Math 205, Summer II, 2016

Week 3a:

Chapter 4, Sections 2, 3 and 4

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4.1, 4.2: $\mathbf{R}^n$ and Vector Spaces

4.3 Subspaces/Nullspace

4.4 Linear Combinations, Spanning

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1. Vector addition; scalar multiplication

in $\mathbf{R}^2$. Vectors are $\vec{x} = (x, y)$, with x, y real numbers.

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.

parallelogram law:

$(2, 1) + (1, 4) = (3, 5)$ (picture!)

$k(x, y) = (kx, ky)$, scaling factor

$3(2, 1) = (6, 3)$; $-\frac{1}{2}(2, 1) = (-1, -\frac{1}{2})$. (sketch)

zero vector: $\vec{0} = (0, 0)$. additive inverse: $-\vec{x} = -(x, y) = (-x, -y)$.

distributive rules (2)

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standard unit vectors: $\vec{i} = (1, 0)$, $\vec{j} = (0, 1)$.

Linear combination property: $\vec{x} = (x, y) = (x, 0) + (0, y) = x\vec{i} + y\vec{j}$.

$\mathbf{R}^3$: $\vec{x} + \vec{y}$, $k \cdot \vec{x}$,

$\vec{0} = (0, 0, 0)$; $-\vec{x} = -(x, y, z) = (-x, -y, -z)$.

standard unit vectors $\vec{i}, \vec{j}, \vec{k}$.

$\vec{x} = (x, y, z) = (x, 0, 0) + (0, y, 0) + (0, 0, z) \ = x\vec{i} + y\vec{j} + z\vec{k}$.

$\mathbf{R}^n: \quad \vec{x} + \vec{y} = (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n)$

$= (x_1 + y_1, \ldots, x_n + y_n).$

$k \cdot \vec{x} = k(x_1, x_2, \ldots, x_n) = (kx_1, \ldots, kx_n)$

$\vec{0} = (0, 0, \ldots, 0), \quad -\vec{x} = -(x_1, x_2, \ldots, x_n)$

$= (-x_1, -x_2, \ldots, -x_n).$

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V any collection, elements $\vec{v} \in V$

called **vectors.**

Formula for vector plus and scalar mult.

Usually scalars are real $k \in \mathbf{R}$;

may also have $k \in \mathbf{C}$, complex.

$(V, +, \cdot)$ is a **Vector Space** if the

vector + and scalar mult satisfy 10 rules:

2 closure rules

4 rules for + (including $\vec{0}$; and $-\vec{x}$)

2 rules for $\cdot$ (including $1 \cdot \vec{v} = \vec{v}$)

2 distributive laws. (2+4+2+2 = 10 or 2+8.)

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Examples: $\mathbf{R}^n$; $V = M_{2\times 2}(\mathbf{R})$;

$F(a, b) =$ functions: f+g, k·f, 0, -f.

$P_k =$ polynomials with real coef. degree $< k$.

Problem

Is $P =$ polynomials of degree exactly 2

a vector space?

Solution. (closure?), zero vector?

(try $p_1(x) + p_2(x) = (x^2 + 3x) + (-x^2 - 2)$.)

Problem Express the solutions of

$A\vec{x} = \vec{0}$ as a subset of $\mathbf{R}^4$

for $A = \begin{pmatrix} 1 & 3 & -2 & 1 \\ 3 & 10 & -4 & 6 \\ 2 & 5 & -6 & -1 \end{pmatrix}$

Solution:

$$A \to \begin{pmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -8 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$= A_R$ in Reduced Row Echelon Form (RREF).

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Last Steps: The bound variables are x_1 and x_2

(since there are leading 1's in col 1 and in col 2);

so $x_3 = s$ and $x_4 = t$ are the free variables.

Use the ith row of A_R to solve for the ith bound variable:

$x_1 - 8x_3 - 8x_4 = 0$, so $x_1 = 8s + 8t$, and

$x_2 + 2x_3 + 3x_4 = 0$, so $x_2 = -2s - 3t$; so

$(x_1, x_2, x_3, x_4) = (8s + 8t, -2s - 3t, s, t)$, for $s, t \in \mathbf{R}$.

Note 1: We reduce A rather than $A^{\#} = (A|\vec{0})$,

but the equations correspond to the augmented matrix $(A_R|\vec{0})$.

Note 2: (Pre-view of 4.4) Using vector addition and scalar mult

in $\mathbf{R}^4$, we say that the solutions ("Nullspace") are "spanned" by the

coefficient vectors $(8, -2, 1, 0)$ and $(8, -3, 0, 1)$

and that these two vectors are a "spanning set" for the solution space. Properties of the free variables

x_3 and x_4 will show that these two vectors are

"independent", so we say that these two vectors

are a "basis" for the solutions.

So in the homogeneous system $A\vec{x} = \vec{0}$, with

$(x_1, x_2, x_3, x_4) = (8s + 8t, -2s - 3t, s, t)$, for $s, t \in \mathbf{R}$,

we find the coefficient vector of each parameter

$\vec{x} = (8s, -2s, s, 0) + (8t, -3t, 0, t)$

$= s(8, -2, 1, 0) + t(8, -3, 0, 1)$, which

expresses each solution as a linear

combination of the vectors $\vec{u}_1 = (8, -2, 1, 0)$

and $\vec{u}_2 = (8, -3, 0, 1)$, so we say that $\vec{u}_1$ and $\vec{u}_2$ **span**

the vector space of solutions; or that $\{\vec{u}_1, \vec{u}_2\}$ is a **spanning set**

for the space.

4.3. Subspaces

Recall that a Vector Space V may be any collection with

formulas for vector plus and scalar mult, $V = (V, +, \cdot)$ with 10 rules.

In practice, in Math 205, Vector Spaces

will usually be subsets of one of four standard

Examples: $\mathbf{R}^n$; $V = M_{m\times n}(\mathbf{R})$;

$F(a, b) =$ functions: f+g, k·f, 0, -f.

$P_n =$ polynomials with real coef. degree $< n$.

We usually assume the 10 rules for these four vector spaces (the rules are easier to verify than to remember). If S is a subset of one of these standard spaces V, and we want S itself to be a vector space using the same formulas for $+$ and scalar mult as in V,

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eight of the rules in S follow from the corresponding rules in V. For example, if every vector in S is in V, and

$\vec{u}, \vec{v} \in S$, then $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ since $\vec{u}, \vec{v} \in V$ and this is one of the 10 rules we're assuming for our known, standard example V.

So to establish that our collection S is a vector space we only need to check the last two rules, in which case we say that S is a

Subspace of V. The two subspace conditions are the

closure rules (1) for every $\vec{u}, \vec{v} \in S$, check $\vec{u} + \vec{v} \in S$ (closure under vector $+$),

and (2) for every $\vec{u} \in S$, and every scalar $c \in \mathbf{R}$, check $c\vec{u} \in S$ (closure under scalar mult).

Checking that S is NOT a subspace is easy - for example, if S is all $(x, y) \in \mathbf{R}^2$ so that $x + y = 1$, we have $y = 1 - x$, for any real x and

so $(1, 0), (0, 1) \in S$ (taking $x = 1, x = 0$).

But $(1,0)+(0,1)=(1,1)$ is not in S (why? check sketch of S.).

So this subset is not a subspace of $\mathbf{R}^2$.

We don't need any other reason, but can

also check closure under scalar mult.

Since $c\vec{u}$ must be in S for every

scalar, we must have, for $c=0, 0\cdot\vec{u}=\vec{0}$, with $\vec{0}=(0,0)$ in S.

But $(0,0)$ is not in S (sketch!).

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By contrast T= all $(x,y)\in\mathbf{R}^2$ so

that $x+y=0$, is a subspace (check).

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3. **Sect 4.4.**

For the formal definition of **span** and **spanning set** we take vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ in a vector space V. For the most part, we think of V as being (1) $\mathbf{R}^n$; (2) $F(a, b)$; or, (3) a subspace of either (a) Euclidean n-space or (b) of k-differentiable functions.

A vector $\vec{v} \in V$ is called a **linear combination** of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ if there are scalars $c_1, c_2, \dots, c_k$ so that $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$.

The **span** of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ is the collection of all $\vec{v} \in V$ so that $\vec{v}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$.

The Span is a subspace of V. If Span $= V$, we say that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ is a **spanning set** of V.

Examples: (1) $\vec{i}, \vec{j}$ span R^2;

(2) $\{\vec{i}, \vec{j}, \vec{k}\}$ is a spanning set for R^3;

(3) the coef vectors obtained from the RREF of A span the solutions of $A\vec{x} = \vec{0}$.

Problem

Does the vector $\vec{v} = (3, 3, 4)$ belong to the subspace $\text{Span}(\vec{v}_1, \vec{v}_2)$ of $\mathbf{R}^3$, for $\vec{v}_1 = (1, -1, 2), \vec{v}_2 = (2, 1, 3)$?

Solution

We have $\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Observe that this is the linear system with augmented matrix $(\vec{v}_1 \vec{v}_2 | \vec{v}) = \begin{pmatrix} 1 & 2 & | & 3 \\ -1 & 1 & | & 3 \\ 2 & 3 & | & 4 \end{pmatrix}$.

Row reduction gives $\begin{pmatrix} 1 & 2 & | & 3 \\ 0 & 3 & | & 6 \\ 0 & -1 & | & -2 \end{pmatrix}$.

We see that this is a consistent system ($2 = r = r^{\#} = 2$), so YES, $\vec{v}$ is in the span.

We are often asked to continue by finding an explicit linear combination, solving for c_1, c_2.

For this, we continue to the RREF. We find $c_2 = 2, c_1 = -1$, and $\vec{v} = -\vec{v}_1 + 2\vec{v}_2$.