Resilient Backbones in Hexagonal Robot Formations

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Abstract Achieving consensus in distributed robot networks is a challenging task when the network contains non-cooperative robots. The conditions of robustness in communication networks are very restrictive and difficult to adapt to robot networks where the communication links are based on proximity. In this paper, we present a new topology network that is suitable for triangular lattices. We introduce sufficient conditions on hexagonal formations to offer resilience up to F non-cooperative robots. Using our framework, a resilient backbone can be designed to connect multiple points or to cover a given area while maintaining a robust communication network. We show theoretical guarantees for our proposed hexagonal formation and its variations. Different scenarios in simulations are presented to validate our approach.

1 Introduction

The communication network of a distributed system is the essential mechanism for coordination. Robots can achieve agreements by only using nearest neighbor communication [1, 2, 3], but such systems rely on the fact that all robots in the network are cooperative. As a consequence, large networks, composed of thousands of robots, are susceptible to failure when one or a small number of robots are non-cooperative and sharing wrong information. These non-cooperative robots can be attackers (e.g. a malicious outsider controlling a few robots and trying to manipulate the whole network) or defective (e.g. a robot with a malfunctioning location sensor).

In the networking literature, robust networks [4, 5, 6, 7] have been developed to achieve consensus in the presence of F-malicious agents. However, those approaches require high connectivity and specific conditions that are difficult to satisfy in proximity-based networks. Increasing the algebraic connectivity of the commu-

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nication network increases the network robustness [8, 9], but it also conglomerates the robots when the communication radius is fixed.

Mobile Robots in lattice-based configurations have been widely studied in robot formations [10, 11]. These lattice-based configurations offer a notion of order and modularity in large systems that bring properties such as scalability, reconfigurability, and redundancy. Organizing these modular systems with interactive components allows us to group large number of robots and to understand their capabilities without checking each individual robot. In this way, we can create a link between robust networks and robot formations. In a previous work [12], we presented a triangular formation for robot networks that can achieve resilient consensus in the presence of a single non-cooperative robot. However, a robust formation for any number of non-cooperative robots is not a straightforward extension. The work in [13] suggests an underlying structure in the communication graph to design of robust formations. In this paper, we propose a design method for robot formations in a triangular lattice. This approach guarantees resilience up to F non-cooperative robots using a fixed communication radius.

The main contribution of this paper is proposing a new hexagonal formation that guarantees resilient consensus for robot networks that contain up to F non-cooperative robots. In contrast to related works, this is the first work that offers and expandable robot formation in the plane while maintaining the resilience properties.

2 Preliminaries on Resilient Consensus

Consider a network composed of a set of nodes $\mathcal{V} = \{1, 2, ..., n\}$ that are represented by points in a planar space $\mathbf{x}_i \in \mathbb{R}^2$, for all $i \in \mathcal{V}$. Every node is equipped with a communication module that allows it to communicate with other nodes. The ability to communicate with adjacent nodes defines the set of connections $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Therefore, the network is modeled as an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The neighbors of node i are $\mathcal{N}_i = \{j | (i, j) \in \mathcal{E}\}$. In this setup, where agents share information among them. An agent is said to be *cooperative* if it applies the consensus update rule and communicates its value to its neighbors at every time-step.

Definition 1 (Non-Cooperative Agent). An agent is *non-cooperative* if it applies a different update rule at any time step.

In a setup where each node has a different scalar value, exponential convergence to a common value can be achieved by computing the average of the neighbors and sharing the result [2, 1, 14, 15]. However, a single non-cooperative node can avoid convergence for the whole team [12].

2.1 Resilient consensus

A *robust network* is defined as a network that can reach consensus, even in the presence of F non-cooperative nodes. Neither the identity nor the strategy of the malicious nodes is known. A known method that achieves consensus by converging to a weighted average is the *Weighted Mean-Subsequence-Reduced (W-MSR)* algorithm [4, 16].

The W-MSR algorithm [4] consists of three steps, executed at time t. First, node i creates a sorted list, from smallest to largest, with the received values from its neighbors $\mathcal{N}_i$. Second, the list is compared to $x_i[t]$, and if there are more than F values that are larger than $x_i[t]$, the F largest values are removed. The same removal process is applied to the smaller values. The remaining neighbors, without removed values, are denoted by $\mathcal{R}_i[t]$. Third, node i updates its value with the following rule:

$$x_i[t+1] = w_{ii}[t]x_i[t] + \sum_{j \in \mathcal{R}_i[t]} w_{ij}[t]x_j[t], \qquad (1)$$

where $w_{ij} > 0$, and $\sum_j w_{ij}[t] = 1$. Yet, for this method to work in the presence of malicious nodes, the network must satisfy certain topological conditions, which we detail below.

Definition 2 (r-reachable). A nonempty vertex set $\mathcal{S} \subset \mathcal{V}$ is r-*reachable* if $\exists i \in \mathcal{S}$ such that $|\mathcal{N}_i \backslash \mathcal{S}| \geq r$, $r \in \mathbb{Z}_{\geq 0}$, that is, if it contains a node that has at least r neighbors outside that set.

Definition 3 (r-robust graph). A graph $\mathcal{G}$ is r-*robust* if for each pair of disjoint sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{V}$ at least one is r-reachable.

Using the WMSR algorithm, a robust network is able to achieve asymptotic consensus even in the presence of F malicious nodes as it is shown in Theorem 1.

Theorem 1 ([4]). *Consider a time-invariant network modeled by a digraph $G = (V, E)$ where each normal node updates its value according to the W-MSR algorithm with parameter F. Under the F-local malicious model, resilient asymptotic consensus is achieved if the topology of the network is $(2F + 1)$-robust. Furthermore, a necessary condition is for the topology of the network to be $(F + 1)$-robust.*

Although these recent works provide a rigorous study of the topological characteristics that are necessary to provide resilience against a number of malicious agents [17, 4], they do not consider the physical constraints that real-world systems often have, such as limited or non-adjustable communication radii. Hence, it is not clear how the methods are applicable in real settings, and it is still an open question if their implementations are suitable to distributed actuator/sensor networks.

2.2 F-Elementals

The work in [13] introduces the concept of F-elemental graphs that satisfy the $2F+1$-robust conditions with the minimum number of nodes.

Definition 4 (F-elemental graph). A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an *F-elemental graph*, $F \in \mathbb{N}$, if it satisfies that:

1. The number of vertices is $|\mathcal{V}| = 4F + 1$.
2. There is a set $\mathcal{V}' \subset \mathcal{V}$ of $2F$ vertices that are connected to all vertices in $\mathcal{V}$.
3. The set $\mathcal{V} \backslash \mathcal{V}'$ forms a connected subgraph.

Theorem 2 ([13]). *Given an upper bound of non-cooperative agents F, if a communication graph is an F-elemental graph, it is $2F+1$-robust.*

The elemental graphs can be the starting point to design a resilient network because they can be extended. The work in [16] presents a method to increase the number of vertices in a r-robust graph by continually adding vertices with incoming edges from at least r nodes in the existing graph:

Theorem 3 ([16]). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an r-robust graph. Then the graph $\mathcal{G}' = (\mathcal{V} \cup \{i\}, \mathcal{E} \cup \mathcal{E}'\})$ is r-robust if $|\mathcal{N}_i| \geq r$, where i is a new vertex added to $\mathcal{G}$ and $\mathcal{E}'$ is the edge set related to i.*

3 Robot Formations on Triangular Lattices

In this paper, we focus on robot formations that are based on triangular lattices. A triangular lattice of triangle side ℓ can be obtained through linear combinations with integer coefficients of the base vectors $\mathbf{v}_1 = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]^T$ and $\mathbf{v}_2 = \left[\begin{smallmatrix} 1/2 & \sqrt{3}/2 \end{smallmatrix}\right]^T$. The triangular lattice is the set

$$\mathbb{L} = \{\ell(a\mathbf{v}_1 + b\mathbf{v}_2) |\, a, b \in \mathbb{Z}\}, \tag{2}$$

where ℓ is the scale factor of the lattice. We call the robots by their index, denoting the robot set by $\mathcal{R} = \{1, 2, ..., n\}$. The robots are located in the lattice, where their associated location is denoted by $\mathbf{x}_i \in \mathbb{L}$, for all $i \in \mathcal{R}$. Each robot is equipped with a communication transceiver and is able to interchange messages with robots that are located within a radius R. Since the communication radius R is always scaled by ℓ, without lose of generality, we assume that $\ell = 1$ for simplicity in this manuscript. The robot team forms a communication network which is defined as follows.

Definition 5 (Communication network). Given a set of robots $\mathcal{R}$ and a communication radius R, the *communication network* of $\mathcal{R}$ is an undirected graph $\mathcal{G}_R = (\mathcal{V}, \mathcal{E}_R)$, where the vertices are the robots $\mathcal{V} = \mathcal{R}$ and the edges are

$$\mathcal{E}_R = \{(i, j) \,|\, i, j \in \mathcal{V} \wedge \|\mathbf{x}_i - \mathbf{x}_j\| \leq R\}.$$

The placement of the robots in the lattice can also be modeled as a graph. We define the lattice graph as follows.

Definition 6 (Lattice graph). A *lattice graph* $\mathcal{G}_\ell = (\mathcal{V}, \mathcal{E}_\ell)$ of a robot set $\mathcal{R}$ is a graph, where the vertices are the robots $\mathcal{V} = \mathcal{R}$ and the edges are

$$\mathcal{E}_\ell = \{(i,j) \mid i,j \in \mathcal{V} \wedge \|\mathbf{x}_i - \mathbf{x}_j\| \leq \ell\}.$$

A set of robots $\mathcal{S} \subseteq \mathcal{R}$ are *lattice neighbors* if its associated lattice graph is connected. Giving a robot set $\mathcal{S}$ and a robot $i \notin \mathcal{S}$, the robot i is a *lattice neighbor of a set* $\mathcal{S}$, if there exists a robot $j \in \mathcal{S}$ such that $\|\mathbf{x}_i - \mathbf{x}_j\| = \ell$. We denote the set of *lattice neighbors of a set* $\mathcal{S}$ by

$$\mathcal{L}_\mathcal{S} = \{i | i \in \mathcal{R} \backslash \mathcal{S}, j \in \mathcal{S} \ \wedge \|\mathbf{x}_i - \mathbf{x}_j\| = \ell\}.$$

3.1 Hexagonal formations and variations

In a triangular lattice, forming hexagons presents advantages in terms of communication for wireless or proximity-based networks. We want to focus on hexagonal configurations and variations.

Definition 7 (p-hexagonal formation). A set of robots on a lattice, $\mathcal{H}_p$, is said to be in a *p-hexagonal formation* if there exists a circle of radius $p \in \mathbb{N}_{\geq 2}$ that surrounds all robots without empty lattice points within the circle.

The number of robots in a p-hexagonal formation is given by

$$\begin{aligned} |\mathcal{H}_p| &= 1 + 6\sum_{i=1}^{p} i \\ &= 3p^2 + 3p + 1 \end{aligned} \tag{3}$$

We denote the lattice neighbors of a p-hexagonal formation by $\mathcal{L}_{\mathcal{H}_p}$. Every p-hexagonal formation has a centroid and it is defined as follows.

Definition 8 (centroid). A robot $i \in \mathcal{H}_p$ is a centroid if the center, $\mathbf{c} \in \mathbb{L}$, of the circle of radius p that encloses all robots in $\mathcal{H}_p$, is equal to the location of robot i, i.e. $\mathbf{x}_i = \mathbf{c}$.

The p-hexagonal formations are suitable to form resilient communication networks. The following theorem shows the relationship between the size of the formation and its robustness against non-cooperative agents.

Theorem 4. *A p-hexagonal formation $\mathcal{H}_p$ is $2F+1-$robust with*

$$F = \begin{cases} 3 & \text{if} \quad p = 2, \\ \left\lfloor \frac{3p(p+1)}{4} \right\rfloor & \text{if} \quad p \geq 3, \end{cases} \tag{4}$$

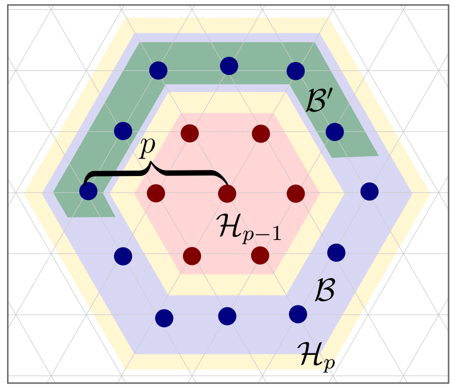


Fig. 1 A $p-$hexagonal formation, for $p = 2$, which is $2F + 1$-robust, $F = 3$, represented by the yellow region. The robots are the blue and red disks. $\mathcal{H}_p$ can be divided into: a concentric $(p - 1)-$hexagonal formation, $\mathcal{H}_{p-1}$ (red region) and its boundary $\mathcal{B}$ (blue region). The boundary $\mathcal{B}$ also contains a subset $\mathcal{B}'$ (green region).

if the robot communication radius is $R = 2p - 1$.

Proof. Initially, we proceed show that the communication network of a $p-$hexagonal formation contains an F-elemental graph. The number of robots in an F-elemental graph is $(4F + 1)$. The set $\mathcal{H}_p$ has enough robots to create an F-elemental graph as it satisfies the inequality

$$\begin{aligned} 4F + 1 &\leq |\mathcal{H}_p| \\ &\leq 3p^2 + 3p + 1, \end{aligned}$$

for any $p \geq 2$ and F given by (4).

The set of robots in the $p-$hexagonal formation, $\mathcal{H}_p$, can be separated into two disjoint subsets, a concentric internal $(p - 1)$-hexagonal formation, $\mathcal{H}_{p-1}$, and a boundary set $\mathcal{B} = \mathcal{H}_p \backslash \mathcal{H}_{p-1}$. Let us consider a subset $\mathcal{B}' \subset \mathcal{B}$ such that its number of robots is $|\mathcal{B}'| = (4F + 1) - |\mathcal{H}_{p-1}|$ and the robots in this set are adjacent in the lattice. We illustrate these sets in Figure 1.

The communication network of the robot set $\mathcal{H}_{p-1} \cup \mathcal{B}'$ is $2F + 1$-robust, since it it can form an F-elemental graph (satisfying the conditions of Def. 4). The set $\mathcal{H}_{p-1} \cup \mathcal{B}'$ contains $4F+1$ vertices (satisfying condition 1). The maximum distance $\|\mathbf{x}_i - \mathbf{x}_j\|$ of any pair of robots $i \in \mathcal{H}_{p-1}$, $j \in \mathcal{H}_p$ is $2p - 1$, and hence, if the communication radius is $R = 2p - 1$, every robot in $\mathcal{H}_{p-1}$ will be communicated with every robot in $\mathcal{H}_p$. We can also verify that the number of elements in $\mathcal{H}_{p-1}$ is greater or equal than $2F$ as

$$\begin{aligned} |\mathcal{H}_{p-1}| &\geq 2F \\ \left\lfloor \frac{3p^2 - 3p + 1}{2} \right\rfloor &\geq F, \end{aligned}$$

where F is given by (4). Therefore, any subset $\mathcal{V}' \subseteq \mathcal{H}_{p-1}$ of $|\mathcal{V}'| = 2F$ satisfies the condition 2 of the F-elemental graph. Since $R > 1$ and the robots in $\mathcal{B}'$ are adjacent in the lattice, they form a connected subgraph. Then, the robots in $\mathcal{H}_{p-1} \backslash \mathcal{V}'$ are also connected to $\mathcal{B}'$, satisfying condition 3.

We can extend the F-elemental graph of the robot set $\mathcal{H}_{p-1} \cup \mathcal{B}'$ to include all remaining robots in $\mathcal{B}\backslash\mathcal{B}'$. Lets pick a robot $i \in \mathcal{B}\backslash\mathcal{B}'$ which is adjacent in the lattice to a robot in $\mathcal{B}'$. Since robot i has more than $2F+1$ neighbors (it is connected to all vertices in $\mathcal{H}_{p-1}$ and, at least one of the vertices in $\mathcal{B}'$), the communication network of the robot set $\mathcal{H}_{p-1} \cup \mathcal{B}' \cup \{i\}$ is also $2F+1$-robust by Theorem 3. We can repeat the logic process of robot i to continue adding, one by one, all the remaining robots in $\mathcal{B}\backslash\mathcal{B}'\backslash\{i\}$. As a result, the network $\mathcal{H}_p = \mathcal{H}_{p-1} \cup \mathcal{B}$ is also $2F+1$-robust. □

The lattice neighbors of a p-hexagonal formation $\mathcal{L}_{\mathcal{H}_p}$ have a minimum number of communication neighbors in $\mathcal{H}_p$ as it is stated in the following lemma.

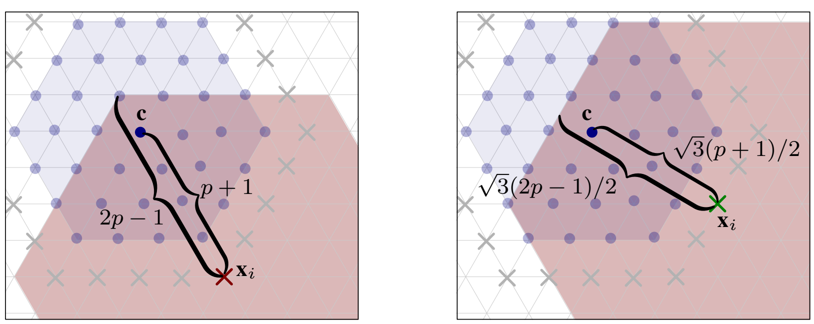
Lemma 1. *Let $i \in \mathcal{L}_{\mathcal{H}_p}$ be a lattice neighbor of a p-hexagonal formation $\mathcal{H}_p$. If the communication radius is $R = 2p-1$, the number of neighbors of i in $\mathcal{H}_p$ satisfies,*

$$3p^2 - p - 1 \leq |\mathcal{H}_p \cap \mathcal{N}_i| \leq 3p^2 + p - 2.$$

Proof. Let $\mathbf{c} \in \mathbb{L}$ be the centroid location of $\mathcal{H}_p$. The lattice neighbors of $\mathcal{H}_p$ are in the set $\mathcal{L}_{\mathcal{H}_p} = \mathcal{H}_{p+1}\backslash\mathcal{H}_p$ which are located over a hexagon of radius $p+1$. In a hexagon of radius $p+1$, the distance from the centroid to any hexagon vertex is $p+1$, and the distance from the centroid to any side is $\frac{\sqrt{3}}{2}(p+1)$. Since, these distances are the minimum and the maximum of any point on the hexagon,we can say that euclidean distance of a neighbor $i \in \mathcal{L}_{\mathcal{H}_p}$ to the centroid $\mathbf{c}$ satisfies

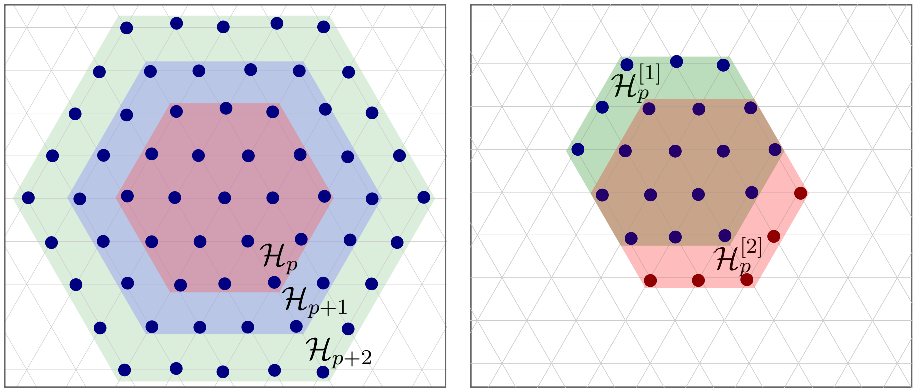
$$\frac{\sqrt{3}}{2}(p+1) \leq \|\mathbf{x}_i - \mathbf{c}\| \leq p+1.$$

We illustrate the closest and the farthest points in Figure 2. In the farther case, where the node i is a hexagonal vertex, the communication radius of robot i covers $\mathcal{H}_p$ but



(a) Farther lattice neighbor of $\mathcal{H}_p$ (b) Closer lattice neighbor of $\mathcal{H}_p$

Fig. 2 The farther and the closer neighbors of the centroid of $\mathcal{H}_p$. The blue disks represent the robots in $\mathcal{H}_p$. The x marks represent the lattice neighbors of $\mathcal{H}_p$. The red and blue X represent the farther and closer neighbors respectively. The red shadowed area represents the communication coverage of the neighbor for $R = 2p-1$.



(a) p-hexagonal expansion. (b) Two adjacent p-hexagonal formations.

Fig. 3 Variations of the p-hexagonal formation with $p = 2$.

the last two lines of the opposite sides (see Figure 2a). The number of non-neighbor robots is $|\mathcal{H}_p \backslash \mathcal{N}_i| = 2(2p+1)$. In the closer case, the neighbor i only has an opposite side of the hexagon of $\mathcal{H}_p$ (see Figure 2b), the number of uncovered nodes is $|\mathcal{H}_p \backslash \mathcal{N}_i| = 2(p+1)+1 = 2p+3$.

Therefore, the number of neighbors of i in $\mathcal{H}_p$, i.e. $|\mathcal{H}_p \cap \mathcal{N}_i|$, is bounded by

$$\begin{aligned} |\mathcal{H}_p| - 2(2p+1) \leq & |\mathcal{H}_p \cap \mathcal{N}_i| \leq |\mathcal{H}_p| - (2p+3) \\ 3p^2 - p - 1 \leq & |\mathcal{H}_p \cap \mathcal{N}_i| \leq 3p^2 + p - 2. \end{aligned}$$

We propose two variations of the p-hexagonal formation. The first one is the *p-hexagonal expansion*, where the robots maintain the same communication radius $R = 2p-1$, but the radius of hexagon is increased to $p+k$, $k \in \mathbb{N}$. The expanded hexagon is illustrated in Figure 3a. An expanded hexagon is also $2F+1$-robust as it is shown in the following corollary.

Corollary 1. *A (p+k)-hexagonal formation, $k \in \mathbb{N}$, is $2F+1$-robust, where F satisfies* (4)*, if the the communication radius is $R = 2p-1$.*

Proof. We use induction to show this proposition. For $k = 1$, a $(p+1)$-hexagonal formation, $\mathcal{H}_{p+1}$, contains a concentric formation $\mathcal{H}_p$, which is $2F+1$-robust (from Theorem 4). Since each robot $i \in \mathcal{H}_{p+1} \backslash \mathcal{H}_p$, satisfies $|\mathcal{H}_p \cap \mathcal{N}_i| \geq 3p^2 - p - 1$ (from Lemma 1) and the inequality $3p^2 - p - 1 \geq 2F+1$ holds for any $p \geq 2$, the communication network of $\mathcal{H}_p \cup \{i\}$ is also $2F+1$-robust (by Theorem 3). In this way, adding one by one each of the nodes in $\mathcal{H}_{p+1} \backslash \mathcal{H}_p$, we show that the set $\mathcal{H}_{p+1}$ is also $2F+1$-robust.

Assuming that the $(p+l)$-hexagonal formation is $2F+1$-robust, we can show that the $(p+l+1)$-hexagonal formation is also $2F+1$-robust. For every robot $i \in$

$\mathcal{H}_{p+l+1} \backslash \mathcal{H}_{p+l}$, there exists a non concentric formation $\mathcal{H}_p \subset \mathcal{H}_{p+l}$ with a centroid robot j such that $\|\mathbf{x}_i - \mathbf{x}_j\| \leq p + 1$. Therefore, the robot i satisfies the minimum number of neighbors in $\mathcal{H}_{p+l}$. In this way, adding one by one each of the nodes in $\mathcal{H}_{p+l+1} \backslash \mathcal{H}_{p+l}$, we demonstrate that the set $\mathcal{H}_{p+l+1}$ is also $2F + 1$-robust. □

Extending the hexagonal formation is a fundamental part to design resilient formations. In the following lemma, we show that any robust network that contains a hexagonal formation can be extended by plugging an adjacent hexagonal formation.

Lemma 2. *Let $\mathcal{S}$ be a set of robots with communication radius $R = 2p - 1$, which contains at least one hexagonal formation $\mathcal{H}_p^{[1]} \subset \mathcal{S}$ and its communication network is $2F + 1$. If a hexagonal formation $\mathcal{H}_p^{[2]}$ is joined to $\mathcal{S}$, such that the centroids $i \in \mathcal{H}_p^{[1]}$ and $j \in \mathcal{H}_p^{[2]}$ are lattice neighbors, the extended robot set $\mathcal{S}' = \mathcal{S} \cup \mathcal{H}_p^{[2]}$ is also $2F + 1$-robust, where F satisfies* (4)*.*

Proof. We show that the additional nodes in $\mathcal{S}' \backslash \mathcal{S}$ satisfy the minimum number of neighbors $(2F+1)$ in $\mathcal{S}$ (Theorem 3). Since each node $i \in \mathcal{S}' \backslash \mathcal{S}$ is a lattice neighbor of $\mathcal{H}_p^{[1]}$, it contains at least $3p^2 - p - 1$ neighbors in $\mathcal{S}$ by Lemma 1. The inequality $3p^2 - p - 1 \geq 2F + 1$ holds for any $p \geq 2$, therefore the communication network of $\mathcal{H}_p \cup \{i\}$ is also $2F + 1$-robust (by Theorem 3). In this way, adding one by one each of the nodes in $\mathcal{S}' \backslash \mathcal{S}$, we show that $\mathcal{S}'$ is $2F + 1$-robust. □

Lemma 2 shows that a set $\mathcal{S}$ with some properties can be extendable. The basic case of a set $\mathcal{S}$ that satisfies these properties is the p-hexagonal formation $\mathcal{H}_p$ as well as its expansions. This lemma is the key to create the resilient backbones that are described in the following section.

4 Resilient Backbones

We want to extend the p-hexagonal formations in such a way that a designer can assemble a resilient network by defining what we call a resilient backbone and its associated extended hexagonal formation.

Definition 9 (Resilient backbone). A *resilient backbone* is a set of robots $\mathcal{C}$ that forms a connected lattice graph and each robot $i \in \mathcal{C}$ is the centroid of a p-hexagonal formation $\mathcal{H}_p^{[i]}$.

Definition 10 (Extended hexagonal formation). An *extended hexagonal formation*, $\mathcal{Q}_p(\mathcal{C})$, is a set of robots that contains a resilient backbone $\mathcal{C}$ and its associated p-hexagonal formations $\mathcal{H}_p^{[i]}$, for all $i \in \mathcal{C}$, i.e.

$$\mathcal{Q}_p(\mathcal{C}) = \cup_{i \in \mathcal{C}} \mathcal{H}_p^{[i]}.$$

A formation $\mathcal{Q}_p(\mathcal{C})$ and its resilient backbone is illustrated in Figure 4.

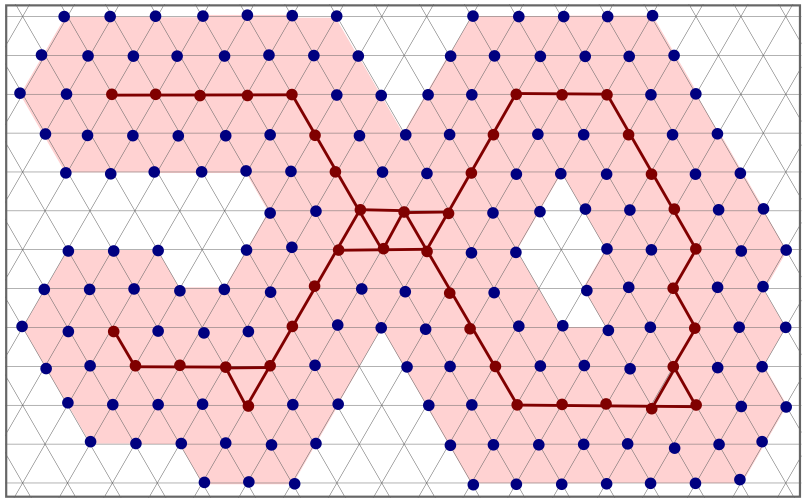


Fig. 4 An extended hexagonal formation $\mathcal{Q}_p(\mathcal{C})$. The robots in its associated resilient backbone $\mathcal{C}$ are represented by the red dots.

Theorem 5. *An extended hexagonal formation $\mathcal{Q}_p(\mathcal{C})$ is $2F+1$-robust, where F satisfies* (4)*, if the communication radius is $R = 2p-1$.*

Proof. By definition, an extended hexagonal formation can be decomposed into p-hexagonal formations $\mathcal{Q}_p(\mathcal{C}) = \cup_{i\in\mathcal{C}}\mathcal{H}_p^{[i]}$. Lets arbitrarily choose a centroid $i \in \mathcal{C}$ and define a set $\mathcal{C}' = \{i\}$. The extended hexagonal formation of $\mathcal{C}'$ is a hexagonal formation, and therefore it is $2F+1$-robust (from Theorem 4). Let $j \in \mathcal{C}\backslash\mathcal{C}'$ be a lattice neighbor of $\mathcal{C}'$, we can say that $\mathcal{Q}_p(\mathcal{C}'\cup\{j\})$ is $2F+1$-robust by Lemma 2. We extend the robust set as $\mathcal{C}' = \mathcal{C}' \cup \{j\}$. Since all robots in $\mathcal{C}$ are lattice neighbors, and $\mathcal{C}$ is a finite set, we can continue adding lattice neighbors from $\mathcal{C}$ to $\mathcal{C}'$ until completing $\mathcal{C}' = \mathcal{C}$. The formation $\mathcal{Q}_p(\mathcal{C}')$ is $2F+1$-robust, therefore $\mathcal{Q}_p(\mathcal{C})$ is also $2F+1$-robust. $\square$

Checking if a communication network is $2F+1$-robust is a NP-Hard problem [4, 16]. However, we can check if a set of robots form a extended hexagonal formation in polynomial time. Algorithm 1 can check if a set of robots $\mathcal{R}$ form an extended hexagonal formation. Line 1 computes a set $\mathcal{S}_i$, for all $i \in \mathcal{R}$, that contains all the surrounding robots in a radius p. Since we check every pair of robots, it takes $\mathcal{O}(|\mathcal{R}|^2)$ computational time. Line 2 identifies the set of centroids $\mathcal{C} \subset \mathcal{R}$ by checking if the the number of surrounding robots is equal to the number of robots of a p-hexagonal formation. Checking every robot, the computational time is $\mathcal{O}(|\mathcal{R}|)$. Creating the adjacency matrix of the set $\mathcal{C}$ can be computed in $\mathcal{O}(|\mathcal{C}|^2)$ time. Checking if a graph is connected (Line 4) can be computed by running a Deep First Search (DFS) algorithm with complexity $\mathcal{O}(|\mathcal{C}|^3)$. Therefore in a large network where the centroids most of the robots are centroids, the time complexity of the algorithm is $\mathcal{O}(|\mathcal{R}|^3)$.

Algorithm 1: CheckExtendedHexagonalFormation($\mathcal{R}, p$)

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S_i = {j | j ∈ R ∧ ‖x_i − x_j‖ ≤ p}, ∀i ∈ R.
C = {i | i ∈ R ∧ |S_i| = |H_p|}                ▷ Identify the centroids of the robot set.
G_ℓ = computeLatticeGraph(C)
return isConnected(G_L) ∧ Q_p(C) = R
```

5 Simulations

Previous works [9, 8] have shown that the r-robustness is related to the algebraic connectivity of the communication graph. In [9], the algebraic connectivity is used to increase the robustness of the graph, but the downside of this approach is that the robots tend to conglomerate when the communication radius is fixed. In this section, we evaluate a critical configuration where the algebraic connectivity is not high. We evaluated three different scenarios in a network of 266 robots and $R = 3$. The robot formation is illustrated in Figure 5. Since this formation is an extended hexagonal formation with $p = 2$, it is resilient up to $F = 3$ non-cooperative robots. In the first scenario, the network has three non-cooperative robots $\{0, 3, 8\}$, and the robots

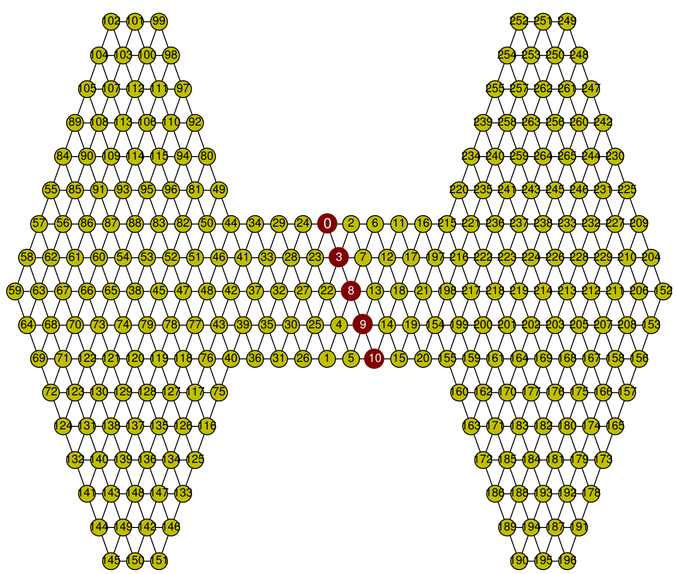
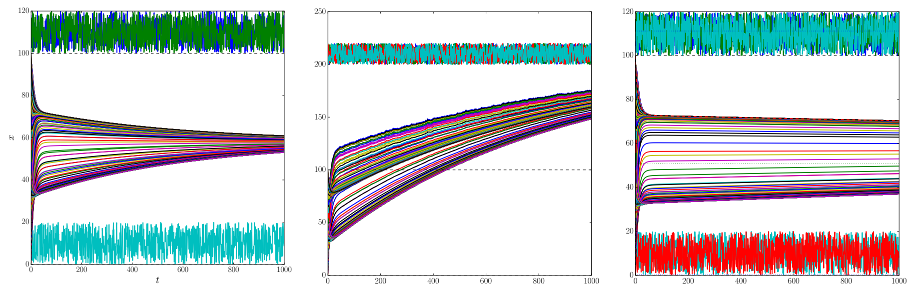


Fig. 5 A set of 266 robots in a extended hexagonal formation. The lines represent the edges of the lattice graph.



(a) F=3 and WMSR(F=3) (b) F=5 and WMSR(F=3) (c) F=5 and WMSR(F=5)

Fig. 6 Consensus for three scenarios in the same robot formation.

apply the WMSR update rule with $F = 3$. We can see in Figure 6a that the robots successfully achieve consensus. In the second scenario, we illustrate that the consensus value can be manipulated if we increase the number of attackers $\{0, 3, 8, 9, 10\}$. In the third scenario, the robots use WMSR with $F = 5$, but the cooperative robots are not able to achieve consensus as it is shown in Figure 6b. Therefore, the first scenario shows that the extended hexagonal formation can achieve consensus in the presence of three malicious robots corroborating our theoretical analyses. Our analyses presents sufficient conditions for robust communication networks as long as we maintain up to F non-cooperative robots in the network. We also showed an strategy of the non-cooperative robots to avoid resilient consensus when the required conditions are not satisfied.

6 Conclusions and Future Work

In this paper, we proposed a new way to design robot formations that guarantee consensus in the presence of non-cooperative robots. We developed a framework based on hexagonal formations and some variations of it. Our formations can guarantee that the properties of robust graphs are fulfilled. Our simulations validated some critical scenarios were the algebraic connectivity is low but the robots still can achieve consensus if the proposed conditions are satisfied. As a future work, we want to study network reconfiguration, different types of lattices, and higher dimensions.

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