Statement of research activities.

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Introduction

My research concentrates on the study of stochastic partial differential equations (SPDEs). Essentially, they are partial differential equations (PDEs) perturbed by a random component called the *noise*. They appear in applications in a wide range of fields, among which I can mention:

- Physics. For instance, SPDEs appear in the propagation of heat above an irregular source of temperature and the movement of a DNA molecule in a fluid. A specific example that has become more and more popular recently, is the Kardar-Parisi-Zhang (KPZ) equation, modeling the behavior of growth surfaces. It arises in a wide class of situations, such as the movement of galaxies or polymer models.

- Biology. For instance, they appear in cell cluster growth models, or when studying the concentration of bacteria in a fluid medium. These have direct connection to the KPZ equation mentioned above. SPDEs also appear in models for population dynamics in a random environment.

- Finance. Most of the mathematical finance models (such as Black-Scholes) are based on stochastic differential equations, the sisters of SPDEs. SPDEs themselves typically appear in optimization of portfolio, or in models for interest rates.

What is an SPDE ?

In most mathematical models, the quantities of interest are functions of both a time variable, t and a space variable x. The quantity of interest is then represented by a function, say $u(t, x)$. In several instances, a model can be stated in the form of a partial differential equation (PDE): an equation for the unknown function u that involves its partial derivatives, both with respect to time t and space x. One of the most common PDEs is the *heat equation*, modeling the diffusion of heat from a source. The second common example is the *wave equation* modeling the propagation of waves. They are stated (in dimension 1) as:

$$\text{Heat equation:} \qquad \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad\qquad \text{Wave equation:} \qquad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0.$$

Before solving such equations, we need to specify a domain on which they hold, some boundary conditions and some initial conditions.

Stochastic partial differential equations (SPDEs) are PDEs into which we introduce a random component. In my research I focus on understanding models based on the heat or the wave equation. The random component, known as the *noise*, arises in the form of a random (generalized) function of time and space. The most commonly used one is the so-called *space-time white noise*. It is a random function $\dot{W}(t, x)$ which is a Gaussian process. It is known as *white* in space and time, intuitively because the noise at time t and position x is independent of the noise at time s and position y (for $t \neq s$, $x \neq y$). Informally, the covariance of the space-time white noise is given by

$$E[\dot{W}(t, x)\dot{W}(s, y)] = \delta_0(t - s)\delta_0(x - y),$$

where δ_0 denotes Dirac's delta function. It is typically used as a standard model in cases where the random component doesn't have a specific known behavior. Another type of noise considered is the *spatially-colored* noise, for which the noise at two different positions depend one on another. Its covariance is given by

$$E[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s)f(x-y),$$

for some appropriate positive-definite function f.

In order to get an SPDE, we introduce the random noise $\dot{W}$ described above into the equation. Typically, the easiest case is to consider an equation with *additive* noise. For example, a stochastic heat equation with additive noise would be:

$$\text{Stochastic heat equation:} \qquad \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \dot{W} = 0.$$

In my research, I mostly consider nonlinear SPDEs. In these equations, we consider a *multiplicative* noise, where the noise multiplies a function of the solution. For example, a stochastic nonlinear wave equation would be:

$$\text{Stochastic nonlinear wave equation:} \qquad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sigma(u)\dot{W} = 0.$$

The mathematical difficulties brought in by Stochastic Partial Differential Equations are multiple. These equations are usually understood in integral form, either in *weak* or *mild* form. Indeed, classical random noises such as the space-time white noise are not differentiable functions. As a consequence, the solution would not be differentiable either and the only formal way to define a solution is through integrals. Hence, the study of existence and uniqueness for these equations requires the construction of specific mathematical tools to handle integrals with respect to a random noise: *stochastic integrals.* The study of these integrals then provides the right functional space in which the solution to the equation will live. Once we know that a solution exists (and is unique), the randomness of $\dot{W}$ requires the use of different tools from probability and analysis in order to determine and study properties of solutions. Among the properties that I typically studied, I can mention:

- Moments of the solution. Since the equation involves a random component, the solution will be random as well. Among the first things we can study about a random object are its moments. They describe different type of *average* behavior of the solution. If moments are known, a lot can be said about the random behavior of the solution.

- Continuity of the solutions: a small change in t and/or in x leads to a small change in the values of $u(t,x)$, despite the randomness. Quantifying those changes (Hölder continuity) is sometimes a challenging question.

- Physical properties of the solution, such as intermittency, chaos or fractal behavior. This is the most important part of my research and is described in more detail below. In short, physical properties aim at describing the qualitative behavior of the solution as a random dynamical process. In particular, intermittency is the fact that the solution can take very large (unexpected) values provided we wait long enough. Chaos describes the fact that the solution is very sensitive to changes in the initial conditions or parameters.

Below, you will find a more specific presentation of the problems in which I have been interested until now.

Framework

We will start by saying a few words about methodologies for existence and uniqueness to help understand the techniques towards our objectives. Before we can get interested in the properties of the solutions to SPDEs, we need to make sure that these solutions actually exist and, if possible, are unique for a given equation. In general, in my research, I have been interested in equations of the following form

$$Lu_t(x) = \sigma(u_t(x))\dot{W}_t(x), \qquad (t > 0, x \in \mathbb{R}^d) \tag{1}$$

where t represents the time variable, x represents the space variable, L is a second-order differential operator and $\dot{W}$ is the random noise. We consider the initial conditions to be deterministic bounded measurable functions. Typical examples are the cases where $L = \partial_t - \Delta$, the heat operator, or $L = \partial_{tt} - \Delta$, the wave operator (where Δ stands for the Laplacian operator). More generally, we can also consider $L = \partial_t - D$, where D is a more general differential operator, such as the generator of a Lévy process.

Several schools have developed when it comes to understanding the formal meaning of equation (1). We will in general consider the equation using the perspective of *martingale-measure stochastic integration* developed by John Walsh [43]. This approach consists in considering the noise $\dot{W}_t(x)$ to be a *martingale-measure* $\{\dot{W}_t(x), t \geqslant 0, x \in \mathbb{R}^d\}$; namely, an object that is a martingale with respect to the time variable t and a measure with respect to the space variable x. Walsh's theory then guarantees under certain conditions that we can integrate *predictable* random processes with respect to the martingale measure.

We can use Duhamel's principle and understand the solution to (1) as a *random-field* solution; namely, a family of random variables $(u_t(x),\, t \geqslant 0, x \in \mathbb{R}^d)$ such that $(t,x) \mapsto u_t(x)$ is a function from $\mathbb{R}_+ \times \mathbb{R}^d$ into $L^2(\Omega)$, that is continuous and solves an integral form of (1), namely the *mild-form* equation:

$$u_t(x) = U_t^{(0)}(x) + \int_0^t \int_{\mathbb{R}^d} \Gamma_{t-s}(y-x)\sigma(u_s(y))W(ds\,dy), \tag{2}$$

where Γ is the Green function associated to the differential operator (the solution to the homogeneous PDE), $U^{(0)}$ is the contribution of the initial conditions (which depends on the operator) and the integral is understood as the integral with respect to a martingale measure in the sense of Walsh [43].

The advantage of considering Walsh's approach is that it allows to fix $t > 0$ and $x \in \mathbb{R}^d$ and study the random variable $u_t(x)$ for itself, or to fix two values $x_1, x_2 \in \mathbb{R}^d$ and study the relationship between $u_t(x_1)$ and $u_t(x_2)$. These are crucial features when it comes to understanding physical properties of the solutions to SPDEs.

The nonlinear stochastic wave equation.

The first question that naturally arises when dealing with SPDEs is the question of existence (and uniqueness) of the solution. This research has been started during my Ph.D. studies. I have been interested in the study of the nonlinear stochastic wave equation. Namely,

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \sigma(u)\dot{W}, \tag{3}$$

where σ is a Lipschitz function, $\dot{W}$ is a spatially correlated noise and the dimension of the spatial variable is $d \geqslant 1$, with an emphasis on $d \geqslant 4$. The equation with $d \geqslant 4$, even though it is mainly of pure mathematical interest, presents difficulties that do not arise in dimensions $d \leqslant 3$, nor with the heat equation. This is mainly because the *fundamental solution* (or Green function) of the wave equation in high dimensions is not a function but a Schwartz distribution.

Existence and uniqueness.

The question of existence and uniqueness of a random-field solution to (3) has been addressed in [43] for the 1-dimensional case, in [26] for the 2-dimensional case and in [25] for the 3-dimensional case. Our results regarding existence and uniqueness for higher dimensions (under certain restrictions) are presented in [15].

The general idea to address existence is to start from (2) and define a Picard iteration scheme, by $u_t^{(0)}(x) = U_t^{(0)}(x)$ and, by induction,

$$u_t^{(n)}(x) = U_t^{(0)}(x) + \int_0^t \int_{\mathbb{R}^d} \Gamma_{t-s}(y-x)\sigma(u_s^{(n-1)}(y))W(ds\,dy). \tag{4}$$

It remains to show that each $u^{(n)}$ is well-defined and that the sequence converges to a limit u which will be the solution. In the 1-dimensional and 2-dimensional cases, the fundamental solution Γ of the wave operator is a function and Walsh's stochastic integral is directly used. In higher dimensions, Γ becomes a Schwartz distribution (nonnegative for $d = 3$) and an extension of the stochastic integral with respect to martingale measures is required for (4) to be well-defined. This was first done in [25] for nonnegative distributions. Our paper [15] extends it to general Schwartz distributions for the purpose of stating existence and uniqueness for (3). The main idea is to use a Fourier representation of the Green function, which is a measurable function, namely

$$\mathcal{F}\Gamma_t(\xi) = \frac{\sin(t|\xi|)}{|\xi|},$$

in any dimension d. The fact that the Green's function is non-negative does not allow for standard bounds, but a careful use of the convolution properties of the Fourier transform was used to extend the definition of stochastic integral. This extension still has some limitations. The stochatic process $\sigma(u_s(y))$ must be a real-valued process with spatially homogeneous covariance. Moreover, we do not have any result concerning moments of order larger than 2 for the integral process.

Hölder regularity.

Once existence and uniqueness have been established, a problem that arises naturally is the question of the Hölder regularity of the solution. To address this question, we need estimates for moments of any order of the solution, in order to apply Kolomogorov's continuity theorem (this was done in [42] for the linear equation). In our case, such estimates cannot be obtained from moments results for the stochastic integral. As mentioned above, we do not have such results in the case where Γ is not nonnegative.

In [15], we have been able to obtain results on higher moments of $u_t(x)$ and, from these, Hölder regularity in the specific case where σ is affine, i.e. the *hyperbolic Anderson model.* This problem was studied for the first time in [29] in spatial dimension 3. The main idea is to write $u_t(x)$ as a sum of iterated stochastic integrals. Indeed, with constant initial conditions, the Picard scheme yields $u^{(0)} \equiv C$ and, setting $v_t^{(n)}(x) = u_t^{(n)}(x) - u_t^{(n-1)}(x)$, we obtain

$$u_t(x) = \sum_{n=0}^{\infty} v_t^{(n)}(x). \tag{5}$$

In the case where σ is affine, each $v^{(n)}$ can be represented as a multiple stochastic integral of order n and its moments are known. Hölder-continuity can be established term by term. In fact, the series expansion (5) corresponds to the Wiener-chaos expansion of u. Hölder-continuity of the solution when σ is not affine remains an open problem.

Note that a similar technique was used in my paper [14] to obtain precise moment representations in the case of the *parabolic* Anderson model. The idea is to represent each multiple stochastic integral $v^{(n)}$ using local times of Brownian motion. This allows to represent the sum as an exponential and leads to an alternate proof of the Feynman-Kac formula for moments of the Parabolic Anderson model under more general assumptions.

Note that following some questions of Carl Mueller and Hakima Bessaih, it appears that an equation that would be of interest in application is an equation of the form

$$Lu_t(x) = \sigma\left(\nabla u_t(x)\right) \dot{W}_t(x).$$

This equation considers a multiplicative noise but with respect to a *friction*-type term, which actually makes more sense, physically speaking, than (3). Existence and uniqueness of a solution to this nonlinear wave equation is currently not known.

Itô-Taylor expansions.

Inspired by the ideas used to obtain (5) in the case where σ is affine, in [13] and [16], we have been interested in equations of the form

$$Lu = \sigma(u)\dot{W}, \tag{6}$$

where σ is a Lipschitz (non-affine) and analytic function, $\dot{W}$ is a noise white in time and possibly correlated in space and L is a second-order differential operator, typically the heat or the wave operator. In that case, (5) still holds, but the terms $v^{(n)}$ are not directly multiple stochastic integrals. Nevertheless, using the Taylor expansion of the function σ, we have obtained a representation of the solution as a series of multiple stochastic integrals of order up to n, plus a known remainder term. Namely,

$$u_t(x) = \sum_{i=0}^{n} \sum_{\beta \in \mathcal{A}^{(i)}} \pi_\beta I_\beta(t,x) + \sum_{\beta \in \mathcal{A}^{(n+1)}} J_\beta(\kappa_\beta(u))(t,x), \tag{7}$$

where $\mathcal{A}^{(n)}$ is a set of multi-indices of order n, the $(I_\beta,\ \beta \in \mathcal{A}^{(n)})$ are iterated stochastic integrals depending on the multi-index β, $(\pi_\beta,\ \beta \in \mathcal{A}^{(n)})$ are real-valued constants that depend on the function σ, the $(J_\beta,\ \beta \in \mathcal{A}^{(n+1)})$ are integral operators applied to the functions $(\kappa_\beta(u),\ \beta \in \mathcal{A}^{(n+1)})$ which depend on σ and its derivatives. Such an expansion is known as an *Itô-Taylor expansion*. The first sum in (7) is the expansion itself and the second sum is the remainder. This truncated expansion of order n is similar to the Taylor expansion for the solution of an ordinary differential equation. In our case, we use Itô formula iteratively, rather than the fundamental theorem of calculus. Notice that this expansion applies to both parabolic and hyperbolic equations.

A natural question arising is the one of the convergence of (7) as the order n goes to infinity. Such results exist for nonrandom models as well as Stochastic Differential Equations. We are currently not able to address this question by a direct estimate of the remainder term in (7). If, however, we consider the heat equation on a compact domain D rather than on the whole real domain, we are able to prove that

$$u_t(x) = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{\beta \in \mathcal{A}^{(i)}} \pi_\beta I_\beta(t,x), \tag{8}$$

almost-surely up to a stopping time τ with $P(\tau > 0) = 1$. The idea is to use a perturbation of the equation with a complex-valued parameter, as well as analyticity arguments and complex analysis techniques (see [16]). This method is inspired by results obtained by Ben Arous [7] for stochastic differential equations (SDEs). For reasons related to the complex-valued nature of our approach, it does not apply to the equation on the whole domain as is.

Convergence (8) remains an open problem in general, even in simple cases. Nevertheless, we conjecture that (8) should hold in some sense (namely, in $L^2(\Omega)$ or a.s.) up to a deterministic finite time T. So we hope to prove that $P(\tau \geqslant T) = 1$. Typically, studying simpler cases such as $\sigma(u) = u(1-u)$ (i.e. a polynomial) helps. This example appears in models for the dynamics of populations competing for resources. Assuming (8) holds, one could obtain

$$u_t(x) = \sum_{i=0}^{\infty} \sum_{\beta \in \mathcal{A}^{(i)}} \pi_\beta I_\beta(t,x). \tag{9}$$

We can actually prove that the right-hand side of (9) satisfies (6) if we assume convergence of the series. If true, equation (9) would provide an explicit expression for the solution to both the heat and wave equations and would open new directions to address the questions of moments and Hölder-regularity.

One application of such Itô-Taylor expansions is to estimate errors in numerical schemes for SPDEs. A typical quantity of interest would be the moments $\mathbb{E}[u_t(x)^p]$ rather than $u_t(x)$ itself. A series representation for the former could be deduced from (7) or (8), but may also be obtained more directly. For instance, Chen and Dalang obtain such series for $p = 2$ in [10]. However, what is the most relevant regarding numerical simulations is to understand how the series depends on the different components of the problem (most importantly initial conditions). Hence, exact expressions are crucial. Discussions with Arnulf Jentzen started regarding this question. Some of the general ideas behind [14] and [16] within the slightly different DaPrato-Zabczyk framework were used by Arnulf Jentzen to prove weak convergence of some Galerkin numerical schemes. It led to the preprint [17] for which I mostly contributed in the form of ideas for the proofs.

Intermittency for parabolic and hyperbolic equations.

As mentioned in the introduction, parabolic and hyperbolic nonlinear stochastic partial differential equations arise in several instances as models for physical systems. One of the most important, that has been the subject of extensive research recently, is the Kardar-Parisi-Zhang (KPZ) equation [37] (see (11) below). First of all, let us describe the mathematical model that we are studying. We will be interested in the following family of parabolic equations:

$$\frac{\partial}{\partial t} u_t(x) = \kappa \Delta u_t(x) + \sigma(u_t(x)) \dot{W}_t(x), \tag{10}$$

where $t > 0$, $x \in \mathbb{R}^d$, σ is a Lipschitz function, $\dot{W}$ is a noise that is white in time and possibly correlated in space, and κ is a constant. We notice that the results below are not restricted to the Laplacian operator Δ in (10). In general, one can consider the generator L of a Lévy process instead of Δ.

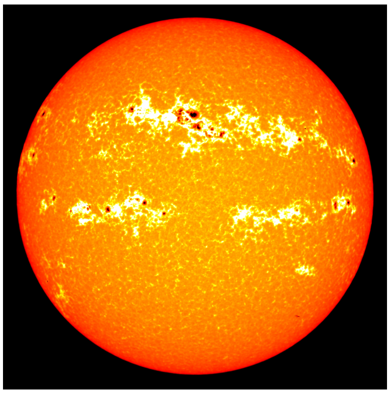
Namely, when $\sigma(u) = \lambda u$, it corresponds to the continuous version of the *parabolic Anderson model*, which has been the subject of a wide literature, initiated by [9]. However, the most important application of this equation is its connection to the KPZ equation (in dimension $d = 1$ here):

$$\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial t^2} - \left(\frac{\partial h}{\partial x}\right)^2 + \dot{W}(t,x), \quad (t > 0, x \in \mathbb{R}). \tag{11}$$

The Hopf-Cole transformation $h_t(x) = \log(u_t(x))$ shows informally (if $\dot{W}$ was smooth) that if u solves the stochastic heat equation (10), then h solves the KPZ equation (11). This connection in the case of white noise has been made formal recently by M. Hairer, using his theory of regularity structures in the extremely important paper [33].

Intermittency

Among physical properties of the solution, one of the most important regards *intermittency* of the solution. The object that equation (10) models is said to be *physically intermittent* if it exhibits the following behavior: As t gets large, for some values of x, the value of $u_t(x)$ takes very large values, untypical of the expected behavior of the process: they are known as *peaks*. Those peaks will be concentrated on small x-domains (known as *islands*). Characterizations of this phenomenon appear in [8] and [9]. Figure 1 illustrates intermittent processes: the left picture illustrates the intermittent levels of energy at the surface of the sun. We see that high levels of energy concentrate on small portions of the surface. The second picture illustrates a simulation of the solution to the nonlinear stochastic heat equation on the compact interval $[0, 1]$ with $u_0(x) = \sin(\pi x)$ as initial condition.



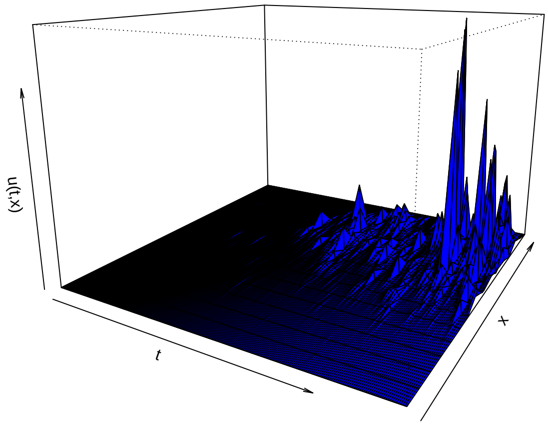


Figure 1: Left: intermittent levels of energy at the surface of the sun (source: NASA.gov); Right: intermittent solution to the nonlinear stochastic heat equation on $[0, 1]$ with $u_0(x) = \sin(\pi x)$.

A random-field is known to be *(mathematically) weakly intermittent* if the following condition is satisfied

$$0 < \gamma(p) := \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in\mathbb{R}^d} \mathbb{E}[|u_t(x)|^p] < \infty \qquad \text{for all } p \geqslant 2. \tag{12}$$

The value $\gamma(p)$ is known as the *moment Lyapunov exponent* of u. In most situations considered in my research, the moments do not depend on x, which is in general not the case for the solution to (10). A related notion is *full intermittency*, namely the convexity of the moment Lyapunov exponent as a function of p. In other words, a random field is *fully intermittent* if

$$\gamma(1) < \frac{\gamma(2)}{2} < \cdots < \frac{\gamma(p)}{p} < \cdots$$

Notice that the nonstrict inequalities are immediate from Jensen's inequality. Also, we can prove that full and weak intermittency are equivalent under the condition that $\gamma(1) = 0$. Full intermittency implies that if $p > q$, then the order of the $L^p(\Omega)$-norm is significantly larger than the order of the $L^q(\Omega)$-norm. This implies existence of large values for u with positive probability.

Formal arguments connecting mathematical and physical intermittency can be found in [8], [38] and [44]. Typically, if one considers the two processes B_t (a Brownian motion) and $\exp(B_t - \frac{t}{2})$, the first one is not intermittent, but the second one is, although they are both martingales. Essentially, the first one is a *sum* of i.i.d. random variables, whereas the second one is a *product* of i.i.d. random variables. These properties are believed to be characteristic of intermittency and, more generally, of the *KPZ (or Tracy-Widom) universality class* as opposed, for instance, to the *Gaussian universality class* of the Central Limit Theorem. (See [41] for more on the KPZ universality class.)

Intermittency is widely studied in particular because it is believed to be connected to *chaos*: due to the peaking behavior, small changes in the initial condition should lead to major changes for the solution. One of the general objectives of my research is to contribute to a better understanding of intermittency and, in particular, which characteristics of the KPZ equation predicted by physicists are exhibited by Stochastic Partial Differential Equations models such as the stochastic heat equation (10) or the stochastic wave equation (3).

Stochastic Young's inequality and position of the peaks

Two of the main papers studying intermittency for (10), [8] and [31], present similar results, but the techniques used to obtain them are significantly different. Indeed, Bertini and Cancrini [8] (similarly as a lot of the literature) strongly use the Feynman-Kac representation for the solution to the heat equation (10) and its moments in the case where $\sigma(u) = \lambda u$ (Parabolic Anderson Model). This representation states that

$$u_t(x) = \mathbb{E}_X^x \left[u_0(X_t) \exp\left(\lambda \int_0^t \int_{\mathbb{R}^d} \delta(X_s - y) W(ds, dy) \right) \right], \tag{13}$$

where X is a Lévy process generated by the operator in (10) (Brownian motion if it is the Laplacian Δ), independent of W; δ is Dirac measure; and $\mathbb{E}_X^x$ is the expectation with respect to X (not W) under the condition $X_0 = x$. Unfortunately, representation (13) is only valid in the case where $\sigma(u) = \lambda u$ and it relies extensively on the very specific form of the equation. Some results for equations involving a non-linearity σ appear in [36]. Another form of Feynman-Kac representation exists for moments of u, rather than for u itself, namely

$$E[u_t^p(x)] = \mathbb{E}_X^x \left[\left(\prod_{i=1}^p u_0(X_t^{(i)}) \right) \exp\left(\lambda^2 \sum_{j,k=1}^p \mathbf{1}_{j \neq k} \int_0^t f\left(X_s^{(j)} - X_s^{(k)} \right) ds \right) \right], \tag{14}$$

where $(X^{(j)})_{j=1,\dots,p}$ is a sequence of independent copies of the process X above and f is the spatial correlation of the noise. The Feynman-Kac formula for moments exists in more general instances, even if (13) doesn't hold.

In the spirit of the approach used by Foondun and Khoshnevisan [31], we aimed to avoid using the Feynman-Kac representation, and develop a different route to obtain similar results using (continuous) techniques of analysis, since these apply in a wider range of examples. In [31], the authors only cover the case of an initial condition bounded away from 0.

In [23], together with Davar Khoshnevisan, we proved mathematical intermittency for equation (10) in the case where $d = 1$, $\dot{W}$ is space-time white noise and the initial condition has compact support. The idea is to introduce the following family of norms for a random field $(Z_t(x) : t \geqslant 0, x \in \mathbb{R})$. For $\beta \geqslant 0$ and $p \geqslant 1$, and an appropriate function $\theta : \mathbb{R} \to \mathbb{R}_+$, we defined the norm $\mathcal{N}_{\beta,p,\theta}$, by

$$\mathcal{N}_{\beta,p,\theta}(Z) := \sup_{t \geqslant 0} \sup_{x \in \mathbb{R}} e^{-\beta t} \theta(x) E[|Z_t(x)|^p]. \tag{15}$$

Further, we developed stochastic Young-type inequalities for stochastic convolutions with respect to the norms above. We consider the random-field defined by

$$(\Gamma \star Z\dot{W})_t(x) := \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) Z_s(y)\, W(ds, dy). \tag{16}$$

The stochastic integral in (16) is actually a *stochastic convolution*, and it appears in the mild-form of the solution (2). The Young-type inequality then states that

$$\mathcal{N}_{\beta,p,\theta}(\Gamma \star Z\dot{W}) \leqslant C\, \|\Gamma\|_{L^1_\beta([0,T], L^2_\theta(\mathbb{R}))}\, \mathcal{N}_{\beta,p,\theta}(Z), \tag{17}$$

where

$$\|\Gamma\|_{L^1_\beta([0,T], L^2_\theta(\mathbb{R}))} := \int_0^\infty dt\, e^{-\beta t} \int_{\mathbb{R}} dx\, \theta(x) \Gamma_t^2(x).$$

The inequalities (17) as well as similar lower bounds for slightly different norms are then used with $\theta(x) = \mathbf{1}_{\{|x|>\alpha t\}}$ to prove that

$$\overline{\lambda}(p) := \inf\left\{\alpha > 0 : \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|>\alpha t} \log \mathbb{E}[|u_t(x)|^p] < 0\right\}$$

and

$$\underline{\lambda}(p) := \sup\left\{\alpha > 0 : \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|>\alpha t} \log \mathbb{E}[|u_t(x)|^p] > 0\right\}$$

satisfy $0 < \underline{\lambda}(p) \leqslant \overline{\lambda}(p) < \infty$ for a large class of initial conditions including compact support functions. The fact that $\underline{\lambda}(p) > 0$ shows that there will exist high peaks and, hence, the solution is weakly intermittent. On another hand, the inequality $\overline{\lambda}(p) < \infty$ shows that there will not be any peaks for positions outside $\overline{\lambda}(p)\, t$, hence the position of the farthest peak from the origin grows linearly with time.

In the case where $L = \Delta$ and u_0 has compact support, a careful quantitative study allows us to obtain explicit bounds on $\underline{\lambda}(p)$ and $\overline{\lambda}(p)$ giving estimates on the speed of propagation of the peaks. We conjectured that the upper bound was sharp, which was proved by Chen and Dalang [10] via explicit computations of the moments. Despite the large litterature on intermittency, very little else was known on the position of peaks before, except [32] for the spatially discrete model. Since the publication of this paper, a large class of speed propagation results for different examples have been established by different authors following the type of ideas developed above. We also mention that these results apply as well to the wave equation for which we can obtain exact values for $\underline{\lambda}(p)$ and $\overline{\lambda}(p)$ showing that the farthest peaks travel at the speed of the traveling waves of the deterministic wave equation, see [22].

Measure-valued intial conditions

In [24], we used the stochastic Young-type inequality (17) for a different purpose, thereby illustrating that these inequalities can be used for a wide range of purposes. We proved existence and uniqueness of a weak solution to (10) in the case where the initial condition u_0 is measure-valued. This allows to formally define the solution to (10) when for instance $u_0 = \delta_0$, the Dirac measure.

Together with Mathew Joseph and Davar Khoshnevisan ([20]), we then extended the result to prove existence of a mild solution for the stochastic heat equation with measure-valued initial condition, such as $u_0 = \delta_0$. In [24], we had only proved existence of a generalized-function-valued solution. A careful estimate on moments of the solution for small values of t allows to show that measure-valued initial conditions are turned into real-valued functions for $t > 0$, similarly as in the deterministic case. However, the presence of the noise does not allow to obtain any smoothing property for solutions.

Chaotic behavior of the equation and study of the intermittent islands.

In [18] and [19], together with Mathew Joseph, Davar Khoshnevisan and Shang-Yuan Shiu, we studied the chaotic behavior of (10) driven either by space-time white noise ([18]) or spatially-colored noise ([19]). Indeed, for a fixed time t, we are able to show that if u_0 is bounded away from 0, then $\sup_{x\in\mathbb{R}} |u_t(x)| = \infty$, whereas $|u_t(\cdot)|$ remains bounded if u_0 has compact support. This shows that a modification in the initial condition can lead to a totally different behavior of the solution. This is different from the deterministic case, in which both solutions would remain bounded for finite t, whether the initial condition has compact support or not. The result doesn't require that t is large. Hence, this chaotic behavior appears before the onset of mathematical intermittency. One of the main tools of the paper is a comparison principle for solutions to the stochastic heat equation due to Mueller [40]. The idea is then to show that if we divide the real line in carefully chosen sub-intervals centered in x_j, then $(u_t(x_j), j \in \mathbb{N})$ forms a sequence of random variables approximately independent one from another. More formally, we have to define a coupling that admits the independence property, while remaining close to the actual solution. A large deviation argument allows to conclude our estimate.

As a corollary to the results above, we were able to give explicit estimates on the rate of blow-up of $\sup_{x\in\mathbb{R}} u_t(x)$ in the case where u_0 is bounded away from 0. Namely, in the space-time white-noise case, if we assume that σ satisfies $a \leqslant \sigma(u) \leqslant b$, we have

$$\sup_{x\in[-R,R]} u_t(x) \sim \frac{(\log R)^{1/2}}{\kappa^{1/4}}, \tag{18}$$

but when $\sigma(u) = \lambda u$, then

$$\sup_{x\in[-R,R]} u_t(x) \sim \exp\left(\frac{(\log R)^{2/3}}{\kappa^{1/3}}\right). \tag{19}$$

This gives evidence of the strongly different behavior of the equation under these two assumptions. The main explanation behind these different behaviors comes from the difference in the growth of the moments. However, since we consider any positive time, not only the asymptotic growth of moments as $t \to \infty$ matters, but also how they depend on the order p. Namely, when σ is bounded, $\mathbb{E}[|u_t(x)|^p] \sim \exp(p\log(p)t)$, but when $\sigma(u) = \lambda u$, then $\mathbb{E}[|u_t(x)|^p] \sim \exp(p^3 t)$. The quantitative behavior of the supremum strongly depends on these sharp estimates. Moreover, if we consider $\dot{W}$ to be a colored noise with bounded covariance function f (smooth noise), together with $\sigma(u) = \lambda u$, we can prove that the supremum behaves like

$$\log \sup_{x\in[-R,R]} u_t(x) \sim (\log R)^{1/2},$$

independently of κ (compare with the second estimate above).

Since the parameter κ essentially relates to the time parameter as t^{-1}, (18) suggests a scaling relation of the form $\log(x) \sim t^\gamma$ with $\gamma = 1/2$ in (18) (which suggest a *Gaussian* universality class) and $\gamma = 2/3$ in (19) (which suggest the *KPZ* universality class).

In the case of a spatially-colored noise, similar results are obtained, see [19]. Namely, if $\sigma(u) = \lambda u$, we prove that for a wide variety of noises we have

$$\sup_{x\in[-R,R]} u_t(x) \sim \exp\left(\frac{(\log R)^{\psi}}{\kappa^{2\psi-1}}\right).$$

The exponents relation $\psi \leftrightarrow 2\psi - 1$ is characteristic of the KPZ-universality class as outlined in [6]. For space-time white noise, the result above shows that $\psi = 2/3$. If we consider a smooth noise, we obtain an exponent $\psi = 1/2$ (independence of κ). Moreover, we are able to prove that any exponent between 1/2 and 2/3 can be attained by carefully chosing the correlation of the noise.

In [21], using the techniques used to estimate the supremum above, we can study the correlation-length of the solution $u_t(x)$ above. Namely, we are able to find which minimal distance $|x - y|$ is to be considered in order for $u_t(x)$ and $u_t(y)$ to be approximately independent. From these estimates, we are able to prove estimates on the size of the intermittent islands, namely that intermittent islands in the interval $[-R, R]$ are functions of $\log R$, where the functions are known and depend on the behavior of the function σ.

In an ongoing project, together with Davar Khoshnevisan, we aim at generalizing the results above to spatial dimensions larger than 2, starting with the stochastic Young's inequality (17).

SPDEs driven by fractional noise: a new form of intermittency.

One of the objectives in the study of the intermittency phenomenon is to understand how this phenomenon is impacted by a change in the noise. In order to understand this, a natural family to study are Stochastic Partial Differential Equations driven by a noise that is *fractional in time*. Intuitively speaking, such a noise behaves like a fractional Brownian motion in time with Hurst parameter H. The case $H = 1/2$ corresponds to the case of a noise that is white in time, which we discussed earlier. Fractional Brownian motion is a Gaussian process, but since its increments are not independent, it is not a martingale. In particular, when considered in the settings of SPDEs, the approach of Walsh with martingale measure integration cannot be the proper tool to define a solution for this type of noise. Instead, some Malliavin Calculus techniques have to be used. This change makes the study of intermittency drastically different: most of the techniques described above do not apply to the case of fractional noise and have to be adjusted.

More specifically, we are mainly interested in the following parabolic equation

$$\frac{\partial}{\partial t}u_t(x) = \frac{\kappa}{2}\Delta u_t(x) + \lambda u_t(x)\dot{W}^H_t(x), \qquad (t > 0, x \in \mathbb{R}^d) \tag{20}$$

where $\dot{W}$ is a noise that is fractional in time with Hurst exponent $H > 1/2$ and possibly correlated in space, with correlation function f. Informally, we can write

$$\mathbb{E}[\dot{W}^H(t,x)\dot{W}^H(s,y)] = |t-s|^{2H-2}f(x-y).$$

Notice that we restrict our attention to the case $\sigma(u) = \lambda u$. This allows to obtain an exact Wiener chaos expansion of the solution u as in (5) and is central in the proof of existence.

We will also be interested in the hyperbolic equation

$$\frac{\partial^2}{\partial t^2}u_t(x) = \kappa^2\Delta u_t(x) + \lambda u_t(x)\dot{W}^H_t(x), \qquad (t > 0, x \in \mathbb{R}^d, d \leqslant 3) \tag{21}$$

where $\dot{W}$ is the same noise as above. For simplicity, we will consider in this presentation that the initial condition is $u_0 \equiv 1$ and, for the hyperbolic case, the initial velocity is $v_0 \equiv 0$.

These two equations apply in similar situations as the ones described earlier, but in a setting where it is reasonable to assume some (positive) correlation between the time increments of the noise. We have only looked into the case where $H > 1/2$, where the increments are positively correlated. This noise happens to be smoother than white noise and existence of solutions is well understood.

Similarly as described earlier, we consider a mild-form of the equation, namely

$$u_t(x) = 1 + \lambda \int_0^t \int_{\mathbb{R}^d} \Gamma_{t-s}(y-x)u_s(y)W^H(\delta s\, \delta y), \tag{22}$$

where Γ is the Green function and, now, the integral is considered in the sense of Skorohod. The *Skorohod integral* is defined in the framework of Malliavin Calculus as the adjoint operator of the Malliavin derivative. It is also known as the *divergence operator*. We do not want to get into technicalities of Malliavin Calculus here. One inportant aspect of the Skorohod integral is that it allows to integrate *non-predictable* processes, as is the case of fractional noise. In the case of space-time white noise with adapted integrands, it corresponds to the Walsh integral. As regards the existence, for the heat equation, we consider the approach of [5], and for the wave equation we consider [2].

Feynman-Kac formulas.

In order to establish intermittency for a random field, one needs to carefully study the moment Lyapunov exponents for all orders p, which must be non-trivial. Hence, a first step to establish intermittency is a good understanding of the moments of the solution. In the case of white noise in time, a lot of results about moments can be obtain via a strong use of the martingale property of the noise, for instance using Itô formula (see [14]) or Burkholder's inequality (see [23]).

In [35], Hu and Nualart obtain a Feynman-Kac type representation for the moments of the solution to (20) of the following form

$$E[u_t^p(x)] = E^B\left[\exp\left(H(2H-1)\sum_{i,j=1}^{p}\mathbf{1}_{i\neq j}\int_0^t ds\int_0^t dr\,|s-r|^{2H-2}f(B_r^i-B_s^j)\right)\right], \tag{23}$$

where $(B^{(i)})_{i=1,\dots,k}$ are k independent copies of (standard) Brownian motion and E^B is expectation with respect to Brownian motion (compare with (14)). In principle, this representation should allow one to obtain estimates on the behavior of the moments and obtain their Lyapunov exponents. Such a program based on the actual Feynman-Kac representation for the solution itself (similar to (13)) has been studied simultaneously to our work in [11]. The authors use large deviations arguments and obtain an exact limit behavior of $E[u_t^p(x)]$ as $t\to\infty$ and, thus, the Lyapunov exponents.

However, this approach is not optimal on two aspects: First, it allows to find the Lyapunov exponents, but thus far it did not provide a way to obtain estimates on the moments of order p for finite fixed time t. This has since been expanded and some results provide such estimates (see for instance [12]). Such estimates are necessary if we aim to understand the physical properties, such as the position of peaks, the size of the peaks or the size of the islands. Secondly, as mentioned earlier, using a Feynman-Kac formula is an approach that is very suitable for the stochastic heat equation, but it does not apply to the stochastic wave equation or to equations where $\sigma(u)\neq\lambda u$.

Dalang, Mueller and Tribe came up with a different approach in [29]. They studied (21) with white noise in time. They obtain a Feynman-Kac-type formula to represent the moments, in which not only the space integral, but both space and time integrals are replaced by an expectation with respect to a suitable stochastic process. For the heat equation, Brownian motion is replaced by a process which is piecewise deterministic and changes behavior at Poisson jump times. On each interval where it is deterministic, it behaves as $\sqrt{t}Z$, where Z is a standard normal random variable. Notice that the latter process has $N(0,t)$ marginal distribution, similarly as Brownian motion. Yet, it is not a Markov process. The advantage of this method of representation of moments is that it generalizes to the stochastic wave equation, by simply changing the distribution of the underlying process. For instance, in dimension $d=1$, where $\Gamma_t(x)=c\mathbf{1}_{[-t,t]}(x)$ for some constant c, one considers tU, where U is a uniform random variable on $[-1,1]$.

We do not give a careful statement of this representation here, since it is lengthy and requires a large amount of notation. Such a moment representation has been extended by Raluca Balan [1] to the case of the stochastic heat equation with fractional noise in time, but for the second moment only. In [28], the authors used the representation of moments described above to obtain sharp moment estimates and prove intermittency for the stochastic wave equation with white noise in time and a bounded correlation f in space. They also mention the heat equation. One advantage of this approach (versus the standard Feynman-Kac formula) is that obtaining careful estimates on the behavior of the piecewise deterministic process is much easier than with Brownian motion.

Intermittency and fractional noise

In collaboration with Raluca Balan [3, 4], we used the approach described above in order to develop a study of intermittency for the stochastic heat and wave equations driven by fractional noise. We obtained weak intermittency via an upper bound on the Lyapunov exponents for the moments of order p and a lower bound on the Lyapunov exponent for the moments of order 2, in the case where the inital conditions are constant. We managed to obtain an upper bound, for any fixed time t, on the moments of order p directly using the Wiener-chaos expansion of the solution u (see (5)), as well as the fact that $L^2(\Omega)$ and $L^p(\Omega)$ norms are equivalent on each Wiener chaos. (Notice that this doesn't contradict the fact that they are not equivalent as a whole, since the constants depend on the order of the Wiener chaos.) These results are under more general assumptions than the results of [11] since they do not require the existence of a Feynman-Kac formula for the solution. In the particular case of a noise that is *white in space* ($f = \delta_0$) and fractional in time with Hurst exponent H, we obtain

$$(i)\ E[u_t^p(x)] \leqslant c\exp\left(cp^3 t^{4H-1}\right) \qquad \text{and} \qquad (ii)\ E[u_t^p(x)] \leqslant c\exp\left(cp^{3/2}t^{H+1/2}\right),$$

for (i) the stochastic heat equation (20) and (ii) the stochastic wave equation (21). Notice that the behavior of the heat equation is different than the one of the wave equation, except for $H = 1/2$ (white noise).

Using a more general, but similar, representation of moments as the one of Dalang-Mueller-Tribe [29] (based on the ideas of [1] for the parabolic case), we obtain a corresponding lower bound in the case where $p = 2$. This suggests that the order in t is sharp. The order in p of these estimates matches the results of space-time white noise (see p.10) and they do not depend on H. In a recent paper, Hu, Huang, Nualart and Tindel [34] proved that our exponents are sharp by establishing a lower bound for all order p. Their approach uses a Feynman-Kac representation of the solution and, thus, does not apply to the wave equation. In the latter case, it is still unclear at this time if the exponents are sharp or not.

Notice that since $H > 1/2$, both exponents of the time variable are larger than 1 and so the Lyapunov exponent as defined in (12) would be infinite. Since we would like intermittency to hold for fractional noise too, this shows that the standard mathematical definition of intermittency must be extended in order to cover our situation. This is why we introduce the notion of $\rho-$intermittency: we say that a random field is *weakly* $\rho-$*intermittent* if

$$0 < \gamma_\rho(p) := \limsup_{t\to\infty} \frac{1}{t^\rho} \log \mathbb{E}[|u_t(x)|^p] < \infty \qquad \text{for all } p \geqslant 2, \tag{24}$$

for an appropriate positive parameter ρ. We point out that a $\rho-$intermittent stochastic process also develops very high peaks concentrated on spatial islands, for the same qualititative reasons as with regular Lyapunov exponents. In the case where $\rho > 1$, the peaks would typically be larger than in the case where $\rho = 1$. Indeed, with $H > 1/2$, the correlation of the noise typically makes it remain large for a longer period of time once it becomes large, thus creating higher peaks.

Obtaining a representation of the moments of all order, similarly as in [29] in the case of fractional noise is still being investigated.

A different type of fractional noise.

Recently, together with my Ph.D. student, Mackenzie Wildman, we have been interested in studying SPDEs under a different type of noise, which shares some properties with fractional Brownian motion, but doesn't require as much technical tools. The idea is to find a way to handle some properties similar to fractional Brownian motion, but within the framework of the Walsh stochastic integral. Avoiding the tools of Malliavin Calculus makes it a more approachable type of noise, which can for instance be easily introduced to young Graduate Students or talented undergraduate students.

The motivation for the choice of noise comes from the representation of fractional Brownian Motion as a Riemann-Liouville process. Namely, we consider the process defined by

$$X_H(t) := \int_0^t (t-s)^{H-1/2}\, dB_s,$$

where $(B_s)_{s\geqslant 0}$ is a Brownian motion. The process $(X_H(t))$ is a fractional Brownian motion conditioned to vanish at $t=0$. Here, the fractional Brownian motion is represented as an Itô integral. Still, the dependence on t makes it difficult to handle, since the process that will be generated by replacing X_H in an equation of the form (10) will not be previsible. Yet, informally using a change of variable, we can rewrite X_H in the form

$$X_H(t) = \int_0^t s^{H-1/2} dB_s^{(t)},$$

where $B_s^{(t)}$ is a Brownian motion, running backwards, conditioned to satisfy $B_t = 0$. From there, the difficulty is the same. In order to make the noise tractable in an Itô framework, we define

$$M_H(t) := \int_0^t s^{H-1/2} dB_s,$$

where $(B_s)_{s\geqslant 0}$ is a regular Brownian motion. The process M_H is not a fractional Brownian motion anymore, but it is still Gaussian and shares the same variance. The covariance structure is different, but it has a similar order of magnitude for the moments and similar long-term behavior. However, it doesn't have stationary increments.

Now, the idea is to define a space-time noise M_H based on the process above, using the following representation: for every function φ with appropraite integrability properties, we define

$$\int_0^t \int_{\mathbb{R}} \varphi(s,y) M_H(ds\, dy) := \int_0^t \int_{\mathbb{R}} \varphi(s,y) s^{H-1/2} W(ds\, dy),$$

where the integral is understood in the sense of Walsh and W is a space-time white noise. We used the same notation M_H since the number of variables makes it unambiguous. This generates a noise that is colored in time and white in space. We then study the stochastic heat equation driven by the noise M_H. Namely,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sigma(u)\dot{M}_H, \tag{25}$$

where $t > 0$, $x \in \mathbb{R}$, under some nice initial conditions. Notice that for $H = 1/2$, this simply comes back to the equation driven by space-time white noise.

So far, we have been able to establish existence and uniqueness of a mild solution to (25) under the assumption that $H > 1/4$. Moreover, we have studied Hölder continuity of the solution both in space and time. We proved that the solution is Hölder continuous of order not larger than α in time and 2α in space, where α is given as a function of H by

$$\alpha = \begin{cases} 1/4 & \text{if } H \geqslant 1/2, \\ H - 1/4 & \text{if } 1/2 > H > 1/4. \end{cases}$$

One interesting observation is that making the noise smoother in time, by choosing $H > 1/2$ does not improve the order of continuity of the solution. This is consistent with the results observed in [34].

We aim at continuing to study the solution to (25) and its properties, in particular as regards intermittency.

An application to Mathematical Finance.

The stochastic process $(M_H(t))_{t\geqslant 0}$ defined in the previous section was inspired by some work done, together with Mackenzie Wildman, in the framework of Mathematical Finance, more precisely in option pricing theory. This work was inspired by the late Vladimir Dobric. He introduced, together with Francisco Ojeda [30] a diffusion process that aims at approximating fractional Brownian motion. Namely, they proposed to use the process defined by the stochastic differential equation

$$dV_t^H = t^{H-1/2}\, dB_t + (2H-1)\frac{V_t^H}{t}\, dt,$$

with $V_0^H = 0$. One can easily prove that another way to define this process is given by

$$V_t^H = t^{2H-1} \int_0^t s^{1/2-H}\, dB_s.$$

In [30], the authors provide a different definition based on fractional Brownian motion, which explains the relationship between this process and fraction Brownian motion. A time-change argument allows to show that the definition above is satisfied as well. This is the one we keep in this description.

The objective of using this process is to be able to obtain a model of fractional-type for the underlying asset of a financial derivative security, while keeping the ability to work in the framework of Itô integrals. One of the disadvantages of using fractional Brownian motion to model financial markets is that it doesn't allow to build a risk-neutral measure using standard approaches. The purpose of this work is to use the process V^H in lieu of Brownian motion in the Black-Scholes option pricing theory. Namely, one considers a financial asset modeled by

$$dS_t = S_t \left(\mu\, dt + \sigma\, dV_t^H\right).$$

From there, we follow a standard approach to try and obtain an option pricing formula for a call option. However, it turns out that Novikov's condition is not satisfied and, thus, we are not able to directly define a risk-neutral measure as in Black-Scholes theory. The issue is the non-integrability of the drift around $t = 0$. In order to remedy to this, we introduce the process defined by

$$dV_t^{H,\varepsilon} = t^{H-1/2}\, dB_t + (2H-1)\frac{V_t^H}{t}\mathbf{1}_{[\varepsilon,\infty)}(t)\, dt.$$

Under the process $V^{H,\varepsilon}$, we can establish the existence of a risk-neutral measure Q^ε and establish an option pricing formula for a call option provided the parameter ε is not too small. The only dependence on ε in the pricing formula is on the initial value S_0^ε. Thus, in practice, the choice of ε doesn't impact the pricing algorithm.

We developed some parameter estimation method based on the quadratic variation of the process V^H, which is independent of ε. Some numerical results are illustrated and, in some instances, happen to improve the pricing results provided by the standard Black-Scholes formula. Convergence of the sequence of measures $(Q^\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0$ is an open problem at this time.

Stochastic Geometry and SPDEs

This project started together with Joe Yukich and Pierre Calka aims at identifying connections between problems from stochastic geometry and SPDEs. We consider the question of the behavior of the convex hull of a set of random points in a given domain. For instance, one can consider uniformly distributed points in a ball or points distributed in the whole plane according to a Gaussian measure. The number of such points follows a Poisson process of rate λ. Properties of the convex hull, such as asymptotic behavior and central limit theorems (as $\lambda \to \infty$) have been studied by Joe Yukich and Pierre Calka. Their results prove that the asymptotic behavior of the convex hull (as $\lambda \to \infty$) is similar to the asymptotic behavior of the solution (as a suitable scaling parameter increases) to a Burgers'equation with random initial condition, namely

$$\text{Burgers' equation:} \qquad \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0$$

(see [39]). Moreover, space-time scaling exponents behave similarly as the KPZ scaling exponents mentioned earlier. Altogether, this led us to believe that there might be a deeper connection between this model of stochastic geometry and certain SPDEs, most likely PDEs with random initial conditions, which are similar to SPDEs with additive noise. It is proved that a certain curve, characteristic of the convex hull of the set of random points satisfies a differential equation similar to Burgers' equation. Namely, the equation exactly corresponds to Burgers' in the first-order approximation. One of the difficulty in the identification between the two problems is that one of them has a natural time parameter (Burgers' equation) and the other one doesn't. Thus, the question of finding the correct time parameter and its physics interpretation is the center of our current research on this question. Such a connection between this model and the KPZ equation could add this class of stochastic geometry problems to the big picture showing the KPZ universality class as a relevant model in several other areas of Probability such as SPDEs, random matrices, interacting partical systems, etc.

This project is ongoing and we cannot claim any publication yet.

Interdisciplinary contributions.

The paper [45] is a contribution that I made to a project in Signal Processing from a group of colleagues at Lehigh who needed some large deviations arguments in order to establish some of their results. I contributed as an external role to help them with concluding this specific proof.

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Note

All my research papers are available online from Math Reviews, together with a list of citations via the Math Reviews database on

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`www.lehigh.edu/~dac311`

or my Google Scholar webpage:

`https://scholar.google.com/citations?user=IMsGqpQAAAAJ&hl=en&oi=ao`

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