Modular Robot Formation and Routing for Resilient Consensus

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***Abstract*— We consider a team of mobile robots with limited communication range tasked to coordinate in large environments. The robots need to maximize the covered area and achieve consensus in the presence of non-cooperative robots. Existing works in robot networks showed that a team can achieve resilient consensus by maintaining an r-robust communication network. However, the communication range of each robot needs to be large to satisfy the conditions of r-robustness. This paper relaxes this requirement on the communication range by leveraging the robot motion. Specifically, we design modular dynamic formations where subsets of robots move along simple closed curves. These modules can be linked together to assemble larger formations. We derive conditions based on periodic robot connections for individual and interconnected modules. We present module designs that satisfy the sufficient conditions in a lattice space. Simulations are provided to support our theoretical results.**

I. Introduction

We are interested in deploying teams of mobile robots with limited communication range in large and complex environments to execute tasks such as surveillance, monitoring, and patrolling. The consensus problem arises in these scenarios when each member of the team has a different estimation of a global variable or state collected through local sensing. The robots exchange information to provide an overall estimate of the state [1], [2]. This is commonly accomplished by allowing individual robots to communicate their states with others in their communication range.

There are a myriad of algorithms solving consensus problems [2], [3], [4], [5]. In practice, robots may become 'non-cooperative' by exchanging corrupted information with neighbors due to internal failures or external disruption. Existing methods overcome the effect of these non-cooperative robots by increasing the connectivity of the team [6], [7]. It has been shown that any network that is resilient against a certain number of non-cooperative robots can retain its resiliency if an arbitrary subset of the communication links becomes *periodic* [6]. Therefore we want to allow some robots to disconnect periodically from their neighbors, travel around, and seek additional connections. Their motion should be carefully planned, such that all the connections are preserved in a periodic fashion, and the robots are able to reach consensus while covering a large area that needs to be monitored persistently. The routing strategy presented in [6] requires to manually define the motion of each robot. Hence, if the environment or the number of robots changes, new trajectories need to be defined to satisfy the periodic robustness.

When robots move around, they come in and out of communication range with other team members. Each robot may not have a consistent set of neighbors to communicate with throughout the duration of the task. While the communication links between robots vary over time, the network maintains *intermittent connectivity*. One of the earliest works considering this concept is [8], where the full connectivity of a team of robots is periodically regained by letting the robots travel far from each other's communication range but regularly gather in certain locations. In [9], periodically occurred communication links between disconnected subgroups are generated to intermittently connect the whole group. It was shown in [10] that robots deployed on pre-defined routes can form a periodic time-varying communication network, which can be mapped to an equivalent static network where standard graph analyzing tools can be used. All these works have focused on *connecting* a network over time, while for reaching resilient consensus, the topology must meet certain robustness conditions, which will be elaborated in Sec. II.

Once the connectivity of the communication topology is known, existing algorithms presented in [11], [12], [13], namely the Weighted Mean-Subsequence-Reduced (W-MSR) algorithm, provides a method for the state values of all cooperative robots to converge to the convex hull of their initial values. The results in [13], [14], [15], [16] provide methods to evaluate whether a communication topology satisfies certain sufficient robustness conditions. When the W-MSR rule is followed by each cooperative robot, the team is resilient to a finite number of non-cooperative robots. Such methods are either computationally inefficient [13], [15], for those require the traversal of all nodes in the graph, or strongly depending on high algebraic connectivity of the network, which drives robots to form a tight cluster [14], [16]. Since monitoring robots must spread out to cover a large area, such tight clusters are not suitable for most applications.

Alternatively, both [7], [17] proposed design strategies by creating well-defined repeating structures that can be composed to create scalable static networks. Such strategies avoid the problem of robots clustering into tight formations but necessitate a large communication range as the number of non-cooperative robots increases. Heterogeneous sets of robots are introduced in [18], [19], [20] to build robotic networks satisfying the robustness conditions. Such methods require a certain number of always-trusted robots that are not universally available in real-world applications.

In this paper, we propose a modular design for robot

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formations and routing strategy to create a periodic communication topology with desired robustness. The idea is to design modular closed-loop paths for subsets of robots which can be composed to create larger formations. We characterize the sufficient conditions for reaching consensus and provide examples realized on triangular lattices. In contrast to the related work, our modular formations can be adapted and scaled to different types of environments without requiring large communication ranges.

II. Preliminaries

In this section we introduce definitions and preliminary results on resilient consensus. Consider an undirected communication graph formed by a set of n robots $\mathcal{V} = \{1, 2, ..., n\}$. The time varying connections (capability of communication) between robots are described with the edge set $\mathcal{E}[t]$, where $\mathcal{E}[t] \subseteq \mathcal{V} \times \mathcal{V}$ for any $t \in \mathbb{Z}_{\geq 0}$. The communication topology is represented by a sequence of graphs $\mathcal{G}[t] = (\mathcal{V}, \mathcal{E}[t])$. We denote the neighbors of node i at time t as $\mathcal{N}_i[t] = \{j | (i, j) \in \mathcal{E}[t]\}$.

Each robot has a scalar value $z_i[t] \in \mathbb{R}$ at time step t. This value represents an estimation variable, heading direction, inter-robot distance or any local variable that involves global coordination. The initial value is denoted by $z_i[0]$. At each time step, robots are able to update their values based on their own values and the values from their neighbors. We say that the robot network *achieves consensus* when all robots converge to a similar value, i.e. $z_1[t] \approx ... \approx z_n[t]$, when t goes to infinity.

Some robots in the team may fail to follow the update rules and send out erroneous information to its neighbors. These robots are referred to as *non-cooperative* ones. To avoid the whole team being affected or manipulated by those non-cooperative robots, a particular consensus algorithm has been introduced in [12]. Let F denote the maximum possible number of the non-cooperative robots in any robot's neighborhood. The weighted mean subsequence reduced (W-MSR) algorithm [12] requires that at every time step t, every cooperative robot i creates a sorted list of all the values it received from its neighbors, and compare them with $z_i[t]$. The W-MSR algorithm removes F highest and lowest values in the list before computing the updated value $z_i[t + 1]$.

For time varying networks, a sliding-time-window version of the W-MSR algorithm, abbreviated as SW-MSR, was introduced in [6] to update $z_i[t]$ based on the most recent values received from all robots that have communicated with i within the last k steps. After removing F values in the sorted list, the set of the values left is denoted by $\mathcal{R}_i[t]$. The update rule of $z_i[t]$ is given by $z_i[t + 1] = w_{ii}[t]z_i[t] + \sum_{j \in \mathcal{R}_i[t]} w_{ij}[t]z^k_{ij}[t]$, where $z^k_{ij}[t]$ denotes the most recent value of $z_j[t]$ that has been received by i during $(t-k, t]$. w_{ii} and w_{ij} are weights that are lower-bounded by some $\alpha \in (0, 1/2)$ and satisfy

$$w_{ii}[t] + \sum_{j \in \mathcal{R}_i[t]} w_{ij}[t] = 1, \forall i \in \mathcal{V}.$$

An important sufficient condition for both W-MSR and SW-MSR to converge asymptotically towards consensus is established on the r*-robustness* of the topology.

Definition 1 (r-robust graph)**.** *A static graph $\mathcal{G}$ is r-*robust *if for any pair of nonempty, disjoint subsets of $\mathcal{V}$, at least one subset of them, denoted as $\mathcal{S}$, contains a node $i \in \mathcal{S}$, such that $|\mathcal{N}_i \backslash S| \geq r$.*

For a time-varying topology, we define its robustness as follows.

Definition 2 (Time-varying r-robust graph)**.** *Given $k \in \mathbb{Z}_{>0}$, a dynamic graph $\mathcal{G}[t]$ is (k, r)-robust if the union $\cup_{\tau=0}^{k} \mathcal{G}[t-\tau]$ is r-robust for all $t \geq k$.*

It was shown in [12] that resilient consensus could be achieved through the W-MSR algorithm if the underlying communication topology is r-robust. The existing results for the time-varying network are reviewed in the following.

Theorem 1 ([6])**.** *Given a communication network $\mathcal{G}[t]$, resilient asymptotic consensus is achieved by the SW-MSR update rule in the presence of F non-cooperative robots in any robots' neighborhood, if $\exists\, k \in \mathbb{Z}_{>0}$, such that $\mathcal{G}[t]$ is $(k, 2F + 1)$-robust.*

Determining whether a given graph is r-robust or not is a coNP-complete problem [13], however, there are sufficient conditions that we can use to efficiently construct large scale graphs that are r-robust:

Theorem 2 ([21] F-elemental graph)**.** *A static graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $4F + 1$ nodes is $(2F + 1)$-robust, if it satisfies the following:*

- *$\exists \mathcal{S} \subset \mathcal{V}$, such that $|\mathcal{S}| = 2F$ and $\mathcal{N}_i \cup \{i\} = \mathcal{V}$ for all $i \in \mathcal{S}$.*
- *$\mathcal{V}/\mathcal{S}$ is a connected graph.*

A graph with the above property is called F-elemental graph.

Theorem 3 ([11])**.** *Suppose $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an r-robust static graph. Consider a new vertex v^*, and $\mathcal{E}^* \subseteq \{(v^*, j) | j \in \mathcal{V}\}$. Then $\mathcal{G}^* = (\mathcal{V} \cup \{v^*\}, \mathcal{E} \cup \mathcal{E}^*)$ is also r-robust if $|\mathcal{N}_{v^*}| \geq r$.*

III. Formulation of Communication Links

We consider the problem of deploying a team of robots to monitor a given environment, where each robot measures a scalar value, such as temperature, time, speed of an object, or distance between two objects. The group as a whole must perform consensus on the measured value. The robots are assumed to have the same communication range, R, and the same constant speed, $\bar{v}$. There are mainly two aspects: *(i)* coverage of the environment and *(ii)* connectivity maintenance of the communication network for resilient consensus.

We approach this problem by considering multiple non-overlapping and non-crossing sub-spaces, each containing a closed path of the same shape with no self-intersection. The paths are described by rotating, translating, shifting phase, or reversing the direction of the same arc-length parametrized function $\boldsymbol{\gamma} : [0, L) \mapsto \mathbb{R}^2$, where L is the arc-length of the paths. We consider a robot team composed of N sub-teams with the same number of robots. Each sub-team has n robots and is allocated to a single path to continuously circulate along it. In the rest of this paper, we use I and J to index the sub-teams and their paths, and i and j to index the robots.

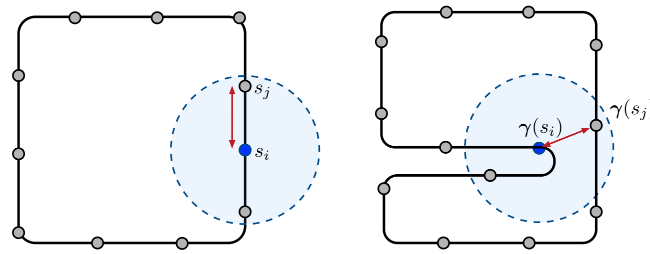
The robots in the same sub-team move with the same speed and direction along the path, so they keep the same order. Let the position of a robot i on the path I be denoted by $s_i \in [0, L_I)$ based on the arc-length parametrization, and the Cartesian coordinate be $\boldsymbol{\gamma}_I(s_i)$. Without loss of generality, we define $\boldsymbol{\gamma}_I$ of this path in a way, such that the robots' motion is along the positive direction of the arc length. That is to say, after a small amount of time δt, robot i's position on the path will be $s_i + \bar{v}\delta t$. If a pair of robots (i, j) are on the same path I, we say i is $\tilde{s}_{ij}$ ahead of j, with $\tilde{s}_{ij} = s_i - s_j$. (For simplicity in the rest of this paper, we use $\tilde{s}_{ij} = (\tilde{s}_{ij} \mod L_I)$ for all the s_i and s_j on the loop path indexed I.) The arc length between i and j is then $\min(\tilde{s}_{ij}, \tilde{s}_{ji})$. The Euclidean distance between i and j is $||\boldsymbol{\gamma}_I(s_i) - \boldsymbol{\gamma}_I(s_j)||$.

We assume that the robots on the same path are indexed along their moving direction, such that robot $i+1$ is the closest robot *ahead of* robot i. Since all robots move at the same speed, the arc length between any two consecutive robots remains constant. We assume that all robots in the same sub-team are *uniformly distributed* along the path, such that the arc length between two consecutive robots is always Δ, satisfying $\Delta \leq R$. The time that one robot takes to finish a cycle on the path is denoted by T. Each path, together with the circulating robots on it, is defined as follows.

Definition 3 (Circulating module)**.** *A* circulating module*, or* module*, is a tuple of a closed path with no self-intersection and a team of robots. All robots are uniformly distributed on the path with a constant arc-length spacing $\Delta \leq R$ and are moving along the path in the same direction at a constant speed.*

A. Inter-robot connection

Inter-robot communication occurs when the Euclidean distance between a pair of robots drops below the communication range R. Note that this condition may be true for robots that are in the same module or in different modules. For the robots in the same module I, we always have $||\boldsymbol{\gamma}_I(s_i) - \boldsymbol{\gamma}_I(s_j)|| \leq \min(\tilde{s}_{ij}, \tilde{s}_{ji})$. Since the arc length between robots in the same module is constant, robots i and j in the same module are connected *at every time step* if $\min(\tilde{s}_{ij}, \tilde{s}_{ji}) \leq R$, which we call a *persistent* connection. For pairs with an arc-length greater than R, and pairs in the different modules, the communication links may temporarily created when they enter certain sections of their paths. The proximity, in this case, is generated due to the geometry of the modules. Since communication is only possible at a certain timing, we call this a *temporal* connection. Fig.1 illustrates both types of inter-robot connections. If both robots enter this section periodically, and they have a common period, the temporal connection between them is also periodic. Since robots in the same module have the same period of motion, all inter-robot connections within the same module are periodic.



(a) Persistent connection (b) Temporal connection

Fig. 1. Examples illustrating the two types of inter-agent connection. (a) The arc length of i and j is no greater than the communication range. All robots in the light blue circle remain connected at all times when they circulate the path. (b) The arc length between i and j is longer than the communication range, but they are able to communicate temporally, and this connection occurs every round of their circulation.

Preserving persistent connections across non-consecutive robots requires a large communication range, and is not always feasible. In this paper, we require persistent connections only between the closest neighbors on the path (i.e. between robot i and robot $i \pm 1$), such that every module is guaranteed to be *connected*. The distances between robots may be affected if the robots fail to maintain the required speed. Whether such uncertainty in the distances between robots will disrupt scheduled connections is subject to the communication range and the geometry of the modules.

B. Inter-module connection

A robot i in module I can communicate with robot j from a different module J if at some time step, both i and j find themselves in communication range with each other. If there exists at least one such pair of robots, we say module I and J are *periodically connected*. In the rest of this section, we describe how this inter-module connection is formed.

Given a pair of circulating modules I and J, let $\mathbf{l}_1 = [s_1, s_1 + l]$ be a segment on path I, and $\mathbf{l}_2 = [s_2, s_2 + l]$ be a segment on path J with the same arc length l. Similarly to Sec. III, if $s_i + l \geq L$, then $\mathbf{l}_i = [s_i, s_i + l]$ represents the segment of $[s_i, L) \cup [0, (s_i + l \mod L)]$. We define *interfacing section* as follows.

Definition 4 (Interfacing section)**.** *Consider a pair of circulating modules I and J with segments $\mathbf{l}_1$ and $\mathbf{l}_2$ respectively. We say $\mathbf{l}_1$ and $\mathbf{l}_2$ are* interfacing sections *if $||\boldsymbol{\gamma}_I(s_1 + \sigma) - \boldsymbol{\gamma}_J(s_2 + l - \sigma)|| \leq R$ for all $\sigma \in [0, l]$, where R is the communication range.*

When a robot i in module I enters the interfacing section, the subset of robots in module J that will be able to communicate with i can be determined. Robots on I and J may travel in reverse directions or the same direction when they are on the interfacing sections depending on how the modules are arranged. Fig. 2 illustrates the interfacing sections with robots traveling on them in reverse directions.

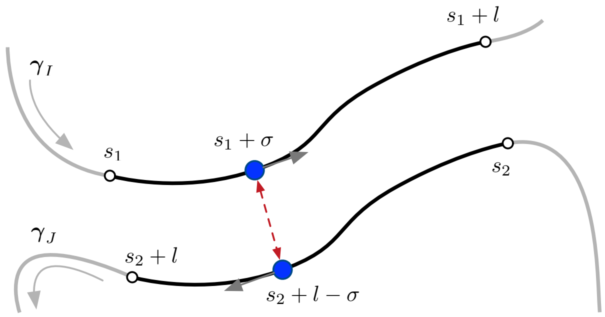


Fig. 2. Illustration of the interfacing sections.

Lemma 1. *Consider a pair of modules I and J with interfacing sections $\mathbf{l}_1$ and $\mathbf{l}_2$, a robot i in module I, and a robot j in module J. At the time when robot i arrives at s_1, if robot j is at $s_j \in [s_2 - l, s_2 + l]$, then i and j are guaranteed to connect temporally before j reaches $s_2 + l$.*

Proof. Since all robots are moving at the same speed, when robot j arrives at $s_j + \delta s$, robot i will arrive at $s_1 + \delta s$. Since $\mathbf{l}_1$ and $\mathbf{l}_2$ are interfacing sections, and $s_j \in [s_2 - l, s_2 + l]$, there exists some $\delta s \in [0, l]$, such that $s_j + \delta s = s_2 + l - \delta s$, and therefore $||\boldsymbol{\gamma}_I(s_1 + \delta s) - \boldsymbol{\gamma}_J(s_2 + l - \delta s)|| \leq R$. □

Since all modules have the same period, the connection between i and j will be periodic, and I and J are periodically connected. For any robot i in module I, it may periodically connect with a number of robots from module J. Since robots in the same module are uniformly distributed along the path, finding this number is straightforward.

Definition 5 (Interfacing index)**.** *The* interfacing index *of robot i in I with module J, denoted by $D_i(J)$, is the number of robots in J that i periodically connects with. Module I's* interfacing index *with module J is $D_I(J) = \min\{D_i(J)\}$ for all robots in module I.*

If both modules have the same period, a robot i in module I connects with the same subset of robots in module J at every cycle, then calculating $D_I(J)$ is straightforward by using the arc length of the interfacing section and the spacing of J.

Lemma 2. *Consider two identical circulating modules I and J with interfacing sections $\mathbf{l}_1$ and $\mathbf{l}_2$ with the arc length l. The path of each module has the same length L. The arc length spacing of each module is Δ. If $L \geq 2l$, then the interfacing index satisfies $D_I(J) \geq \lceil \frac{2l}{\Delta} \rceil - 1$.*

Proof. According to Lemma. 1, a robot i entering $\mathbf{l}_1$ at s_1 will have temporal connections with all robots located between $[s_2 - l, s_2 + l]$. The section has an arc length of $2l$. If $2l$ is an integer multiple of Δ, there are $\lceil \frac{2l}{\Delta} \rceil$ or $\lceil \frac{2l}{\Delta} \rceil + 1$ robots on this section. Otherwise there are $\lceil \frac{2l}{\Delta} \rceil$ or $\lceil \frac{2l}{\Delta} \rceil - 1$ robots on this section. Since both modules have a common period, all the temporal connections are periodic. Therefore robots i will have periodic connections with at least $\lceil \frac{2l}{\Delta} \rceil - 1$ robots in module J. □

In the rest of this paper, we design circulating modules that can be connected to one another. The interfacing indices are used to discuss the connection between those modules.

IV. General Properties of Modular Formations

A. Periodic communication topology

A formation composed of multiple modules with the same period T will have a communication topology that varies with the time period of T. Therefore we have $\mathcal{G}[t + T] = \mathcal{G}[t]$ for all $t \geq 0$. At any time step t, the collection of all communication links that existed in the last T time units is described by $\mathcal{G}^T[t] = \cup_{\tau=0}^{T} \mathcal{G}[t - \tau]$. It is easy to see that $\mathcal{G}^T[t]$ remains unchanged for all $t \geq T$. Therefore, we drop the time argument and write $\mathcal{G}^T[t]$ as $\mathcal{G}^T = (\mathcal{V}, \mathcal{E}^T)$. Furthermore, we define $\mathcal{N}_i^T = \{j | (i, j) \in \mathcal{E}^T\}$ to be the set of neighbors that can communicate with i within one period of time.

According to Theorem 1, a time-varying communication topology $\mathcal{G}[t]$ with a period T achieves resilient consensus in the presence of F non-cooperative robots if $\mathcal{G}^T = \cup_{\tau=0}^{T} \mathcal{G}[t - \tau]$ is $(2F+1)$-robust. Therefore, for all dynamic formations with periodic communication topology, we consider the robustness of the union graph $\mathcal{G}^T$. If the graph union $\mathcal{G}^T$ is $(2F+1)$-robust, we consider the formation to be a $(2F+1)$-*robust formation*.

In the rest of this section, we propose a construction of $(2F+1)$-robust formations. We start from a core formation fortified by the concept of F-elemental graph (see Theorem 2), and we expand it by adding modules in such a way that preserves the $(2F+1)$-robustness.

Definition 6 (Core formation)**.** *We call a composition of identical circulating modules as a $(2F+1)$-*core formation*, if their overall periodic communication topology is $(2F+1)$-robust.*

B. Construction of $(2F+1)$-core formations

An F-elemental graph contains $4F+1$ nodes. If a module contains at least $4F+1$ robots, an F-elemental graph may be generated within a single module. Let $\mathcal{V}_I = \{1, 2, ..., n\}$ represent the set of robots in a single module I, the communication topology of module I is denoted as $\mathcal{G}_I[t] = (\mathcal{V}_I, \mathcal{E}_I[t])$, with $\mathcal{E}_I[t] \subseteq \mathcal{V}_I \times \mathcal{V}_I$, for all t. The collection of connections within one period is captured by $\mathcal{G}_I^T = (\mathcal{V}_I, \mathcal{E}_I^T)$ with $\mathcal{E}_I^T = \cup_{\tau=0}^{T} \mathcal{E}_I[t - \tau]$. A sufficient condition to generate an F-elemental graph within this single module is described as follows.

Lemma 3. *A communication topology $\mathcal{G}_I^T = (\mathcal{V}_I, \mathcal{E}_I^T)$ is a $(2F+1)$-core formation, if $|\mathcal{V}_I| \geq 4F+1$, and $\mathcal{G}_I^T$ is a complete graph.*

Proof. Choose an arbitrary subset $\mathcal{V}_F \subseteq \mathcal{V}_I$ that contains $4F+1$ nodes. Denote the topology of this subset as $\mathcal{G}_F^T$. Since $\mathcal{G}_I^T$ is a complete graph, $\mathcal{G}_F^T$ is a complete graph as well. According to Theorem 2, $\mathcal{G}_F^T$ is F-elemental and therefore is $(2F+1)$-robust. For all $v^* \in \mathcal{V}_I \setminus \mathcal{V}_F$, we have $|\mathcal{N}_{v^*} \cap \mathcal{V}_F| \geq 4F+1$. Since $\mathcal{G}_F^T$ is a sub-graph of $\mathcal{G}_I^T$, according to Theorem 3, adding any node from $\mathcal{V}_I/\mathcal{V}_F$ back to the F-elemental graph will yield a new $(2F+1)$-robust graph. The same logic is repeated until all nodes in $\mathcal{V}_I \setminus \mathcal{V}_F$ are added back, and the final topology $\mathcal{G}_I^T$ is $(2F+1)$-robust as well. □

When a standard module contains less than $4F+1$ robots, we have to construct an F-elemental graph with multiple modules. A sufficient condition is presented in the next lemma.

Lemma 4. *Consider two modules, I and J, with their communication topology $\mathcal{G}_I[t] = (\mathcal{V}_I, \mathcal{E}_I[t])$ and $\mathcal{G}_J[t] = (\mathcal{V}_J, \mathcal{E}_J[t])$ identical to each other. The period of both modules is T. $|\mathcal{V}_I| = |\mathcal{V}_J| < 4F+1$. The overall communication topology is $\mathcal{G}_{IJ}[t]$. Module J's interfacing index with module I is $D_J(I)$. Then there exists a sub-graph $\mathcal{G}_F^T \subseteq \mathcal{G}_{IJ}^T$ that is an F-elemental graph if $\mathcal{G}_I^T$ is a complete graph, and $D_J(I) + |\mathcal{V}_I| \geq 6F$.*

Proof. First of all, since $D_J(I) \leq |\mathcal{V}_I|$, we have $\mathcal{V}_I \geq 3F$. Since I and J are identical modules, $|\mathcal{V}_{IJ}| = 2|\mathcal{V}_I| \geq 6F > 4F+1$.

Now we construct $\mathcal{G}_F^T$ by selecting a subset $\mathcal{S}_J$ of b consecutive robots along the path of J, with $b = 4F+1-|\mathcal{V}_I|$. We index the b robots as $1_J, 2_J, ..., b_J$. Without loss of generality, we index the first $D_J(I)$ robots from module I that robot 1_J is able to connect with as $1_I, 2_I, ..., D_J(I)_I$. Since I and J are identical, robot 2_J will connect to $2_I, 3_I, ..., (D_J(I)+1)_I$, and so on.

Therefore, robots indexed as $b_I, (b+1)_I, ..., D_J(I)_I$ are those who periodically connect with all the b robots in $\mathcal{S}_J$. Since $b = 4F+1-|\mathcal{V}_I|$ and $D_J(I)+|\mathcal{V}_I| \geq 6F$, the set $\mathcal{S}_F = \{b_I, (b+1)_I, ..., D_J(I)_I\}$ contains at least $2F$ nodes.

Let $\mathcal{V}_F = \mathcal{V}_I \cup \mathcal{S}_J$, $\mathcal{E}_F^T$ be the collected connections over one period. We have $|\mathcal{V}_F| = 4F+1$. There are at least $2F$ nodes in $\mathcal{S}_F \subseteq \mathcal{V}_F$, each is connected to all the nodes in $\mathcal{V}_F$, since $\mathcal{G}_I^T$ is a complete graph. $\mathcal{V}_F \setminus \mathcal{S}_F$ is periodically connected since it is composed of three periodically connected sub-sets: $\mathcal{S}_1 = \mathcal{S}_J$, $\mathcal{S}_2 = \{1_I, 2_I, ..., (b-1)_I\}$, and $\mathcal{S}_3 = \{(D_J(I)+1)_I, (D_J(I)+2)_I, ..., |\mathcal{V}_I|_I\}$. $\mathcal{S}_1$ and $\mathcal{S}_2$ are periodically connected since robot 1_J and robot 1_I are periodically connected. $\mathcal{S}_1$ and $\mathcal{S}_3$ are periodically connected since robot 2_J and robot $(D_J(I)+1)_I$ are periodically connected. According to Theorem 2, the topology $\mathcal{G}_F^T = (\mathcal{V}_F, \mathcal{E}_F^T)$ is an F-elemental graph. □

If there is any node in the graph $\mathcal{G}_{IJ}^T$ but not in the sub-graph $\mathcal{G}_F^T$, adding them back following a similar discussion as in Lemma 3 will lead to that $\mathcal{G}_{IJ}^T$ is also a $(2F+1)$-core formation.

Notice that we only consider $(2F+1)$-core formations composed of 1 or 2 identical modules, since any module of at least $2F+1$ robots can form a $(2F+1)$-core formation by itself or by connecting with another identical module. Modules containing $2F$ or fewer robots may form $(2F+1)$-core formations by connecting multiple identical modules but have no guarantee of robustness while we extend the core formation by adding more identical modules. This will be further illustrated in Sec. IV-C.

C. Extending a $(2F+1)$-robust formation

We discuss how to expand a $(2F+1)$-robust formation by adding new modules to the current formation while preserving its $(2F+1)$-robustness.

Lemma 5. *Consider a set of connected modules $\mathcal{R} = \{1, 2..., H\}$. The overall communication topology is represented by $\mathcal{G}_\mathcal{R}[t]$, where $\mathcal{G}_\mathcal{R}^T$ is $(2F+1)$-robust. Deploy a new module $H+1$ next to $\mathcal{R}$, and define $\mathcal{R}' = \{\mathcal{R}\} \cup \{H+1\}$. The communication topology of the new assembly $\mathcal{G}_{\mathcal{R}'}^T$ is $(2F+1)$-robust, if there exists a module $J \in \{1, 2, ..., H\}$, such that $D_{H+1}(J) \geq 2F+1$.*

Proof. Since $D_{H+1}(J) \geq 2F+1$, we have $|\mathcal{N}_{H+1} \cap \mathcal{V}_\mathcal{R}| \geq 2F+1$ holds for all nodes $p \in \mathcal{V}_{H+1}$. Therefore adding any node from $\mathcal{V}_{H+1}$ to the formation $\mathcal{R}$ will create a new $(2F+1)$-robust formation. By repeating adding nodes to the graph until all nodes in $\mathcal{V}_{H+1}$ are added, it will create a new $(2F+1)$-robust formation every time and the final formation with a topology of $\mathcal{G}_{\mathcal{R}(H+1)}^T$ is therefore $(2F+1)$-robust as well. □

Now we consider a set of connected modules, $\mathcal{Q}$. The higher-level graph $\mathcal{G}_\mathcal{Q}$ treats the modules as the nodes and inter-module connections as the edges. The topology of the robots in each module is $\mathcal{G}_I$, and the topology of all the robots in the whole set $\mathcal{Q}$ is $\mathcal{G}_q$. The theorem below examines whether the topology $\mathcal{G}_q$ of all robots is $(2F+1)$-robust.

Theorem 4. *Consider a set of connected modules, $\mathcal{Q}$, that is composed of $H \in \mathbb{Z}_{\geq 2}$ identical circulating modules, each with a periodic communication topology $\mathcal{G}_I^T = (\mathcal{V}_I, \mathcal{E}_I^T)$, for all $I \in \{1, 2, ..., H\}$. Let $\mathcal{V}_\mathcal{Q} = \{1, 2, ..., H\}$, and $\mathcal{E}_\mathcal{Q}$ contains all $(I, J) \in \mathcal{V}_\mathcal{Q} \times \mathcal{V}_\mathcal{Q}$. Let $\mathcal{T}_\mathcal{Q} = (\mathcal{V}_\mathcal{Q}, \mathcal{E}_\mathcal{Q}^\mathcal{T})$ be a spanning tree of $\mathcal{G}_\mathcal{Q} = (\mathcal{V}_\mathcal{Q}, \mathcal{E}_\mathcal{Q})$. Let $\mathcal{V}_q$ be the set of all robots in $\mathcal{Q}$, and $\mathcal{E}_q^T$ the set of collected inter-robot connections within one period. The communication topology of the whole formation $\mathcal{G}_q^T = (\mathcal{V}_q, \mathcal{E}_q^T)$ is $(2F+1)$-robust, if the following conditions hold.*

- *$\mathcal{G}_I^T$ is a complete graph for all $I \in \mathcal{V}_\mathcal{Q}$;*
- *There exists some $\mathcal{T}_\mathcal{Q}$, such that for all $(I, J) \in \mathcal{E}_\mathcal{Q}^\mathcal{T}$, there is $\max\{D_I(J), D_J(I)\} \geq 2F+1$, and $\max\{D_I(J), D_J(I)\} + |\mathcal{V}_I| \geq 6F$.*

Proof. According to Lemma 3 and Lemma 4, there exist some $(I, J) \in \mathcal{E}_\mathcal{Q}^\mathcal{T}$ such that $\mathcal{G}_{IJ}^T$ is a $(2F+1)$-core formation. According to Lemma 5, adding modules satisfying the conditions stated in the theorem will preserve the $(2F+1)$-robustness of the formation. Since every pair of $(I, J) \in \mathcal{E}_\mathcal{Q}^\mathcal{T}$ satisfies the conditions, the overall formation $\mathcal{Q}$ is $(2F+1)$-robust. □

Notice that $D_I(J)$ does not necessarily equals to $D_J(I)$. Although all modules are identical, robots from different modules may enter the interfacing section asynchronously and yields different indexes for different modules.

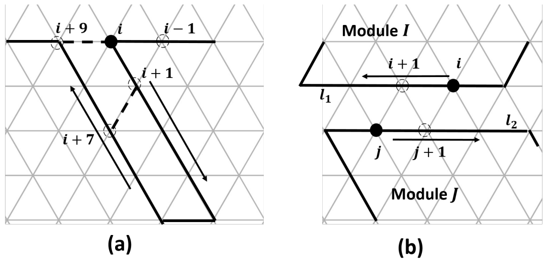


Fig. 3. Examples of periodic connections. (a) shows δi-bridges formed by a *slit* shaped bent on the path. The dash line connection i and $i+9$ is a 9-bridge, and the one connection $i+1$ and $i+7$ is a 6-bridge. (b) shows a pair of interfacing sections with a path length of $4d$. The interfacing indexes will be $D_I(J) = D_J(I) = 9$.

V. Geometric Design

In the previous section, sufficient conditions on circulating modules are provided to ensure resilient consensus on formations composed of identical modules. In this section, we discuss the actual design of the path's shape in a latticed space [22], [23] to satisfy those conditions.

We use a triangular lattice as introduced in [7]. The lattice is defined as $\mathbb{L} = \{d(x\mathbf{v}_1 + y\mathbf{v}_2)|x, y \in \mathbb{Z}\}$, where d is the scale factor of the lattice, which denotes the length of the triangular sides. $\mathbf{v}_1 = [1, 0]^\top$ and $\mathbf{v}_2 = [\frac{1}{2}, \frac{\sqrt{3}}{2}]^\top$ are the base vectors. Let $d = \Delta$, where Δ is the spacing in a circulating module, which is slightly smaller than the robots' communication range R. When all robots arrive at the top of some lattice points, communication is only available between those on the lattice points next to each other. Connecting lattice nodes with straight lines of a length of d creates a lattice grid.

We design the path of a module as a closed curve composed of a limited number of line segments along the grid lines. At time $t = 0$, there is a robot on every lattice node on the path. Let the number of robots in one module to be n, and the robots indexed as $1, 2, ..., n$ along the moving direction of robots. The location of a robot i at t is denoted as $s_i[t]$. Let 1 time step to be the time that a robot takes to move from one lattice node to the next, $s_i[t+1] = s_{i+1}[t]$, for all $i = 1, 2, ..., n-1$, and $s_n[t+1] = s_1[t]$.

Robot i is always connected with $i+1$ and $i-1$. At some lattice node along the path, robot i is in communication range with some robots $i + \delta i, \delta i \in \mathbb{Z}_{\geq 2}$, and we say there exists a δi-communication bridge. As shown in Fig. 3, the dashed lines in part (a) are a 6-bridge and a 9-bridge. The connections across δi-bridges are periodic.

When two identical modules are periodically connected through a pair of interfacing sections, the path length of an interfacing section is always an integer multiple of d, as shown in Fig. 3, part (b). Let the arc length of the interfacing section be $pd, p \in \mathbb{Z}_{\geq 1}$, the interfacing index of one module with another is always $2p+1$.

A. Path features

a) Fully Connected graph: One of the important conditions to construct $(2F+1)$-robust formations is to ensure that in each module, every robot is periodically connected with all other robots. For the n robots in the same module, there should be δi-bridges in the path for all $\delta i = 2, 3, ..., \lceil \frac{n}{2} \rceil$. One method to ensure the δi-bridges is to create a 'slit' shape as shown in Fig. 3. A section of the path with a length of $\lceil \frac{n}{2} \rceil$ and being bent as a slit shape can provide a series of δi-bridges for $\delta i = 2, 3, ..., \lceil \frac{n}{2} \rceil$.

b) Interfacing section: The other important condition to construct $(2F+1)$-robust formations is to provide a sufficient interface number while connecting two modules. For example, as shown in Fig. 3, part (b), a straight line segment of length pd on the path can provide a sufficient interfacing index while serving as an interfacing section if $p \geq F$ and $2p + n + 1 \geq 6F$. If we are connecting multiple identical modules, each module should contain at least 2 non-overlapping sections that can provide sufficient interfacing indexes.

B. Examples

Following the constraint of path features analyzed, we provide some example designs.

a) Unit triangles: A module I_3 with 3 robots on a triangle path with each edge of a unit length d, as shown in Fig. 4, part (a), is always fully connected. It has 3 sections, each with a length of d, that can be interfacing sections. Hence placing another triangle module at one of the 6 locations (shown in the shaded triangles) can establish an interfacing index of 3 between them. Therefore this module with $n_{I_3} = 3$ and $p_{I_3} = 1$ can form large scale formations that is resilient in consensus performance against $F_{I_3} = 1$ non-cooperative robots.

b) Triangles with slits: Fig. 4, part (b) shows an example module I_{15} with 15 robots. Its communication topology is $\mathcal{G}_{I_{15}}[t]$. The robots form a triangle with 2 sections of length $4d$ that can be interfacing sections. Black dash lines show δi-bridges of $\delta i = 2, 3, ..., 6$. The dashed line highlighted

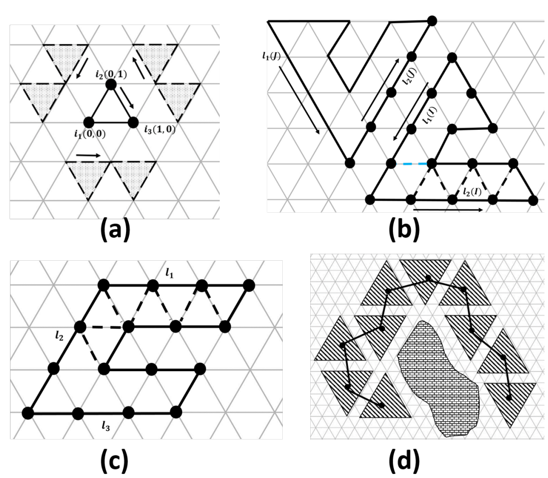


Fig. 4. (a) Unit triangle module I_3; (b) Triangle module I_{15}; (c) U-shape module I_{16}; (d) A formation composed of multiple I_{15} modules is resilient against $F = 4$ non-cooperative robots.

in blue shows a communication bridge that is both a 7-bridge and an 8-bridge. Therefore $\mathcal{G}_{I_{15}}^T$ is fully connected. It can be shown that two I_{15} modules can form a $(2F+1)$-core formation with $F_{I_{15}} = 4$, and connecting multiple identical I_{15} modules can form a formation that is resilient against $F_{I_{15}} = 4$ non-cooperative robots in any robot's neighborhood. Fig. 4, part (d) shows an example formation. Notice that each module can only connect with 2 other modules.

c) U-shape: Fig. 4, part (c) shows a U-shape path with a slit bent on it. This module I_{16} has 16 robots and the path contains 3 sections, each of the length $3d$, that can be interfacing sections. The communication topology of this module is $\mathcal{G}_{I_{16}}[t]$. Dash lines show δi-bridges for $\delta i = 2, 3, ..., 8$. Therefore $\mathcal{G}_{I_{16}}^T$ is fully connected. It can be shown that a single I_{16} is a $(2F+1)$-core formation with $F_{I_{16}} = 3$, and connecting multiple I_{16} modules can form a formation that is resilient against 3 non-cooperative robots. Every module can assembled with 3 other modules.

VI. Simulations

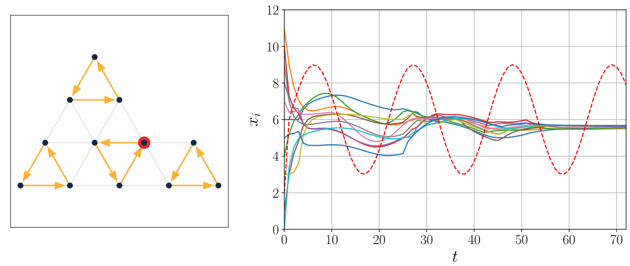
In this section, we present four different dynamical networks where the cooperative robots achieve resilient consensus in the presence of multiple non-cooperative robots[1]. The non-cooperative robots are shown in red circles. The initials of each team were randomly chosen, including the non-cooperative robots. The cooperative robots follow the SW-MSR algorithm, and the non-cooperative robots generate oscillatory and random values. The signal from the non-cooperative robots try to avoid consensus from the cooperative robots. In each scenario, the communication network is periodic and satisfies all the sufficient conditions that we described in the previous sections. The result of the simulations is summarized in Figure 5. We can see that in all the cases, the cooperative robots can achieve resilient consensus using the SW-MSR algorithm. If we compare the convergence rate in panels (c) and (d), we can see that the convergence rate is faster for a small number of modules. This result is expected since the network connectivity is reduced by adding new modules to the network. However, in comparison to methods in the literature where high connectivity is required to satisfy r-robustness, we can achieve r-robustness and still maintaining low connectivity. In panel (d), we show that the modular configuration limits the neighborhood of each robot, allowing up to $F = 4$ non-cooperative robots in every robot's neighborhood.

VII. Conclusions

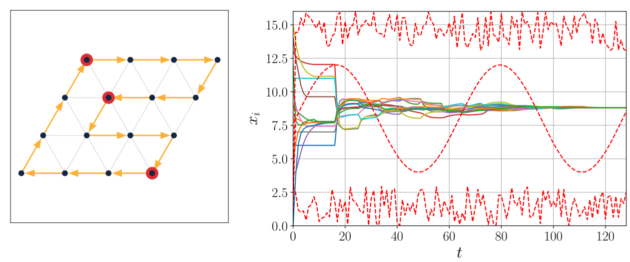
This work proposed a formation and routing strategy to generate robotic communication networks that is resilient in consensus performance by connecting modular circulating routes, each deployed with a sub-group of robots. We showed how circulating modules with a complete communication topology can be composed to form a resilient formation. The modules are connected to each other so that they have a sufficient number of "inter-module" communication, which we call the interfacing index. With the assumption that all robots are moving at the same constant speed, we satisfy the conditions on the connectivity and on the interfacing index by carefully designing the geometry of the modular circulating path.

Our future work will allow modules to have different shapes and different numbers of robots. More generalized routes in environments without lattice will be studied. We also consider the deployment on pre-defined circulating

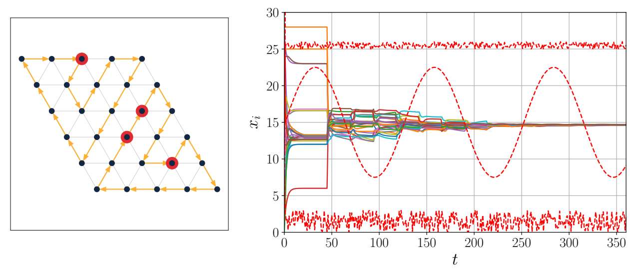
[1]Some of the simulation results are shown in the video available at [https://youtu.be/dj2afGyhBB4]



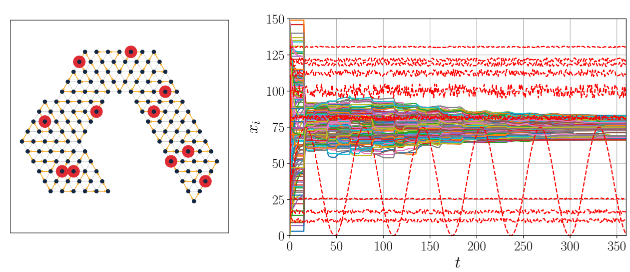
(a) Triangular module (3 robots each)



(b) U-shape module



(c) Large triangular module (15 robots each)



(d) Ten large triangular modules

Fig. 5. Simulations for four different modular dynamical networks. On the left side of each panel, we present the network topology. The dark disks represent the robots and the arrows illustrate their direction of motion. On the right side, the evolution of the scalar variable in time. The solid lines and the dashed lines represent the scalar values of the cooperative and non-cooperative robots respectively.

paths, in which the design variable is now the robots' speed profile so that they achieve required temporal connections.

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