1)$, and every robot has a communication range $R = \rho_\beta(p_{min}(4F+1) + p_{min}(2F+1))$. Let the footprint of such formation be $\mathcal{F}$. If there is a $p_{min}(2F+1)$-formation, with footprint denoted by $\mathcal{F}_0$, which is of the same type as the footprint $\mathcal{F}$ and has its center at some lattice point located at $\mathbf{x}_0$ such that $\mathcal{F}_0 \subset \mathcal{F}$, then any robot located in the footprint of a $p_{min}(4F+1)$-formation with center at $\mathbf{x}_0$ can be added to the communication network of the set of robots $\mathcal{V}$, preserving the robustness.*

Proof. Let $\mathcal{C}$ be the set of robots in the $p_{min}(2F+1)$-formation with footprint $\mathcal{F}_0$, which implies that $|\mathcal{C}| \geq 2F+1$. By the definition of the p-formation, this subset of robots is within a ball of radius $\rho_\beta(p_{min}(2F+1))$ centered at $\mathbf{x}_0$. By the theorem's premise, $\mathcal{F}_0 \subset \mathcal{F}$ therefore $\mathcal{C} \subset \mathcal{V}$, and $p > p_{min}(2F+1)$, leading to

$$R = \rho_\beta\left(p + p_{min}(2F+1)\right) > 2\rho_\beta\left(p_{min}(2F+1)\right), \qquad (22)$$

where we used the linearity of $\rho_\beta(p)$. Then, by Lemma 3, any robot located within the ball centered at $\mathbf{x}_0$ of radius $\rho_\beta(p + p_{min}(2F+1)) - \rho_\beta(p_{min}(2F+1)) = \rho_\beta(p) = \rho_\beta(p_{min}(4F+1))$, which circumscribes the footprint of a $p_{min}(4F+1)$-formation centered at $\mathbf{x}_0$, can be added to the communication network of the set of robots $\mathcal{V}$, preserving the robustness. □

Figure 3 shows an example of the successive application of Theorem 5 to obtain an extended p-formation from a 2-large square formation. Theorem 4 and Corollary 2 can be extended to ensure the robustness of a p-formation for any value $1 \leq F \leq F_{max}(p)$. This extension allows to change the communication range to achieve the desired robustness.

Theorem 6. *Consider a p-formation. Let F be a value such that $1 \leq F \leq F_{max}(p)$. If $p_{min}(4F+1) > p_{min}(2F+1)$ and the communication range of every robot in the formation is given by*

$$R \geq \rho_\beta\left(p_{min}(4F+1) + p_{min}(2F+1)\right), \qquad (23)$$

then the communication graph associated to the p-formation is at least $(2F+1)$-robust.

Proof. Let the p-formation have its central layer $L_0^{\mathbb{L},d}$ at $\mathbf{x}_0$, and let $\mathcal{V}$ be the set of all the robots on the formation. Since $F \leq F_{max}(p)$, and p_{min} is a non decreasing function, then $p \geq p_{min}(4F_{max}(p)+1) \geq p_{min}(4F+1) > p_{min}(2F+1)$. For an F such that $p_4 = p_{min}(4F+1) > p_2 = p_{min}(2F+1)$, select a subset of robots $\mathcal{V}_1 \subset \mathcal{V}$ such that they form a p_4-formation also centered at $\mathbf{x}_0$. By Theorem 4, given the communication range of $R = \rho_\beta(p_4 + p_2)$, then the communication graph of only the robots in $\mathcal{V}_1$ in a p_4-formation is at least $(2F+1)$-robust. We now show that the rest of the robots in $\mathcal{V} \setminus \mathcal{V}_1$ can be added to the formation, preserving the robustness.

Starting at $m = 0$, consider a robot i on the (p_4+m)th layer of the p-formation. Select a subset $\mathcal{C}_i$ of robots within a p_2-formation footprint centered at some lattice point on the (p_4+m-p_2)th layer of the p-formation, so that robot i is on the outermost layer. By Theorem 5, the robots within the concentric p_4-formation footprint can be added to the r-robust formation, preserving the robustness. Based on the construction by layers of the p-formations, this p_4-formation footprint includes the neighbors of robot i in the (p_4+m+1)th layer. Repeating the process with every robot on the (p_4+m)th layer allows every robot on the (p_4+m+1)th layer to join the network. Repeating the process for $m = 0$ to $m = p - p_4 - 1$ allows every robot in the p-formation to joint the network and preserve the robustness. Therefore, the communication graph associated to the p-formation is at least $(2F+1)$-robust. □

The constraint of $p_{min}(4F+1) > p_{min}(2F+1)$ ensures that the communication range $R = \rho_\beta(p_{min}(4F+1) + p_{min}(2F+1))$ satisfies $R \geq \rho_\beta(2p_{min}(2F+1)+1)$, so that a robot in the layer $L^{\mathbb{L},d}_{p_{min}(2F+1)+1}$ is within the range of all the $(2F+1)$ robots in the $p_{min}(2F+1)$-formation footprint. This allows for the extension the formations as discussed in the previous results on this section. If $p_{min}(4F+1) = p_{min}(2F+1)$, an augmented communication range of $R = \rho_\beta(p_{min}(4F+1) + p_{min}(2F+1)+1)$, such that $R \geq \rho_\beta(2p_{min}(2F+1)+1)$, can be used for every robot. The ability to extend the formations is gained at the cost of increasing the communication range.

V. Simulations and results

Our previous work in [16] presents sufficient communication ranges for sparse formations on lattices to satisfy the desired r-robustness. The ranges for *connected formations* R_-, *triangular formations* $R_\triangle$, and *square formations* $R_\square$ are presented, as well as a strategy assigning different communication ranges for different robots. The sum of the square of the communication ranges, which is proportional

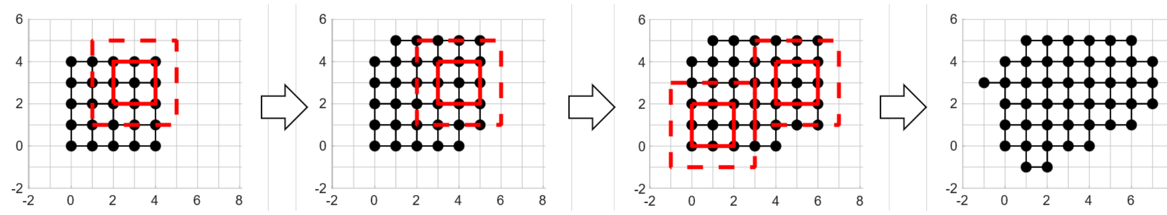

Fig. 3. Extension of a 9-robust 2-large square formation resilient to $F = 4$ malicious robots, following Theorem 5. The $p_{min}(2(4)+1) = 1$-formation footprint is delineated by a solid red line, while the footprint of the $p_{min}(4(4)+1) = 2$-formation is delinated by a dotted line. The formation is extended while preserving the r-robustness by placing new robots at any of the lattice points within the 2-large square footprint around any available 1-large square footprint.

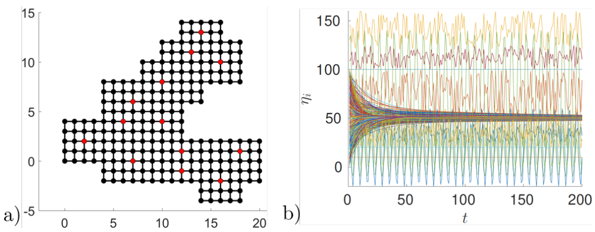

Fig. 4. a) A formation obtained by extending a 9-robust 2-large square formation, resilient against $F = 4$ malicious robots using the local model. Every robot has a communication range of $\rho_S(3)$. b) Asymptotic consensus is achieved in spite of thirteen malicious robots shown in red.

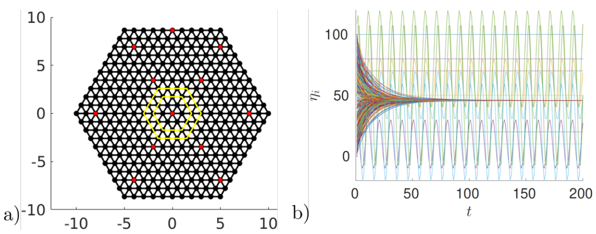

Fig. 5. a) A 10-hexagonal formation that is 19-robust, where each robot has a communication range of $\rho_{\mathcal{H}}(5)$, giving it the maximum robustness of a 3-hexagonal formation. The footprints of the 3-hexagonal and 2-hexagonal formation are delineated in yellow. b) Asymptotic consensus is achieved in spite of the twelve malicious robots following the local model, shown in red.

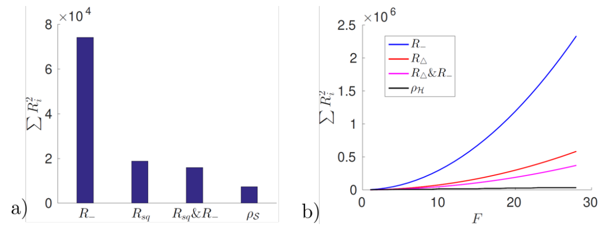

Fig. 6. a) Sum of the square of the ranges for all the robots in the formation from Figure 4 a) using different communication ranges. b) Sums of the square of the communication ranges for all the robots in the formation from Figure 5 a) for different values of F. In both cases, the p-formation range computed in this paper requires the lowest power compared to previous strategies in the literature.

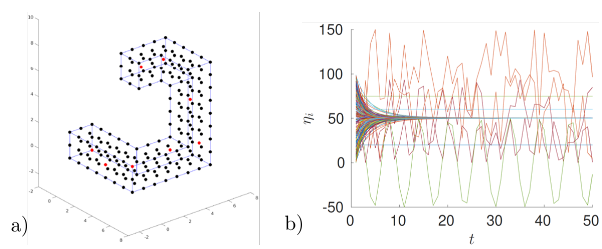

Fig. 7. a) A 13-robust formation of robots obtained by extending a 1-cubic formation, resilient against $F = 6$ malicious robots under the local model. Since $p_{min}(4F+1) = p_{min}(2F+1) = 1$, the extension was enabled by augmenting the communication range of the robots to $R = \rho_{\mathcal{C}}(3)$. b) Asymptotic consensus is achieved in spite of the malicious robots following the local model, shown in red.

to the required power for the communication, was used as a metric of the efficiency to achieve a robustness with a given communication range. This section presents the comparison between the use of the communication ranges for sparse formations and the communication ranges for p-formations. The simulations show that, for a given formation, the results of our present paper require a smaller communication range than previous results in the literature.

Figure 4 shows a formation obtained by extending a large square formation using Theorem 5, as well as the convergence of the consensus algorithm in the presence of malicious robots. Figure 6 a) shows the comparison between the sum of squares of the communication ranges in the extended large square formation using the results for sparse formations and the communication range ρ_S. Figure 5 shows an example of a 10-hexagonal formation constructed by extending a 3-hexagonal formation using Theorem 6, as well as the convergence of the consensus algorithm in the presence of malicious agents. Figure 6 b) shows the comparison between the sum of squares of the communication ranges using the results for sparse formations and $\rho_{\mathcal{H}}$ for the first 28 feasible values of F. The comparisons shows that using the communication range for p-formations requires less power than the other strategies. Figure 7 shows an extended p-cubic formation and its consensus, using the augmented communication range.

VI. Conclusions

In this paper, we present strategies to deploy robots in triangular, square, and cubic lattices. We derive sufficient communication ranges to satisfy a desired r-robustness, allowing the formation to achieve consensus in the presence of malicious robots. The higher number of robots in the vecinity of each robot allows for the use of smaller communication ranges compared to the sparse formations in the literature. Our results optimize the energy usage in the network.

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