Analysis of Consensus Networks Driven by Symmetric-\(\alpha\)-Stable Motion

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Abstract—This manuscript discusses consensus seeking networks in the presence of non-Gaussian stochastic noise. We explore the fundamental principles of the solutions of the systems and we define performance measures similar to the ones used in the case of Gaussian stochastic noise. We outline the technical difficulties for the exact calculation of these measures and we propose estimates. It is argued that the conventional design tools in optimal network synthesis become obsolete, when the stochastic source of perturbation is not Gaussian.

I. INTRODUCTION

The interest in performance and robustness of large scale dynamical networks is a rapidly growing field. Developing tools that analyze the reaction of networked control systems towards external disturbances is crucial for sustainability, from engineering infrastructures to living cells; examples include a group of autonomous vehicles in transportation networks, energy and power networks, metabolic pathways and even financial networks (see for example [1], [5] and references therein).

Researchers approach a system that suffers from exogenous disturbances, by considering mathematical models that involve some source of Gaussian stochastic noise. The resulting dynamics are assumed to reflect the manner with which the nominal system behaves towards uncertainties. The advantage in modeling uncertainty with Brownian motion relies on the elegant theory of stochastic differential equations [2]. Engineers then leverage these powerful results in optimal network synthesis problems that minimize the effect of noise [3], [10], [15], [16].

A. Criticism of the Gaussian Hypothesis

However elegant their analysis may be, systems driven by Gaussian noise are susceptible to considerable criticism on the grounds of being incapable of modeling significant real-world uncertainties, such as large disruptions or shocks. The light-tailed property of the associated normal distributions allows “rare events” to occur either with exponentially small probability or too often. As explained in [18], a shock event in a system should have the following three attributes:

- Lie outside the realm of regular expectation.
- Carry an extreme impact.
- Have likelihood of happening.

This point is illustrated in Figure 1 where we depict output observable fluctuations of our study model (8), around the expected value. The fluctuations are facilitated through an additive source of noise that perturbs the network. When this source generates Brownian motion, the realized process fluctuates amorously around the origin. When the source generates Lévy stable noise of non-Gaussian type, the dynamics get more impulsive. They resemble the reaction of a nominal system towards strong, rare, but not improbable perturbations.

Indeed, \(\alpha\)-stable models (see definitions in §II) are more flexible in modeling such realistic types of disturbances and shocks, at least along the lines of the aforementioned attributes. Systems with such stochastic disturbances, have been studied in other scientific disciplines [7], [13]. To the best of our knowledge, the control community lacks similar efforts, perhaps with the exception of [8].

B. Contribution

Linear consensus networks regard a finite number of interconnected agents that seek to agree on a state of interest. These types of dynamical systems are a simple yet rich benchmark to study the effect general Lévy noise in multi-agent control systems. The mathematical model is introduced in (8) and from now we will refer to it as \(\alpha\)-stable consensus network. The source of uncertainty is an additive noise, parametrized by the stability index \(\alpha \in (0, 2]\). The \(\alpha\)-stable consensus networks constitute a family of stochastic dynamic networks that include networks disturbed by Gaussian
(white) noise as a special case (i.e. $\alpha = 2$). We investigate the model in its fundamental properties and we establish sufficient conditions for the well-posedness of its solutions. Next, we present measures of performance for $\alpha$-stable consensus networks as quantifiers of the cumulative dispersion of the agents' states around the nominal value. We report the few cases of exact calculation of these measures and we establish estimates for the general case. These estimates are explicit functions of the network topology. We numerically validate our estimates and characterize the type of graphs in which the bounds approximate better the actual values. Finally, we discuss elementary network design problems arguing that network synthesis techniques are critically affected by the type of additive noise.

The results of this paper are Propositions 2.1, 5.2 and Theorems 5.3, 6.1 and 6.2. Their proofs are omitted due to space limitations.

II. PRELIMINARIES

Notation: By $\mathbb{R}^n$ we denote the $n$-dimensional Euclidean space, every element $x \in \mathbb{R}^n$ of which is considered as a column vector. The $p$-norm of $x = (x^{(1)}, \ldots, x^{(n)})^T \in \mathbb{R}^n$, is $\|x\|_p = \sqrt[p]{\sum_{j=1}^n |x_j|^p}$.

Elements of Stable Random Variables: A random variable $z$ follows a stable distribution denoted as

$$z \sim S_\alpha(\sigma, \beta, \mu),$$

if there exist parameters $0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$ and $\mu \in \mathbb{R}$, such that its characteristic function is of the form:

$$\phi_z(\theta) = \mathbb{E}[e^{i\theta z}] = \exp\{\sigma^{\alpha}\left(-|\theta|^\alpha + i\theta \omega(\theta, \alpha, \beta)\right) + i\mu \theta\}$$

where $\omega(\theta, \alpha, \beta)$ stands for the function

$$\omega(\theta, \alpha, \beta) = \begin{cases} \frac{\beta |\theta|^{\alpha - 1}\tan \frac{\pi \alpha}{2}}{\pi}, & \alpha \neq 1 \\ -\beta \frac{2}{\pi} \ln |\theta|, & \alpha = 1. \end{cases}$$

The parameter $\alpha$ is the stability of the distribution of $z$ and it basically characterizes its impulsiveness and magnitude. The parameter $\sigma$ is the scale of the distribution and it is closely related to the standard deviation: The larger the scale parameter is, the more spread out the distribution becomes. The parameter $\beta$ is the skewness of the distribution, an indicator of asymmetry, and $\mu$ is the shift parameter that plays the role of the mean value. The next result summarizes basic properties of stable random variables.

**Proposition 2.1:** Let $z \sim S_\alpha(\sigma, \beta, \mu)$. The following properties hold.

1. For any $a \in \mathbb{R}$, $z + a \sim S_\alpha(\sigma, \beta, \mu + a)$.
2. For any $a \neq 0$, $az \sim \begin{cases} S_\alpha(\|a\|\sigma, \text{sgn}(a)\beta, \alpha \mu - 2a \ln(|a|)\sigma \beta), & \alpha \neq 1 \\ S_\alpha(|a|\sigma, \text{sgn}(a)\beta, \alpha \mu - 2a \ln(|a|)\sigma \beta), & \alpha = 1 \end{cases}$

3. If $\alpha < 2$,

$$\mathbb{E}[|z|^p] = \begin{cases} < \infty, & \text{for } 0 < p < \alpha \\ = \infty, & \text{for } p \geq \alpha \end{cases}$$

In addition, if $\mu = 0$, and $\beta = 0$ only if $\alpha = 1$, it holds

$$\mathbb{E}[|z|^p] = c^p \sigma^p,$$

where $c = c(\alpha, \beta, \mu) = \left(\mathbb{E}[|z_0|^p]\right)^{\frac{1}{p}}$ for $z_0 \sim S_\alpha(1, \beta, 0)$. The normal distribution is a special case of stable distribution for $\alpha = 2$, with mean value $\mu$ and standard deviation $\sqrt{2}\sigma$. Stable distributions have in general infinite variance if $\alpha < 2$, and infinite mean if $\alpha \leq 1$.

**Proposition 2.2:** Let $z_i \sim S_\alpha(\sigma, \beta, \mu), i = 1, 2$ be independent. Then

$$z_1 + z_2 \sim S_\alpha(\sigma_1^2 + \sigma_2^2, \beta, 2\mu).$$

If $\beta = \mu = 0$, then $z \sim S_\alpha(\sigma, 0, 0)$ is called symmetric $\alpha$-stable, for which we write $z \sim S\alpha S$. Its characteristic function takes the particularly simple form

$$\phi_z(\theta) = e^{-|\theta|^\alpha}.\tag{1}$$

A finite collection of $\alpha$-stable random variables $z^{(i)} \sim S_\alpha(\sigma, \beta, \mu_i), i = 1, \ldots, d$ can form an $\alpha$-stable vector $z = (z^{(1)}, \ldots, z^{(d)})^T$.

A stochastic process $z = \{z_t, t \geq 0\}$ is stable if all its finite dimensional distributions are stable. An $\alpha$-stable Lévy stochastic process attains the properties:

1) $z_0 = 0$ almost surely
2) $z$ attains independent increments
3) $z_t - z_s \sim S_\alpha((t - s)^{1/\alpha}, \beta, 0)$ for $0 \leq s < t < \infty$.

A vector-valued $\alpha$-stable process $z_t = (z_t^{(1)}, \ldots, z_t^{(d)})^T$, is a family of $\alpha$ stable vectors parametrized by $t \geq 0$.

**Stable Integrals.** Let $(\Omega, \mathcal{F}, P)$ be a probability space and $L^0(\Omega)$ the set of all real random variables defined on it. Let also $(B, \mathcal{B}, m)$ be a measurable space. Take $\beta : B \rightarrow [-1, 1]$ a measurable function and $B_0 \subset B$ that contains sets of finite $m$-measure.

**Definition 2.3:** A $\sigma$-additive set function $M : B_0 \rightarrow L^0(\Omega)$ is a random measure with the properties

1) It is an independently scattered function on sets
2) For each $A \in \mathcal{B}$,

$$M(A) \sim S_\alpha\left(\left(\mathbb{E}[M(A)]\right)^{1/\alpha}, \int_B \beta(y) m(dy) \right).$$

For the remainder of this paper, we will focus on un-skewed distributions, i.e. we will set the skewness density $\beta(x) \equiv 0$.

**Example 2.4:** Let $M$ be an $\alpha$-stable random measure on $([0, \infty), B)$ with $m(dx) = \frac{1}{\alpha} dx$ and skewness density $\beta(x) \equiv 0, 0 \leq x < \infty$. The process $Z = \{Z_t, t \geq 0\}$ defined through $Z_t = M([0, t]), 0 \leq t < \infty$ is a $\alpha S\alpha$ Lévy motion.

The stable integral defined as

$$I(f) := \int_B f(y) M(dy)$$

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are taken over integrands that are members of
\[F_\alpha = \left\{ f \in B : \int_B |f(y)|^\alpha m(dy) < \infty \right\}. \quad (1)\]

**Proposition 2.5:** The integral \( I(f) \) attains the properties:
[a.] \( I(f) \sim S_\alpha(\sigma, 0, 0) \) with \( \sigma^2 = \int_B |f(x)|^\alpha m(dx) \).
[b.] \( I(a_1 f_1 + a_2 f_2) = a_1 I(f_1) + a_2 I(f_2) \), for \( f_1, f_2 \in F_\alpha \), and \( a_1, a_2 \in \mathbb{R} \).

**Example 2.6:** Let the \( \alpha \)-stable random measure \( M \) of Example 2.4. Then \( f(x) = e^{-\lambda x}, \lambda > 0 \), clearly belongs to \( F_\alpha \), for any \( \alpha \in (0, 2] \). The integral
\[\int_0^t e^{-\lambda(t-s)} M(ds) \sim S_\alpha(\sigma, 0, 0)\]
is \( S_\alpha S \), with \( \sigma^\alpha = \frac{1-e^{-\lambda \xi^\alpha}}{\alpha \lambda} \).

**Algebraic Graph Theory:** The vector of all ones is denoted by \( \mathbf{1} \) and the \( n \times n \) centering matrix is shown by
\[M_n := I_n - \frac{1}{n} \mathbf{1}\mathbf{1}^T. \quad (2)\]

An undirected weighted graph \( G \) is denoted by the triple \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, w) \), where \( \mathcal{V} \) is the set of nodes of \( \mathcal{G} \), \( \mathcal{E} \) is the set of links of the graph, and \( w : \mathcal{E} \to \mathbb{R}_+ \) is the weight function that maps links between nodes \( i \) and \( j \) to a non-negative scalar \( a_{ij} \). The matrix \( L = [l_{ij}] \) with
\[l_{ij} = \begin{cases} -a_{ij}, & i \neq j \\ \sum_{j=1}^n a_{ij}, & i = j \end{cases}\]
is called the Laplacian matrix of \( \mathcal{G} \).

**Assumption 2.7:** The coupling graphs of the networks considered in this paper are simple, undirected, and connected.

A direct consequence of Assumption 2.7 is that \( L \) can be represented as \( L = QAQ^T \), where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \) with \( \lambda_i \in \mathbb{R} \) placed in ascending order, \( Q = [q_1 | \ldots | q_n] \) is a matrix the \( i \)-th column of which is corresponds to the eigenvector associated with the eigenvalue \( \lambda_i \) of \( L \). Also \( q_i \) are chosen to satisfy
\[q_i^T q_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}\]

Finally we note that \( \lambda_1 = 0 \) with \( q_1 = \frac{1}{\sqrt{n}} \mathbf{1} \). For given \( L = QAQ^T \) we define the spectral functions:
\[f_{ij}(t) = \sum_{k=2}^n q_{ik} q_{jk} e^{-\lambda_k t} \quad (3)\]
\[g(t) = \sum_{k=2}^n e^{-\lambda_k t} \quad (4)\]
\[G_\alpha = \int_0^\infty g^\alpha(s) ds \quad (5)\]

Note that \( f_{ij} \) clearly belong to \( F_\alpha \). Also, \( q_{ij} \in [0, 1] \) implies

**III. PROBLEM STATEMENT**

Let a network \( n < \infty \) autonomous agents, each of which is represented through a real-valued state \( x^{(i)} \) for \( i = 1, \ldots, n \). The state vector of our multi-agent system at time \( t \) is \( x_t = (x_t^{(1)}, \ldots, x_t^{(n)})^T \). The dynamic system that governs the rate of change of \( x \) is:
\[dx_t = -L x_t dt + dz_t, \quad t > 0 \quad (8)\]
where \( L \) is the graph Laplacian matrix that satisfies Assumption 2.7. The agents execute a consensus algorithm on a network with symmetric couplings. The consensus process is perturbed by \( n \) sources of noise that are powered by \( dz_t = M(dt) \), a multi-dimensional stable measure.

**Assumption 3.1:** The vector term \( dz_t = M(dt) = (M_1(dt), \ldots, M_n(dt))^T \) is a collection of \( n \) independent random measures on \((0, \infty), B([0, \infty]), | \cdot | \), and
\[M_i(t-s) \sim S_\alpha(|t-s|^{1/\alpha}, 0, 0), \quad \text{for } i = 1, \ldots, n \]
is a random measure as in Example 2.4.
The initial condition in (8) \( x_0 = (x_0^{(1)}, \ldots, x_0^{(n)})^T \), is a fixed vector chosen independently of measure \( dz_t \). System (8) is the differential form of an Ornstein-Uhlenbeck process, the integral representation of which is

\[
x_t = e^{-Lt}x_0 + \int_0^t e^{-L(t-s)} dz_s
\]

(9)

Processes of this type have been studied in the past (see for example [12] and [14]) for \( dz_s \), a general stable measure and \(-L\) being Hurwitz (i.e. \( \lim_{t \to \infty} e^{-Lt} \) is the zero matrix).

IV. THE OUTPUT STATISTICS IN \( \alpha \)-STABLE CONSENSUS NETWORKS

Unlike the models discussed in [12] and [14], \(-L\) in (8) is not Hurwitz. On the other hand, the main interest in the study of consensus dynamics regards output observables that is not Hurwitz. On the other hand, the main interest in the study of agents’ state from each other (i.e. \( x^{(i)} - x^{(j)} \)), or deviation of agents’ position and the network average (i.e. \( x^{(i)} - \frac{1}{n} \sum_{j=1}^n x^{(j)} \)). For the latter case, we obtain the linear mapping

\[
y = M_n x
\]

(10)

where \( M_n \) is the centering matrix from (2). Applying this transformation to (8), sets the marginal eigenvalue, unobservable and makes the state transition matrix \( e^{-Lt} \) asymptotically stable. Realization of the first element of \( y_t \) is depicted in Figure 1. Moreover, the stochastic process \( y = \{y_t, t \geq 0\} \) enjoys a number of remarkable properties.

**Proposition 4.1:** Under Assumptions 2.7 and 3.1, the process \( y = \{y_t, t \geq 0\} \) in (10) generated by \( x = \{x_t, t \geq 0\} \) to be the realization of (9), satisfies:

\[
y_t = Q \Phi(t) Q^T y_0 + \int_0^t Q \Phi(t-s) Q^T dz_s,
\]

(11)

where

\[
\Phi(t) = \text{Diag}[0, e^{-\lambda_2 t}, \ldots, e^{-\lambda_n t}]
\]

and \( \{\lambda_i\}_{i=2}^n \) are the eigenvalues of \( L \). Furthermore, \( y_t \) is a stable vector, with \( y_t^{(i)} \) a stable random variable with \( t \)-dependent distribution parameters. Also, \( \overline{y}(t) := \lim_{t \to \infty} y_t^{(i)} \) is a stable random variable with characteristic function

\[
\phi_{\overline{y}(t)}(\theta) = \exp \{-\sigma_j^\alpha |\theta|^\alpha \}
\]

where \( \sigma_j^\alpha = \sum_{i,j} \sigma_{ij}^\alpha \) with

\[
\sigma_{ij}^\alpha = \frac{1}{\alpha} \int_0^\infty |f_{ij}(t)|^\alpha dt.
\]

The result explains that \( y_t \) follows a stable distribution for all \( t \). In addition \( \overline{y} := \lim_{t \to \infty} y_t \) converges in distribution to an \( \alpha \)-stable random vector with stable parameters that depend exclusively on the network topology.

V. PERFORMANCE METRICS

When \( \alpha = 2 \), both \( x = \{x_t, t \geq 0\} \) and \( y = \{y_t, t \geq 0\} \) are Gaussian processes. We recall that \( x \) and \( y \) can be completely determined from their first and second moments, both of which are well-defined. As Proposition 2.1 explains, this does not hold when \( \alpha < 2 \). In this section we will introduce a family of metrics that quantify the performance of the output observable of (8), in terms of the scale at which the distribution of the variable is dispersed around its nominal value (here zero).

**Definition 5.1:** The cumulative scale of an \( \alpha \)-stable vector \( y = (y^{(1)}, \ldots, y^{(m)})^T \), is defined to be

\[
\Sigma_\alpha(y) = \parallel \sigma \parallel_\alpha^\alpha = \sum_{i=1}^m \sigma_i^\alpha
\]

where \( \sigma = (\sigma_1, \ldots, \sigma_m) \), \( \sigma_i \) is the scale of \( y^{(i)} \).

For \( \overline{y} \), the long term output vector of (11), \( \Sigma_\alpha := \Sigma_\alpha(\overline{y}) \) can be trivially expressed in terms of the marginal scale parameters (12),

\[
\Sigma_\alpha = \frac{1}{\alpha} \sum_{i=1}^n \sum_{j=1}^n \int_0^\infty |f_{ij}(t)|^\alpha dt
\]

(13)

In other words, \( \Sigma_\alpha \) is a measure of dispersion of networks’ fluctuation around the moving average. The larger the \( \Sigma_\alpha \), the more impulsive and magnified the fluctuation of the agents around the moving average is. The functions \( f_{ij}, i,j \in \mathcal{V} \) as in (3), represent the network contribution in the form of the steady-state distribution of \( \overline{y} \). The next result asserts, that \( \Sigma_\alpha \) is monotonically decreasing with \( \alpha \).

**Proposition 5.2:** Assume the network dynamics of (8) with the output process (10). Then

\[
\frac{\partial}{\partial \alpha} \Sigma_\alpha < 0.
\]

In other words, the more impulsive the noise, the more fragile the consensus network. Now, for \( \alpha = 2 \), Assumption 2.7 and Property 2.1 yield

\[
\Sigma_2 = \frac{1}{2} \sum_{i,j} \int_0^\infty f_{ij}(t) dt = \frac{1}{2} \sum_{k=2}^n \frac{1}{2\lambda_k} = \frac{1}{2} \mathbb{E}[\|\overline{y}\|_2^2],
\]

(14)

where \( \lambda_k \) are the eigenvalues of \( L \), and the last step is in view of Property 3 of Proposition 2.1. In other words \( \Sigma_2 \) at \( \alpha = 2 \) is intimately related to the cumulative variance of the output \( \overline{y} \) of system 8, i.e. the \( \mathcal{H}_2 \)-norm of the consensus network; a central measure of performance in stochastically driven dynamical systems [15]. The Gaussian case is unique in its kind, in the sense that leads to a closed form expression of (13). No other value of \( \alpha \) offers this elegance, at least for arbitrary graph topologies.

**Theorem 5.3:** Let the dynamical network (8) with Assumptions 2.7 and 3.1 to hold. Consider the process \( y = \{y_t, t \geq 0\} \) as in (11), and the cumulative scale \( \Sigma_\alpha \). If for the spectrum of \( L \) it holds that \( \lambda_2 = \lambda_n \), then

\[
\Sigma_\alpha = \frac{(n-1)(1+(n-1)^{\alpha-1})}{\alpha \sum_{i=1}^n \lambda_i^{\alpha-1}}
\]

where \( \alpha \in (0,2] \) and \( \lambda := \lambda_2 = \lambda_3 = \cdots = \lambda_n > 0 \).

A typical example of a graph with identical non-zero laplacian eigenvalues is the complete graph with uniform weights.
VI. SPECTRAL BASED ESTIMATES OF $\Sigma_\alpha$

The form of stable integrals in (12) is indicative of the extent to which it can be calculated exactly in closed form. For arbitrary network topologies one should rely on estimates and bounds of the network scale $\Sigma_\alpha(\gamma)$. The purpose of this section is to elaborate on (13) and establish upper bounds of $\Sigma_\alpha$. It is desirable to express the estimates as explicit functions of the eigenstructure of $\alpha$ with $\Sigma$.

Theorem 6.1: Assume the $\alpha$-stable consensus network (8) with $\alpha \in (0, 2]$. The following estimates on $\Sigma_\alpha = \Sigma_\alpha(\gamma)$ hold:

If $\alpha \in (0, 1]$,

$$\Sigma_\alpha \leq \frac{1}{\alpha(n-1)^{\alpha}} \sum_{k=2}^{n} ||q_k||_{2,\alpha}^2 \Lambda_{\alpha,1}^{(k)} + \frac{1+n}{\alpha^{\alpha-1}} G_\alpha.$$

If $\alpha \in [1, 2]$,

$$\Sigma_\alpha \leq \min \left\{ d_1 \Lambda_{\alpha,1}^{n-1} \sum_{k=2}^{n} ||q_k||_{2,\alpha}^2 \Lambda_{\alpha,1}^{(k)} + d_2 G_\alpha, \right.$$

$$\left. d_3 \Lambda_{\alpha,1}^{n-1} \sum_{k=2}^{n} ||q_k||_{2,\alpha}^2 \Lambda_{\alpha,1}^{(k)} + d_4 G_\alpha \right\}$$

for $d_1 = \frac{2^{n-1}}{\alpha(n-1)^{\alpha}}$, $d_2 = \frac{2^{n-1} - n - 1}{(1+(n-1)^{1-\alpha})(1+n^{\alpha-1})}$, $d_3 = \frac{1}{\alpha(n-1)^{\alpha}}$, $d_4 = \frac{1}{\alpha^{\alpha-1}(n-1)^{\alpha-1}}$.

Theorem 6.1 establishes upper bounds of $\Sigma_\alpha$ for all $\alpha \in (0, 2]$. The main point that is worth mentioning here is the distinction between noise sources with finite first moments, (i.e. $\alpha \in [1, 2]$) and noise sources with undefined first moment (i.e. $\alpha \in (0, 1]$).

In either case, the bounds are form from two terms: The first term equals the weighted sum of the $\alpha$-norm of the $n-1$ eigenvectors of $L$. The weight of the $k^{th}$ term in this sum is an eigenvalue-based function that essentially measures the deviation of the $k^{th}$ eigenvalue with respect to the rest $n-2$. The second term effectively involves the sum of the inverse non-zero eigenvalues of $L$ that is expressed in integral form.

As $\lambda_2 \uparrow \lambda_n$, $\Lambda_{\alpha,1} \downarrow 0$ and the estimates of Theorem 6.1 coincide with the exact value of $\Sigma_\alpha$ in Theorem 5.3. However, for $\alpha = 2$, the estimates in Theorem 6.1 do not match with the value in (14). This non-negligible discrepancy motivates the additional upper bound of $\Sigma_\alpha$. The following result proposes an estimate of $\Sigma_\alpha$, via a harmless perturbation of the scale parameter of the Gaussian case.

Theorem 6.2: Assume the $\alpha$-stable consensus network (8) with $\alpha \in (1, 2]$. Then

$$\Sigma_\alpha \leq \frac{1}{\alpha} \sum_{k=2}^{n} \frac{1}{2\lambda_k} + \frac{1}{\alpha} \int_{0}^{2} \int_{0}^{\infty} n \cdot w(s) \cdot \ln g(s) \cdot dsdw,$$

where $g(t)$ is as in (4).

This result, although general, is not expected to provide sharp estimates for values of $\alpha$ away from 2.

The spectral bounds of Theorems 6.1 and 6.2, could be leveraged when developing optimal design algorithms, that reform the coupling scheme to withstand the effect of the imposed noise. In order to verify the qualification of the estimates we must validate their efficiency on different network topologies.

VII. NUMERICAL EXPLORATIONS

In this section we present four examples related to (11). The first three, regard elementary network design problems. Their objective is to demonstrate that all basic design strategies (addition/removal of links and re-weighting) are critically affected by the stability parameter, $\alpha$, of the input noise. The fourth example is a validation of the estimates in Theorems 6.1 and 6.2.

Example 7.1: [Design via Expansion] Let an $\alpha$-stable network over $n = 6$ agents, the graph of which is illustrated as $G_1$ in Figure 2. The couplings are of unit magnitude. We have the option to add a new unit-weight link to the network so as to improve its performance. We look for the edge location, that upon connection the new $\Sigma_\alpha$ is minimized. Numerical explorations reveal that the optimal selection is a function of $\alpha$. For $\alpha = 2$ to $\alpha = 1.6655$ the optimal location is a link between nodes 1 and 4 (blue dotted curve). From $\alpha = 1.6655$ to $\alpha = 0.3312$ there appear to be two equivalent alternatives: one is the pair (1,3) and the other is (3,4) (red dashed curves). For stability values below 0.3313 the optimal pair is (1,3).

Example 7.2: [Design via Sparsification] et an $\alpha$-stable network over $n = 10$ agents. The graph is the $G_2$ one in Figure 2. The problem here is to select the one link of the network that, upon removal, increases $\Sigma_\alpha$, the least. Our findings suggest that within the stability range $\alpha = 2$ to $\alpha = 1.8932$, the optimal pair is (2,4) (removal blue dashed curve). From $\alpha = 1.8937$ to $\alpha = 0.1971$ the optimal pair is (8,10) (removal of the red dashed curve). Finally, for $\alpha < 0.1971$ the optimal pair appears to be (2,6) (removal of the green dashed curve).

Example 7.3: [Design via Re-weighting] In this last example, we regard a small network of 4 agents, illustrated as $G_3$ in Figure 2. All but links between nodes (2,3) and (2,5) are fixed and of unit weight. On the other hand, the edges $a_{23}$ and $a_{25}$ are assumed to satisfy $a_{23} = 2 - b$, $a_{25} = b$ for some $b \in (0, 2)$. In other words, keeping the overall network budget constant and equal to $a_{12} + a_{23} + a_{25} + a_{45} = 4$ we seek to calibrate the control parameter $b$ towards the value that minimizes $\Sigma_\alpha$. The simulations are illustrated in Figure 3 where we highlight the dependence of the optimal calibration (the black dots) as a function of $\alpha$.

All three design examples lead to a definitive conclusion: Performance evaluation tools are intimately associated with the particular type of stochastic uncertainty. We conclude this section with a numerical example one the accuracy of the spectral estimates in Theorems 6.1 and 6.2.
Fig. 3: The cumulative scale parameter $\Sigma_\alpha$ as function of the control $b$, in Example 7.3. The different curves correspond to stable noises of various stability parameters. The lowest curve is this of the Gaussian case, $\alpha = 2$. The $\Sigma_\alpha$ curves increase monotonically as $\alpha$ varies from 2 to 0, verifying Proposition 5.2. The sequence of black dots signify the global minimum in each type of noise.

Fig. 4: Simulation of Example 7.4. Graphs with smaller $\lambda_2/\lambda_1$ ratio provide scale estimates closer to the actual value. Estimate 1, regards $\Sigma_\alpha$ with $\alpha \in (0,1]$. Estimate 2, regards $\Sigma_\alpha$ with $\alpha \in [1,2]$. Estimate 3, regards $\Sigma_\alpha$ with $\alpha \in [1,2]$ as in Theorem 6.2.

Example 7.4: We test the scale estimates of Theorems 6.1 and 6.2. We choose two graphs. The first graph has a significantly larger eigenvalue ratio than the second one. The curves are depicted in Figure 3 and are compared with the exact value. Two remarks are due at this point. Firstly, the estimates perform better in graphs with ratio $\lambda_2/\lambda_1$ close to 1. Secondly, as the noise distribution becomes more and more impulsive (smaller values of $\alpha$) the estimates becomes less efficient.

VIII. DISCUSSION

We considered the standard paradigm in networked control systems in the presence of multiple independent $\alpha$-stable sources. We defined measures of performance that quantify network fragility in terms of the scale parameter of distributions that are proven to be explicit functions of the network parameters. We saw that, in general, the exact calculation of these performance measures is quite challenging.

Our spectral estimates of the measures are shown to perform quite well for types of networks satisfying the property $\lambda_1/\lambda_2 \approx 1$. In addition to the complete graph connectivity (where $\lambda_1/\lambda_2 = 1$) another type of graphs that satisfy a ratio $\lambda_1/\lambda_2$ close to 1 is this of expander graphs [17].

In the network design examples, we demonstrated that any optimal synthesis strategy must take into account the type of infused noise. Our conclusion is that network synthesis in $\alpha$-stable network, based on conventional tools results in sub-optimal choices.

Our results, both theoretical and numerical, anticipate to set the ground for a new perspective in the field. All in all, performance analysis and synthesis of general $\alpha$-stable consensus networks calls for a new theoretical framework and design tools.

REFERENCES