1. Let G be the cyclic group of order 18, and let "a" be a generator for G; so that $G =< a >= \{e, a, a^2, \ldots, a^{17}\}$, where e is the identity of G.

   (a) Find all elements of order 6 in G.

   (b) Find a subgroup of order 6 in G, and list all six of its elements.

   (c) Does G have any other subgroups of order 6? Either find one or explain why there aren't any.

2. (a) Find the disjoint cycle decomposition of $(14)(135)(23)(246)$.

   (b) What are the orders of the following permutations? (i) $(123)(456789)$ in S_{15}, and

   (ii) $(abc)(defghjk)$ in $\text{Sym}(\{a, b, \ldots, z\})$

3. Let G_1 be the cyclic group of order 6, $G_1 =< a >$, G_2 be the symmetric group S_4, and G be the direct product $G_1 \times G_2$.

   (a) What is the order of G?

   (b) Find the inverse of $(a^2, (1234))$.

   (c) How many elements of G have order 2? Describe them.

4. Let G be the group whose elements are the triples (a, b, c) of real numbers, $a \neq 0$, with product $(a, b, c) \cdot (d, e, f) = (ad, ae + b, af + c)$ and identity $(1, 0, 0)$.

   (a) Find the inverse of (a, b, c).

   (b) Compute $((a, b, c))^{-1} (1, i, j)(a, b, c)$.

2. Let G be a group with order divisible by 7. Suppose that G has a subgroup H for which the index of H in G is 5 (so H has 5 cosets in G). Prove that G is not simple.

3. Let D_{12} be the dihedral group of order 12, G_2 be the symmetric group S_9, and G be the direct product $D_{12} \times S_9$. Suppose that the rotation subgroup of D_{12} has generator R.

(a) Describe the collection of elements of G whose order divides 9. (Your description doesn't need to include the exact count of these elements.)

(b) Which of these elements have exact order 9?

(c) Suppose that S is a reflection in D_{12}. Describe the collection of elements of G that have exact order 8.

5. Let Z_{12} be a cyclic group of order 12, given as the group generated by a, so $Z_{12} = <a>$. Let $\alpha = a^3$, and $H = <\alpha>$ be the subgroup generated by α. What are the elements of H? Find the cosets of H in Z_{12}.

6. If $n = 3^3 * 5^2$, and $G = \mathbf{Z}_n^\times$ is the group of invertible elements in $\mathbf{Z}_n$, find the order of G. For each of the following, state whether G has the property: (1) G is non-Abelian?; (2) G is cyclic?; (3) G has an element of order 8? Give a reason for your answer (not a proof, just why did you say "yes" or "no").

Answer as many of the following as you can. You need not answer all questions or parts to score well: select the topics that you know.

5. Let G_1 and G_2 be groups, and f be a homomorphism from G_1 to G_2. Let e_1 and e_2 be the identities of G_1 and G_2, respectively. Define the kernel of f and prove that it is a subgroup. If k belongs to the kernel of f, and x is any element of G_1, does $x * k * (x^{-1})$ belong to the kernel of f (either prove that it does, or explain why it does not)?

6. If $n = 3^4 * 5^2$, and $G = \mathbf{Z}_n^\times$ is the group of invertible elements in $\mathbf{Z}_n$, find the order of G.

8. Let r and s be the usual generators for the dihedral group $D_{2n} = <r, s | r^n = s^2 = 1, rs = sr^{-1}>$ with elements r^j and sr^j. Show that every element sr^j has order 2. Find the order of each element in D_{24}. Determine the center of D_{24}.

3. We consider groups of order 12. (a) Find the order of each element in (i) the alternating group A_4 and (ii) the dihedral group D_{12}, both non-Abelian of order 12.

(b) Find the Abelian groups of order 12, up to isomorphism.

(c) Given that any other non-Abelian group of order 12 is isomorphic to Q_{12} $=< a, b : a^3 = 1, b^4 = 1, ab = b(a^{-1}) >$ with "a" of order 3 and "b" of order 4, explain why each of A_4, D_{12} and Q_{12} are non-isomorphic, and count the number of non-isomorphic groups of order 12.

4. Let C(a) $= \{x \in G | x^{-1}ax = a\}$ for $a \in G$. Prove that C(a) is a subgroup of G.

1. Let G be a group of order 351; and suppose that H is a subgroup of G, with order(H)=13, for which H is NOT normal in G. How many elements of order 13 are there in G? Explain why this group G is not simple. (You are given that 351 has prime factorization $351 = 3^3 \cdot 13$.)

4. (a) Let G be a group of order $3^3 \cdot 11 \cdot 13$. For each prime p dividing the order of G, either find a number N_p that is consistent with the information provided by the Sylow Theorems for which $N_p \neq 1$, or show that $N_p = 1$ is the only possible value.

(b) Let G be a finite group of order n, and suppose that H is a subgroup of order m in G. Suppose that no other subgroup of G has order m. Prove that H is normal in G. (Note: the integer m may not be a prime power, so Sylow's Theorems do not apply.)

5. Let G_1 be the cyclic group of order 6, $G_1 =< a >$, G_2 be the symmetric group S_4, and G be the direct product $G_1 \times G_2$.

(a) What is the order of the Sylow 2-subgroups of G?

(b) How many elements of G have order 2? Describe them.

(c) Show that (b) is inconsistent with G having a unique Sylow 2-subgroup.

(d) One of the Sylow 2-subgroups of G has the form $P =< a^3 > \times D_8$, where the dihedral group of order 8 has the form $< (1234), (12)(34) >$. If $g = (g_1, g_2)$ is an element of G, use the fact that S_4 has a normal subgroup V_4 of order 4 to show that $P \cap g * P * (g^{-1})$ has order 8 (notice and explain that V_4 is a subgroup of the given subgroup D_8).

(e) If S is the set of Sylow 2-subgroups of G, and $P\prime$ is a 2nd Sylow 2-subgroup distinct from the one in part (d), describe the orbit of $P\prime$ under the action of P on S by conjugation. Explain how your answer is consistent both with the numerical information from part (d) as given in the proof of the Sylow theorem, and with what we know about the subgroups of S_4.

6. Use the Euclidean Algorithm to find the inverse of $\bar{7}$ in $\mathbf{Z}/(41\mathbf{Z})$. Show steps, not just the number.

Added 2013:

2. (a) Find the order N of the (multiplicative) group of units $G = \mathbf{Z}_n^\times$ in the ring $\mathbf{Z}_n$ when $n = 2013 = 3 \cdot 11 \cdot 61$. How many Abelian groups, up to isomorphism, are there of order N? Recall that (1) for each prime power p^a dividing N, with p^{a+1} not dividing N, we determine all possible cyclic groups of prime power orders $p^{b1}, p^{b2}, \ldots, p^{bt}$ with $a = b1 + b2 + \ldots bt$; and then (2) count the number of ways of picking one sum for each prime.

(b) Write G as a product of cyclic groups of prime power order. For each of the other Abelian groups with the same order as $G = \mathbf{Z}_{2013}^\times$ give a specific prime power that does not occur in this product of G. (For example, if N were divisible by 9; half of the groups could be eliminated by having the specific divisor 9 instead of divisors 3 and 3.)

3. Solve x $= 7$ (mod 25), x $= 5$ (mod 16), x $= 11$ (mod 21)

for $1 \leq x < 8400 = 25 \cdot 16 \cdot 21$ given

$336^{-1} = 16$ (mod 25), $525^{-1} = 5$ (mod 16), and $400^{-1} = 1$ (mod 21).

8. (a) How many non-isomorphic Abelian groups of order $7^3 = 343$ are there? Give a collection of such groups in which every group is non-isomorphic and every isomorphism class is represented.

(b) Do the same for Abelian groups of order 32.

(c) Without listing the groups, how many non-isomorphic Abelian groups are there of order $32 * 7^3$? Describe their structure. Can you describe the ones that have an element of order $8 * 49$?

5. Consider Abelian groups of order $m = 2^5 \cdot 3^4 \cdot 5^2$. (a) Find the number of non-ismorphic Abelian groups of order m.

(b) Give the elementary divisors of one of the groups with the largest number of invariant factors. How many isomorphism classes are of this type?

5. In this problem, we consider A_4 as a subgroup of S_6, using the coset action on a subgroup of order 2. Specifically, set $\alpha = (12)(34)$, and $H = <\alpha>$.

(a) If $\beta = (13)(24)$, $\sigma = (123)$, and $\tau = (124)$, verify that every element of A_4 belongs to exactly one the six disjoint left cosets

$$H, \sigma H, \sigma^2 H, \tau H, \tau^2 H, \beta H.$$

(b) Find the cycle decomposition of the action of σ regarded as an element of S_6, where S_6 is identified with the symmetric group on the above six cosets, and σ acts by left multiplication. (Hint: Since σ must be an element of order 3 in S_6, the only possible answers are that σ remains a 3-cycle - in which case it must fix 3 of the cosets of H - or σ is the product of two disjoint 3-cycles. Notice that there are 3 cosets that are obviously moved by σ - which ones! Then check to see what happens to one of the other cosets.)

7. Prove that the group $GL(2, \mathbf{F}_3)$ of 2-by-2 matrices with entries in the field with 3 elements is non-Abelian. (Recall that the elements of $\mathbf{F}_3$ are the integers 3.)