**Week 2a:**

Chapter 2, Sections 4, 5, 6

Section 2.4: Row Reduction

2.5: Gaussian Elimination

2.6 Inverses

**Linear Systems**

If $A$ is an $m \times n$ matrix with entries $a_{i,j}$,
$\vec{x}$ is the $n$-column vector with entries $x_1, \ldots x_n$,
and $\vec{b}$ is the $m$-column vector with entries $b_1, \ldots b_m$,
the matrix equation $A\vec{x} = \vec{b}$ gives $m$ equations,
each of the form ($i$th row of $A$) $\cdot \vec{x} = b_i$,
which is called a **linear system of $m$ equations in $n$ variables.**

In the equation $A\vec{x} = \vec{b}$, the matrix $A$ is called the **coefficient matrix** of the system, and $\vec{x}$ and $\vec{b}$ are called the vector of unknowns and the right-hand side vector, respectively.

Our preliminary qualitative description of the solutions of these systems (in 2-space and 3-space) suggests three cases: (1) just one unique solution; (2) no solutions; or (3) infinitely many solns.

We say that the system of equations is **consistent** if there is at least one solution; and **inconsistent** if there are no solutions.

Two systems with the same solutions are called **equivalent.**

Our objective is to replace a given system by the simplest possible equivalent system. For the calculations - we don't use the equations, but instead one more matrix $A^{\#} = (A|\vec{b})$, which is called the **augmented matrix** of the system.

**Problem 3.**

For the system

$$\begin{aligned} x + \ y + \ z - w &= 3, \\ 2x + 4y - 3z + 7w &= 2 \end{aligned}$$

determine the coef. matrix $A$, the right-hand side vector $\vec{b}$ and the augmented matrix $A^{\#}$.

Solve the system using elemenary row operations, and write the solution in vector form.

**Solution:**

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 2 & 4 & -3 & 7 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$A^{\#} = \begin{pmatrix} 1 & 1 & 1 & -1 & | & 3 \\ 2 & 4 & -3 & 7 & | & 2 \end{pmatrix}.$$

$$\left( A|\vec{b} \right) \to \begin{pmatrix} 1 & 1 & 1 & -1 & | & 3 \\ 0 & 2 & -5 & 9 & | & -4 \end{pmatrix} \ (r_2 \to r_2 - 2(r_1))$$

$$\to \begin{pmatrix} 1 & 1 & 1 & -1 & | & 3 \\ 0 & 1 & -\frac{5}{2} & \frac{9}{2} & | & -2 \end{pmatrix} \ (r_2 \to \tfrac{1}{2} r_2)$$

$$\to \begin{pmatrix} 1 & 0 & \frac{7}{2} & -\frac{11}{2} & | & 5 \\ 0 & 1 & -\frac{5}{2} & \frac{9}{2} & | & -2 \end{pmatrix} \ (r_1 \to r_1 - r_2)$$

The augmented matrix has been "reduced" to Reduced Row Echelon Form (RREF). For the last step, we notice "leading 1's" in column 1 and 2, so we call the variables corresponding to those columns bound variables - $x$ and $y$ are bound. The other variables are free variables, and we use each row to solve for one bound variable.

$x + \frac{7}{2}z - \frac{11}{2}w = 5$, so
$x = 5 - \frac{7}{2}z + \frac{11}{2}w$ and

$y - \frac{5}{2}z + \frac{9}{2}w = -2$, so
$y = -2 + \frac{5}{2}z - \frac{9}{2}w$.

Finally, we replace the free variables by parameters, $z = r, w = s$, then $(x, y, z, w) =$
$(5 - \frac{7}{2}r + \frac{11}{2}s, -2 + \frac{5}{2}r - \frac{9}{2}s, r, s)$
$= (5, -2, 0, 0) + r(-\frac{7}{2}, \frac{5}{2}, 1, 0) + s(\frac{11}{2}, -\frac{9}{2}, 0, 1)$.

## 1.1 Gauss-Jordan Elimination

We simplify the information in a matrix by using Elementary Row Operations (ERO's). There are 3:

1. $P_{ij} : r_i \leftrightarrow r_j$ means switch rows $i$ and $j$.
2. $M_i(k) : r_i \to kr_i$ means multiply row $i$ by $k \neq 0$.
3. $A_{i,j}(k) : r_i \to r_i, r_j \to (r_j + kr_i)$ means replace (row $j$) by the (linear) combination (row $j$) + $k$(row $i$), leaving row $i$ unchanged.

We say that matrices $A$ and $B$ are **row equivalent** if there is a sequence of ERO's that starts with $A$ and ends with $B$.

We say that a matrix is (row) reduced to simplest form if we have a matrix in reduced row echelon form (RREF) that is row equivalent. We may record the specific row operations, in the order used, that give the row reduction. The **Rank** of a matrix is the number of non-zero rows in a row echelon form of the matrix.

**Problem 4.** Use Gauss-Jordan Elimination to solve

$$\begin{aligned} 2x_1 - \ x_2 + 3x_3 - \ x_4 &= \ 3 \\ 3x_1 + 2x_2 + \ x_3 - 5x_4 &= -6 \\ x_1 - 2x_2 + 3x_3 + \ x_4 &= \ 6 \end{aligned}$$

**Solution:** We use the augmented matrix $A^{\#}$,

with $A^{\#} = (A|\vec{b}) \to \begin{pmatrix} 1 & -2 & 3 & 1 & | & 6 \\ 3 & 2 & 1 & -5 & | & -6 \\ 2 & -1 & 3 & -1 & | & 3 \end{pmatrix}$,

$\to \begin{pmatrix} 1 & -2 & 3 & 1 & | & 6 \\ 0 & 8 & -8 & -8 & | & -24 \\ 0 & 3 & -3 & -3 & | & -9 \end{pmatrix}$,

$\to \begin{pmatrix} 1 & -2 & 3 & 1 & | & 6 \\ 0 & 1 & -1 & -1 & | & -3 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$ (REF),

$\to \begin{pmatrix} 1 & 0 & 1 & -1 & | & 0 \\ 0 & 1 & -1 & -1 & | & -3 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$ (RREF).

[to be continued ... ]

Note 1: the matrix in the next-to-last step is in Row Echelon Form (REF), but not reduced. In Gaussian elimination we may stop at a REF and continue solving using back-substitution. For Gauss-Jordan elimination, the back-substitution steps are also done as row operations on the augmented matrix. Small systems with a unique solution may be done by Gaussian elimination without confusion, since the objective is to find numbers; Gauss-Jordan elimination will be the method-of-choice for finding generators when there are infinitely many solutions, and can always be used.

**Last step in Gauss-Jordan:** From the REF we identify **free variables** and **bound variables.** Again $x_1$ and $x_2$, are bound, the other variables are free variables, and recalling

$$(A|\vec{b})_R = \left(\begin{array}{cccccc} 1 & 0 & 1 & -1 & | & 0 \\ 0 & 1 & -1 & -1 & | & -3 \\ 0 & 0 & 0 & 0 & | & 0 \end{array}\right),$$

the 1st row corresponds to the equation

$x_1 + x_3 - x_4 = 0$, which we solve for

$x_1 = -x_3 + x_4$ and the 2nd row gives

$x_2 - x_3 - x_4 = -3$, so $x_2 = -3 + x_3 + x_4$,

and $(x_1, x_2, x_3, x_4) = (-x_3 + x_4, -3 + x_3 + x_4, x_3, x_4)$.