GORENSTEIN DUALITY AND UNIVERSAL COEFFICIENT THEOREMS

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ABSTRACT. The present paper describes a duality phenomenon for cohomology theories with the character of Gorenstein rings.

CONTENTS

1. INTRODUCTION

We describe a Universal Coefficient Theorem relating homology and cohomology of suitable torsion spaces when the coefficient ring R_* has good homological properties, and we go on to lift this to a highly structured equivalence when the cohomology theory is sufficiently nice. For example, given a cohomology theory $R^*(\cdot)$ whose coefficient ring R_* is Gorenstein of shift a (see Section 2) with $R_0 = \mathbb{Z}_{(p)}$ the statement is as follows. For spaces X with $R_*(X)$ a torsion module (i.e., so that each element is annihilated by some power of the maximal ideal), there is an isomorphism

$$R^*(X) \cong \Sigma^a (R_*(X))^\vee$$

of R_*-modules, where $M^\vee = \mathrm{Hom}(M, \mathbb{Z}/p^\infty)$ is the Pontryagin dual. We will explain that this applies in particular when R_* is a polynomial ring on finitely many generators, and how to find the shift a in that case. In particular it applies to the well-known chromatic Johnson-Wilson theory $BP\langle n \rangle$, to give the striking duality

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phenomenon proved by the first author [4], which was motivated by his work with W.S.Wilson regarding the 2-local ku-homology and ku-cohomology groups of the Eilenberg-MacLane space $K(\mathbb{Z}/2, 2)$.

The proof is based on the homological behaviour of the coefficient ring $BP\langle n\rangle_*$. When [4] was posted on the arXiv, the second author recognized the statement as following from a form of Gorenstein duality and wrote down the proof of a statement in a structured context. The two proofs are based on the same piece of algebra, but in rather different contexts so that their overlap in applicability is actually rather small.

The authors decided that combining the two papers would be to the advantage of both, by giving generality and perspective as well as specific examples. The present paper describes the algebra behind both results, and then develops it in the two contexts: the first gives conclusions in terms of homology and cohomology groups and the second, when it applies, enhances this to a conclusion in the derived category.

2. Gorenstein rings

In this section we remind the reader of some well known facts about a graded commutative local Noetherian ring A with residue field k (see [2] for general background). We will soon apply them in our topological context.

Definition 2.1. We say that A is *Gorenstein* if any one of the following three equivalent conditions hold

(1) A is of finite injective dimension as a module over itself
(2) The Ext groups $\mathrm{Ext}^i_A(k, A)$ are non-zero for finitely many i or
(3) The Ext groups are

$$\mathrm{Ext}^i_A(k, A) = \begin{cases} \Sigma^b k & \text{for } i = n \\ 0 & \text{otherwise} \end{cases}$$

for some b where n is the Krull dimension of A.

The condition that we will make direct use of is the third.

If A is Gorenstein then $a = b - n$ is called the *Gorenstein shift* of A.

Remark 2.2. (i) For some purposes the bigrading of $\mathrm{Ext}^*_A(k, A)$ is significant, but for us only the total degree will be relevant.

(ii) We include the case of an ungraded ring as a graded ring entirely in degree 0. In this case each Ext group is ungraded and the Part 3 necessarily has $b = 0$.

Example 2.3. (i) If $A = k[x_1, \ldots, x_n]$ with all the x_i of positive degree then A is Gorenstein with $b = -\sum_i |x_i|$ and $a = b - n$.

(ii) If $A = K[x_1, \ldots, x_n]$ for an ungraded Gorenstein local ring K of shift c with all the x_i of positive degree then A is Gorenstein of shift $b + c - n$ where $b = -\sum_i |x_i|$.

In this context, the Gorenstein condition automatically implies a duality statement. For this we make use of the injective hull $I(k)$ of the residue field, and the Matlis dualization process for A-modules M defined by

$$M^\vee = \mathrm{Hom}_A(M, I(k)).$$

Example 2.4. (i) If $A = k[x_1, \ldots, x_n]$ as above then $I(k) = \mathrm{Hom}_k(A, k)$ and the Matlis dual has a simple description

$$M^\vee = \mathrm{Hom}_A(M, I(k)) = \mathrm{Hom}_k(M, k)$$

(ii) If $A = K[x_1, \ldots, x_n]$ as above then we can express the duality for A in terms of that for K. Indeed, $I_A(k) = \mathrm{Hom}_K(A, I_K(k))$ and the Matlis dual has a simple description

$$M^\vee = \mathrm{Hom}_A(M, I_A(k)) = \mathrm{Hom}_K(M, I_K(k))$$

With Matlis duality in hand, we can state the Gorenstein duality enjoyed by torsion modules over a Gorenstein ring.

Lemma 2.5. *If A is a Gorenstein graded ring and M is a torsion A-module then*

$$\mathrm{Ext}^i(M, R) = \begin{cases} \Sigma^b M^\vee & \textit{if } i = n \\ 0 & \textit{otherwise} \end{cases}$$

Proof: This is a standard consequence of the Gorenstein condition. To give a proof, one may consider the stable Koszul complex

$$K_\infty^\bullet(\mathfrak{m}) = (A \longrightarrow A[\frac{1}{y_1}]) \otimes_A \ldots \otimes_A (A \longrightarrow A[\frac{1}{y_s}])$$

associated to the maximal ideal $\mathfrak{m} = (y_1, \ldots, y_s)$. The map $K_\infty^\bullet(\mathfrak{m}) \longrightarrow A$ induces a weak equivalence $\mathrm{Hom}_A(M, \cdot)$ in the derived category. Now we use $H_\mathfrak{m}^*(A) = H_\mathfrak{m}^n(A) = \Sigma^b I(k)$ by [2, 13.3.4]. It follows that

$$\mathrm{Ext}_A^*(M, R) \simeq \mathrm{Ext}_A^n(M, \Sigma^b I(k))$$

as required.   □

3. Consequences for cohomology theories

Suppose now that R is a ring spectrum (i.e., an A_∞-ring) so that R_* is a Gorenstein commutative local Noetherian ring of shift a in the sense of Section 2 and $R_0 = K$. We immediately have a universal coefficient theorem for spectra X with $R_*(X)$ torsion.

Theorem 3.1. *If R is a ring spectrum with R_* Gorenstein of shift a then for any X with $R_*(X)$ torsion, there is an isomorphism*

$$R^*(X) \cong \Sigma^a R_*(X)^\vee.$$

Proof: By [12, Corollary, p.257] or [8, IV.4.1], if R is an A_∞ ring spectrum, there is a Universal Coefficient spectral sequence

$$\operatorname{Ext}_{R_*}^{s,t}(R_*X, R_*) \Rightarrow R^{s+t}X.$$

From Lemma 2.5, the spectral sequence must collapse, as it is confined to the single column $s = n$, and the Lemma 2.5 gives the values. $\qquad\square$

Example 3.2. The p-local Johnson-Wilson spectrum [10] $R = BP\langle n \rangle$ has coefficient ring $R_* = \pi_*(R) = \mathbb{Z}_{(p)}[v_1, \ldots, v_n]$, with $|v_i| = 2(p^i - 1)$.

According to Example 2.3 (ii) this is a Gorenstein local ring. Here $b = -D$ where $D = \sum_i |v_i| = 2((p^{n+1}-1)/(p-1)-(n+1))$ is the sum of the degres of the generators and $c = -1$ is the Gorenstein shift of $K = \mathbb{Z}_{(p)} \longrightarrow \mathbb{F}_p$, so that the Gorenstein shift of $R_* \longrightarrow \mathbb{F}_p$ is $a = b+c-n = -D-n-1$, and Matlis duality is $(\cdot)^\vee = \operatorname{Hom}_{\mathbb{Z}_{(p)}}(\cdot, \mathbb{Z}/p^\infty)$. We may apply the general result since, by [1, Corollary 3.5], $BP\langle n \rangle$ can be realised as an A_∞-ring spectrum, so we may apply Theorem 3.1.

Explicitly, if $R = BP\langle n \rangle$ and $R_*(X)$ is torsion, there is an isomorphism of right R_*-modules

$$R^*(X) \cong (R_*(\Sigma^{D+n+1}X))^\vee.$$

It may be worth making explicit the two simplest cases

Example 3.3. If $n = 0$ then $BP\langle 0 \rangle = H\mathbb{Z}_{(p)}$ and $D = n = 0$, we have

$$H^*(X; \mathbb{Z}_{(p)}) = \Sigma^{-1}\operatorname{Hom}(H_*(X; \mathbb{Z}_{(p)}), \mathbb{Z}/p^\infty),$$

whenever $H_*(X; \mathbb{Z}_{(p)})$ is p-torsion.

Example 3.4. If $n = 1, p = 2$ then $BP\langle 1 \rangle = ku$ is 2-local connective K-theory and $D = 2, n = 1$. We thus have

$$ku^*(X) = \Sigma^{-4}\operatorname{Hom}(ku_*(X), \mathbb{Z}/2^\infty),$$

whenever $ku_*(X)$ is $(2, v_1)$-torsion.

We may illustrate this example with $X = K(\mathbb{Z}/2, 2)$ to see that even in this case the duality statement is of considerable interest.

4. An example for connective K-theory, with $X = K(\mathbb{Z}/2, 2)$.

In [13] and [5], Wilson and the first author gave partial calculations of $ku_*(K_2)$, where $K_2 = K(\mathbb{Z}/2, 2)$, in their studies of Stiefel-Whitney classes. In [6], these authors made a complete calculation of $ku^*(K_2)$. Using Theorem 3.1, we can now give a complete determination of $ku_*(K_2)$: it is torsion since it contains no infinite groups or infinite v_1-towers [5].

The work in [5] and [6] was done using the Adams spectral sequence. It is interesting to compare the forms of the two Adams spectral sequence E_∞ calculations. What appears as an h_0 multiplication in one usually appears as an exotic extension (multiplication by 2 not seen in Ext) in the other. We illustrate here with corresponding small portions of each. The portion of $ku^*(K_2)$ in Figure 1 is called A_5 in [6]. Note that in our ku^* chart, indices increase from right to left. Exotic extensions appear in red. One should think of the dual of the ku_* chart as an upside-down version of the chart. The dual of the element in position $(30, 7)$ in Figure 2 is in position $(34, 0)$ in Figure 1.
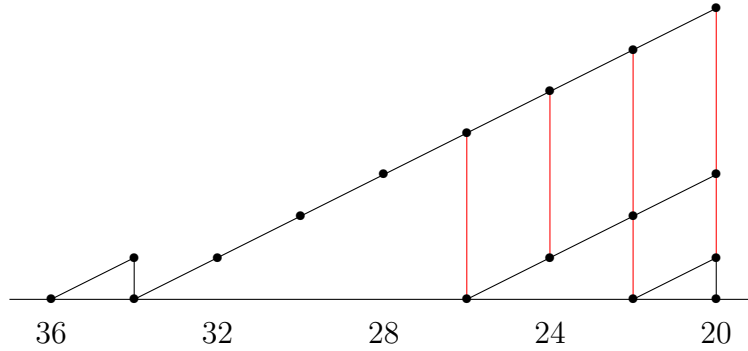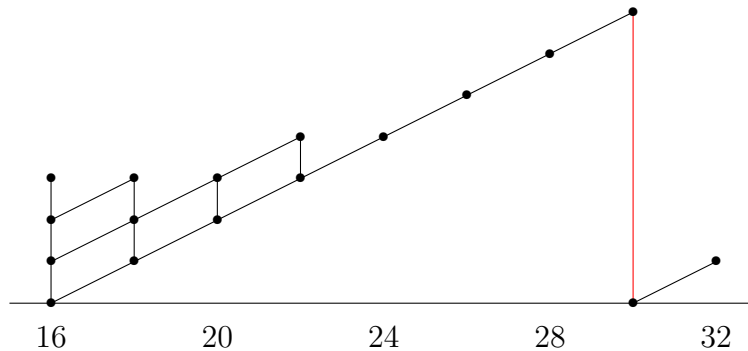
Figure 1. A portion of $ku^*(K_2)$

Figure 2. Corresponding portion of $ku_*(K_2)$

5. Gorenstein ring spectra and Gorenstein duality

We now suppose that the representing spectrum R of our cohomology theory admits the structure of a commutative ring (E_∞-ring). In this case we now describe how the duality statement may be lifted to one at the level of R-modules. When this applies, the duality statement in Theorem 3.1 can be deduced by passage to homotopy. There are many examples where R is a ring with Gorenstein coefficients R_* but where R is not known to be realized as a commutative ring ($R = BP\langle n\rangle$ with $n \geq 3$ for instance), so that the approach of Section 3 gives the best available results. On the other hand there are many examples where R is Gorenstein in a suitable derived sense without the coefficient ring R_* being Gorenstein, and in this case we obtain results without counterparts in the setting of Section 3. The results in this section build on [9] in the context of [7].

5.A. **The Gorenstein condition.** We suppose given a commutative ring spectrum R and a map $R \longrightarrow \ell$. In our examples ℓ will either be the local ring $K = R_0$ or its residue field k. Translating the third condition of Definition 2.1 into the derived category, we obtain the definition of a Gorenstein ring spectrum.

Definition 5.1. [7, 8.1] We say that $R \longrightarrow \ell$ is *Gorenstein of shift* a if there is an equivalence
$$\mathrm{Hom}_R(\ell, R) \simeq \Sigma^a \ell$$
of left R-modules.

Remark 5.2. The paper [7] also requires a finiteness condition before $R \longrightarrow \ell$ can be called Gorenstein. We will impose a slightly stronger finiteness condition later.

Notation 5.3. In all the examples we consider here, ℓ will be an Eilenberg-MacLane spectrum. We will use the same letter ℓ to denote both the classical ring and the Eilenberg-MacLane spectrum, relying on context to determine which is intended at any point.

Example 5.4. We note that if R_* is itself Gorenstein (as in Examples 2.3) it follows that $R \longrightarrow k$ is Gorenstein with the same shift. In particular, this applies to the examples of polynomial rings over k or over a Gorenstein local ring K.

5.B. **Anderson duality and Brown-Comenetz duality.** Suppose R is connective and $K = R_0$. For any injective K-module I and any R-module Y we may define the Brown-Comenetz dual spectrum I^Y to be the R-module defined by the formula
$$[M, I^Y]_*^R = \mathrm{Hom}_K(\pi_*(M \otimes_R Y), I),$$
for R-modules M; this defines I^Y as the representing object since the right hand side is a cohomology theory (I is injective over K). Note the special case $M = R \wedge X$,

when

$$[X, I^Y] = \mathrm{Hom}_K(\pi_*(X \wedge Y), I),$$

Now if K is of injective dimension 1 over itself then we can choose an injective resolution of K-modules

$$0 \longrightarrow K \longrightarrow I \longrightarrow J \longrightarrow 0.$$

(for example if $K = \mathbb{Z}_{(p)}$ then we may take $I = \mathbb{Q}$ and $J = \mathbb{Q}/\mathbb{Z}_{(p)} = \mathbb{Z}/p^\infty$). For an R-module Y, there are Brown-Comenetz duals I^Y and J^Y with respect to I and J, and we define the Anderson dual with respect to K via the cofibre sequence

$$K^Y \longrightarrow I^Y \longrightarrow J^Y.$$

Up to equivalence K^Y is independent of the resolution. One can immediately find a short exact sequence for maps into the Anderson dual.

Corollary 5.5. *There is a short exact sequence*

$$0 \longrightarrow \mathrm{Ext}_K(\Sigma Y_*^R(M), K) \longrightarrow [M, K^Y] \longrightarrow \mathrm{Hom}_K(Y_*^R(M), K) \longrightarrow 0$$

In particular, we have an isomorphism

$$[K, K^R]_* \cong \mathrm{Hom}_K(K, K) = K$$

5.C. **Two finiteness conditions.** There are numerous cases of interest where, $R \longrightarrow \ell$ satisfies the strong condition that ℓ is small over R in the sense that ℓ is finitely built from R. If R is a conventional commutative local Noetherian ring this is equivalent to R being regular.

The relevant results continue to hold under a much weaker condition. We require ℓ to be *proxy small* [7, 4.6] in the sense that there is a small object $\widehat{\ell}$ so that ℓ finitely builds $\widehat{\ell}$ and $\widehat{\ell}$ builds ℓ. If R is a conventional commutative local Noetherian ring this always holds, and we may take $\widehat{\ell}$ to be the Koszul complex associated to any finite generating set for the maximal ideal.

5.D. **Cellularization.** Let $\mathcal{E} = \mathrm{Hom}_R(\ell, \ell)$ and note that for any R-module M, $\mathrm{Hom}_R(\ell, M)$ is a right $\mathcal{E}$-module. We have an evaluation map

$$\epsilon : \mathrm{Hom}_R(\ell, M) \otimes_\mathcal{E} \ell \longrightarrow M.$$

We note that $\mathrm{Hom}_R(\ell, M)$ is built from $\mathcal{E}$ and hence $\mathrm{Hom}_R(\ell, M) \otimes_\mathcal{E} \ell$ is built from ℓ. The evaluation map is thus a map from a ℓ-cellular object to M, and it is a ℓ-equivalence provided ℓ is proxy small ([7, 6.10 and 6.14]). Thus

$$\mathrm{Cell}_\ell M := \mathrm{Hom}_R(\ell, M) \otimes_\mathcal{E} \ell$$

is ℓ-cellularization.

Example 5.6. Suppose R is connective with $R_0 = \mathbb{Z}_{(p)}$ and R_n is a finitely generated $\mathbb{Z}_{(p)}$-module in each degree.

(i) If $K = \mathbb{Z}_{(p)}$ and $R \longrightarrow K$ is an isomorphism in π_0 then we see that K^R is built from $K^K \simeq K$, and so $\mathrm{Cell}_K K^R \simeq K^R$.

(ii) Now suppose $k = \mathbb{F}_p$ and that in π_0 the map $R \longrightarrow k = \mathbb{F}_p$ is projection $\mathbb{Z}_{(p)} \longrightarrow \mathbb{F}_p$ onto the residue field.

Since $\mathbb{Q}^R$ is rational, we find $\mathrm{Cell}_{\mathbb{F}_p} \mathbb{Q}^R \simeq *$. This gives

$$\mathrm{Cell}_{\mathbb{F}_p} \mathbb{Z}_{(p)}^R \simeq \Sigma^{-1} \mathrm{Cell}_{\mathbb{F}_p} (\mathbb{Z}/p^\infty)^R \simeq \Sigma^{-1} (\mathbb{Z}/p^\infty)^R,$$

the desuspension of the Brown-Comenetz dual of R.

5.E. **Algebraic criteria for cellularity.** There are a number of cases where we can give criteria for cellularity by looking at coefficients. Suppose $\mathfrak{m} = \ker(R_* \longrightarrow k) = (y_1, \ldots, y_s)$ and write

$$\Gamma_\mathfrak{m} R := \mathbf{fib}(R \longrightarrow R[\frac{1}{y_1}]) \otimes_R \cdots \otimes_R \mathbf{fib}(R \longrightarrow R[\frac{1}{y_s}]),$$

for the stable Koszul complex and $\Gamma_\mathfrak{m} M = \Gamma_\mathfrak{m} R \otimes_R M$. By construction there is a spectral sequence

$$H_\mathfrak{m}^*(R_*; M_*) \Rightarrow \pi_*(\Gamma_\mathfrak{m} M),$$

so that in particular if $M = R$ and R_* is Cohen-Macaulay, this collapses to an isomorphism

$$H_\mathfrak{m}^n(R_*) = \Sigma^n \pi_*(\Gamma_\mathfrak{m} R).$$

We say that R has *algebraic k-cellularization* if the stable Koszul complex gives the cellularization:

$$\Gamma_\mathfrak{m} R \simeq \mathrm{Cell}_k R.$$

We note that in this case k is proxy small with the unstable Koszul complex $R/y_1 \otimes_R \cdots \otimes_R R/y_s$ as witness.

Lemma 5.7. *If R has algebraic k-cellularization then M is k-cellular if and only if M_* is a torsion R_*-module.*

Proof: If M is k-cellular then $M \simeq \mathrm{Cell}_k M \simeq \Gamma_\mathfrak{m} R \otimes M$, and the spectral sequence for calculating M_* is finite with a torsion E_2-term so M_* is torsion.

Conversely if M_* is torsion then $H_\mathfrak{m}^*(M_*) = H_\mathfrak{m}^0(M_*) = M_*$ and the map $\Gamma_\mathfrak{m} M \longrightarrow M$ is an isomorphism in homotopy and hence a weak equivalence. $\square$

Remark 5.8. If k is a finite field then an R_*-module M_* is torsion if and only if it is *locally finite* in the sense that the submodule generated by an element $x \in M_*$ is a finite set.

Example 5.9. (i) Suppose $R_0 = \mathbb{Z}_{(p)}$ with $R_* = \mathbb{Z}_{(p)}[x_1, \ldots, x_n]$ (for example $R = BP\langle n \rangle$), so that $\mathfrak{m} = (p, x_1, \ldots, x_n)$. In this case $R \longrightarrow k$ is regular and has algebraic k-cellularization.

If is clear that $\Gamma_{\mathfrak{m}} R$ is the $\widehat{k}$-cellularization where

$$\widehat{k} = R/p \otimes_R R/x_1 \otimes_R \cdots \otimes_R R/x_n.$$

In this case $\widehat{k} \simeq k$ so that R has algebraic k-cellularization.

(ii) It is shown in [9, 5.1] that this extends to the case that $R \longrightarrow k$ is proxy regular and has algebraic cellularization if R_* is a hypersurface ring.

(iii) It is proved in [9, 5.2] or [3] that the ring spectrum tmf of 2-local topological modular forms has algebraic k-cellularization.

(iv) The proof of [7, 9.3] shows that for any compact Lie group G the spectrum $R = C^*(BG) = \mathrm{map}(BG, k)$ of cochains on BG has algebraic k-cellularization.

5.F. **Gorenstein duality.** We continue to suppose that R is a connective commutative ring spectrum with $R_0 = K$ a local ring with residue field k. We say that $R \longrightarrow k$ has *Gorenstein duality of shift a* if we have an equivalence

$$\Gamma_k R \simeq \Sigma^a I(k)^R,$$

and we say $R \longrightarrow K$ has *Gorenstein duality of shift a* if

$$\Gamma_K R \simeq \Sigma^a K^R.$$

In this section we recall that under favourable circumstances, if $R \longrightarrow k$ is Gorenstein of shift a then it has Gorenstein duality of shift a, and similarly for $R \longrightarrow K$.

If $R \longrightarrow k$ is Gorenstein of shift a in the sense of 5.1, we have

$$\mathrm{Hom}_R(k, R) \simeq \Sigma^a k \simeq \mathrm{Hom}_R(k, \Sigma^a I(k)^R).$$

Provided the composite $\mathrm{Hom}_R(k, R) \simeq \mathrm{Hom}_R(k, \Sigma^a k^R)$ is an isomorphism of right $\mathcal{E}$-modules (for example if $\mathcal{E}$ has a unique action on k), then we may apply $\otimes_{\mathcal{E}} k$, and provided k is proxy small, Subsection 5.D shows that we have an equivalence

$$\mathrm{Cell}_k R \simeq \Sigma^a \mathrm{Cell}_k I(k)^R$$

of left R-modules.

Similarly

$$\mathrm{Cell}_K R \simeq \Sigma^a \mathrm{Cell}_K K^R$$

Example 5.10. If $R_0 = K = \mathbb{Z}_{(p)}$, $k = \mathbb{F}_p$ and $R_* = K[x_1, \ldots, x_n]$ (for example $R = BP\langle n \rangle$) then $R_* \longrightarrow K$ is Gorenstein of shift $-D - n$, so is $R \longrightarrow K$. The ring $\mathcal{E}$ is exterior on generators of degree $-|x_1| - 1, \ldots, -|x_n| - 1$. This has a unique action on K by the argument of [7, 3.9]. We deduce

$$\Gamma_K R \simeq \Sigma^{-D-n} \Gamma_K K^R \simeq \Sigma^{-D-n} K^R.$$

We now apply Γ_p to both sides, noting $\Gamma_p\Gamma_{\mathbb{Z}} = \Gamma_{\mathbb{F}_p}$ and see

$$\Gamma_{\mathbb{F}_p}R \simeq \Sigma^{-D-n}\Gamma_p K^R \simeq \Sigma^{-D-n-1}(\mathbb{Z}/p^\infty)^R.$$

This is the statement that $R \longrightarrow \mathbb{F}_p$ has Gorenstein duality of shift $-D - n - 1$.

Remark 5.11. It would be preferable to take $k = \mathbb{F}_p$ and argue directly. The map $R \longrightarrow \mathbb{F}_p$ is Gorenstein of shift $-D - n - 1$ and we would like to deduce Gorenstein duality of the same shift. However this would require a unique action of $\mathrm{Hom}_R(\mathbb{F}_p, \mathbb{F}_p)$ on $\mathbb{F}_p$, and the degrees of generators do not make this obvious.

5.G. **The UCT.** Finally we may deduce the duality statement for a commutative ring spectrum with Gorenstein duality.

Theorem 5.12. *Suppose the $R \longrightarrow k$ has Gorenstein duality of shift a. It then follows that if M is k-cellular we have an equivalence*

$$\mathrm{Hom}_R(M, R) \simeq \Sigma^a I(k)^M.$$

Proof: Since M is k-cellular we have

$$\mathrm{Hom}_R(M, R) \simeq \mathrm{Hom}_R(M, \mathrm{Cell}_k R) \simeq \mathrm{Hom}_R(M, \Sigma^a I(k)^R) \simeq \Sigma^a I(k)^M$$

$$\square$$

The special case $M = R \wedge X$ is particular interest especially when cellularity can be detected in terms of coefficients.

Corollary 5.13. *In particular, taking $M = R \wedge X$, we see that if R_*X is a torsion module we have an isomorphism*

$$R^*(X) = \Sigma^a \mathrm{Hom}_K(R_*X, I(k)).$$

Proof: We may calculate

$$R^*(X) = [X, R]_* =$$
$$[R \wedge X, R]_*^R \simeq [R \wedge X, \Gamma_{\mathbb{F}_p}R]_*^R \simeq [R \wedge X, \Sigma^a(\mathbb{Z}/p^\infty)^R]_*^R \simeq [X, \Sigma^a(\mathbb{Z}/p^\infty)^R]_*$$
$$= \Sigma^a\mathrm{Hom}_{\mathbb{Z}}(R_*(X), \mathbb{Z}/p^\infty)$$

$$\square$$

Example 5.14. Suppose $R_0 = K = \mathbb{Z}_{(p)}$, $k = \mathbb{F}_p$ and $R_* = K[x_1, \ldots, x_n]$ (for example $R = BP\langle n\rangle$) with $a = -D - n - 1$ we recover Theorem 3.1. We note that it is only known that $BP\langle n\rangle$ admits the structure of a commutative ring for $n \leq 2$.

Example 5.15. The spectrum $R = \mathrm{tmf}$ of 2-local topological modular forms has $R_0 = K = \mathbb{Z}_{(2)}$ and $k = \mathbb{F}_2$. This has Gorenstein duality in the form

$$\Gamma_{\mathbb{F}_2}\mathrm{tmf} \simeq \Sigma^{-23}(\mathbb{Z}/2^{\infty})^{\mathrm{tmf}}.$$

A result of this type was first proved by Mahowald-Rezk [11]; a proof of precisely the statement here, along with a discussion of alternative approaches can be found in [3, 4.8]. From this we deduce

$$\mathrm{tmf}^*(X) = \Sigma^{-23}\mathrm{Hom}_{\mathbb{Z}_{(2)}}(\mathrm{tmf}_*(X), \mathbb{Z}/2^{\infty})$$

whenever $\mathrm{tmf}_*(X)$ is tmf_*-torsion.

Example 5.16. If G is a finite group, we may take $R = C^*(BG; k)$ with $R_0 = k$. By [7, 10.3] this has Gorenstein duality with shift 0. In the present case the Matlis dual is the vector space dual so that Gorenstein duality takes the form

$$\Gamma_k C^*(BG) \simeq C_*(BG).$$

A plentiful supply of modules comes from G-spaces X, where we may form $M = C_*(EG \times_G X)$; this is obviously torsion since it is zero in negative degrees. Inserting this into the theorem we see that for any G-space there is a spectral sequence

$$\mathrm{Ext}^{*,*}_{H^*(BG)}(H_*(EG \times_G X), H^*(BG)) \Rightarrow H^*_G(EG \times_G X),$$

which is in the form of a Universal Coefficient Theorem relating homology and cohomology of the Borel construction. One could approach this in an equivariant context, where the form would seem more familiar.

5.H. **Variations.** We continue to assume $R \longrightarrow k$ has Gorenstein duality of shift a, and we could also have considered the implications of Gorenstein duality for $R \longrightarrow K$ (which is of shift $a + 1$). This would take the form

$$R^*(X) \sim \Sigma^{a+1}\mathrm{Hom}_K(R_*(X), K)$$

where the symbol $\sim$ indicates that on the right Hom_K and Ext_K will be involved when $\widehat{R}_*(X)$ is not projective over K. The torsion requirement on $R_*(X)$ would now refer not to the maximal ideal but to the ideal $J = \ker(R_* \longrightarrow K)$.

More explicitly, if $R_*(X)$ is J-power torsion, there is a short exact sequence

$$0 \longrightarrow \Sigma^a\mathrm{Ext}_K(R_*(X), K) \longrightarrow R^*(X) \longrightarrow \Sigma^{a+1}\mathrm{Hom}_K(R_*(X), K) \longrightarrow 0.$$

We have focused on connective theories, and hence obtained universal coefficient theorems when $R_*(X)$ is torsion. It is explained in [9] that when R is Gorenstein we may nullify K to form a new theory $\widehat{R}$, which can be thought of as splicing together R and $\mathrm{Cell}_K R$, and which will usually not be connective. For example if $R = ku$, with $a = -4$ then $\widehat{R} = KU$, if $R = \mathrm{tmf}$ with $a = -23$ then $\widehat{R} = \mathrm{Tmf}$, and if $R = C^*(BG)$ then $\widehat{R}$ is the fixed points of the Tate spectrum of G.

In favourable cases when $R \longrightarrow K$ has Gorenstein duality with shift $a+1$ then we obtain an Anderson self-duality statement for $\widehat{R}$.

$$\widehat{R} \simeq \Sigma^{a+2} K^{\widehat{R}}$$

This then gives a Universal Coefficient Theorem

$$\widehat{R}^*(X) \sim \Sigma^{a+2} \mathrm{Hom}_K(\widehat{R}_*(K), K).$$

The symbol $\sim$ has the same meaning as above, but there is now no torsion requirement on $\widehat{R}_*(X)$.

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