

FOR WHICH p -ADIC INTEGERS x CAN $\sum_k \binom{x}{k}^{-1}$ BE DEFINED?

DONALD M. DAVIS

ABSTRACT. Let $f(n) = \sum_k \binom{n}{k}^{-1}$. First, we show that $f : \mathbb{N} \rightarrow \mathbb{Q}_p$ is nowhere continuous in the p -adic topology. If x is a p -adic integer, we say that $f(x)$ is p -definable if $\lim f(x_j)$ exists in \mathbb{Q}_p , where x_j denotes the j th partial sum for x . We prove that $f(-1)$ is p -definable for all primes p , and if p is odd, then -1 is the only element of $\mathbb{Z}_p - \mathbb{N}$ for which $f(x)$ is p -definable. For $p = 2$, we show that if k is a positive integer, then $f(-k - 1)$ is not 2-definable, but that if the 1's in the binary expansion of x are eventually very sparse, then $f(x)$ is 2-definable.

Some of our proofs require that p satisfy one of two conditions. There are three small primes which do not satisfy the relevant condition, but our theorems can be proved directly for these primes. No other prime less than 100,000,000 fails to satisfy the conditions.

1. STATEMENT OF RESULTS

Let $\mathbb{N} \subset \mathbb{Z}_p \subset \mathbb{Q}_p$ denote the natural numbers (including 0), p -adic integers, and p -adic numbers, respectively, with metric $d_p(x, y) = p^{-\nu_p(x-y)}$. Here and throughout, $\nu_p(-)$ denotes the exponent of p in a rational or p -adic number.

The function $f : \mathbb{N} \rightarrow \mathbb{Q}_p$ defined by

$$f(n) = \sum_{k=0}^n \binom{n}{k}^{-1}$$

has been studied in [2], [3], [6], and [7]. Throughout this paper, f will always refer to this function. Cursory calculations suggested that perhaps this f might be continuous in the 2-adic topology and extendable over \mathbb{Z}_2 . For example, `Maple` computes

Proposition 1.1. *If $\nu_2(m+2) = \nu_2(n+2) \geq 4$ and $m, n < 8000$, then*

$$\nu_2(f(m) - f(n)) = \nu_2(m - n) + 1 - 2\nu_2(m + 2).$$

Date: August 6, 2012.

Key words and phrases. binomial coefficients, p -adic integers.

2000 Mathematics Subject Classification: 05A10, 11B65, 11D88.

If $n \equiv 14 \pmod{16}$ and $n < 70,000$, then $\nu_2(f(n)) = 1 - \nu_2(n + 2)$.

If this persisted without the bounds, then f would extend to a continuous function on $\mathbb{Z}_2 \cap \{x : x \equiv 14 \pmod{16}, x \neq -2\}$. However these bounds are not large enough to reveal the problems that can occur.

For technical reasons, some of our results involve the notion of Wieferich primes. Recall that a *Wieferich prime* p is one for which p^2 divides $2^{p-1} - 1$. The only known Wieferich primes are 1093 and 3511. Let LW denote the set of Wieferich primes greater than 3511. The name of this set refers to ‘‘large Wieferich.’’ As of May 2012, it was known that LW contains no integers less than $17 \cdot 10^{15}$.([4])

Others of our results require that p satisfies that for all n such that $1 \leq n \leq p - 2$, $\nu_p(f(n)) \leq 1$. We will call such primes *good*. Paul Zimmermann has tested all $p < 10^8$ and found that in this range there is only one case, $p = 23$ and $n = 12$, in which $\nu_p(f(n)) > 1$, with $\nu_{23}(f(12)) = 2$. Thus all primes less than 100,000,000 are good except for $p = 23$.

We will prove the following theorem in Section 2. Its corollary is the first of our main results.

Theorem 1.2. *a. If $k \geq 1$ and $e > \max\{1, j + \nu_2(j) : 0 < j < k\}$, then*

$$\nu_2(f(2^e - k - 1)) = k + \nu_2(k) - e.$$

b. If p is an odd prime which is not a Wieferich prime, $k \geq 1$, and $e > \max\{1, 1 + \nu_p(j) : 0 < j < k\}$, then

$$(1.3) \quad \nu_p(f(p^e - k - 1)) = \nu_p(k) + 1 - e.$$

c. If $p = 1093$ or 3511 , $k \geq 1$, and $e > \max\{1, 2 + \nu_p(j) : 0 < j < k\}$, then

$$(1.4) \quad \nu_p(f(p^e - k - 1)) = \nu_p(k) + 2 - e.$$

Corollary 1.5. *If $p \notin LW$, then the function $f : \mathbb{N} \rightarrow \mathbb{Q}_p$ is nowhere continuous (in the p -adic topology).*

Proof. We give the proof when $p = 2$. The proof when p is odd is extremely similar, using (1.3) and (1.4).

Let $n \in \mathbb{N}$, and let $\epsilon > 0$ be given. We will show there exists $m \in \mathbb{N}$ such that $d_2(m, n) < \epsilon$ and $d_2(f(m), f(n)) > 1$. Choose e so that $2^{-e} < \epsilon$ and $2^e > n + 1$. By Theorem 1.2, if $L > \max\{j + \nu_2(j) : 0 < j < 2^e - n - 1\}$, then

$$\nu_2(f(2^L - (2^e - n))) = 2^e - n - 1 + \nu_2(n + 1) - L.$$

Choose L large enough that this is less than $\nu_2(f(n))$ and less than 0. Call this value $-r$ with $r \geq 1$. Then $\nu_2(f(2^L - (2^e - n)) - f(n)) = -r$ and so

$$d_2(f(2^L - (2^e - n)), f(n)) = 2^r \geq 2,$$

while $d_2(2^L - (2^e - n), n) = 2^{-e} < \epsilon$. So $m = 2^L - (2^e - n)$ has the desired properties. \square

Since \mathbb{N} is dense in \mathbb{Z}_p , this of course implies that f cannot be extended to a function \bar{f} that is continuous at even one point of \mathbb{Z}_p . We state this, but omit the elementary and standard proof.

Corollary 1.6. *If $p \notin LW$ and $x \in \mathbb{Z}_p$, it is impossible to define $\bar{f}(x)$ so that for all sequences $\langle n_i \rangle$ in \mathbb{N} such that $n_i \rightarrow x$ we have $f(n_i) \rightarrow \bar{f}(x)$.*

We remark that the summand functions $f_k(x) := \binom{x}{k}^{-1} = \frac{k!}{x(x-1)\cdots(x-(k-1))}$ are continuous on $\mathbb{Z}_p - \{0, \dots, k-1\}$.

Although $\bar{f}(x)$ cannot be defined so that it works nicely for all sequences of positive integers approaching x , it might happen that it can be defined so that it is the limit of the most natural sequence of positive integers approaching x , namely the finite partial sums.

Definition 1.7. *For a p -adic integer $x = \sum_{i=0}^{\infty} \epsilon_i p^i$ with $\epsilon_i \in \{0, \dots, p-1\}$, let $x_n = \sum_{i=0}^n \epsilon_i p^i$. Say $\bar{f}(x)$ is p -definable if $\lim f(x_n)$ exists in \mathbb{Q}_p . If so, define $\bar{f}(x) = \lim f(x_n)$.*

The following result points to a similarity and a difference between the prime 2 and the odd primes.

Theorem 1.8. *For all primes p , $\bar{f}(-1)$ is p -definable. If $p = 2$, then $\bar{f}(-1) = 0$, while if p is odd, then $\bar{f}(-1) \equiv 1 \pmod{p}$.*

Proof. We will prove in Proposition 4.7 that $\nu_2(f(2^e - 1)) \geq 2e$, from which the result for $p = 2$ is immediate since $2^e - 1 = (-1)_{e-1}$ in the notation of Definition 1.7. Now let p be an odd prime. We will prove in Proposition 2.6 that, for $e \geq 1$, $\nu_p(f(p^e - 1) - f(p^{e-1} - 1)) \geq e$. Thus $\langle f(p^e - 1) \rangle$ is a Cauchy sequence, and since $p^e - 1 = (-1)_{e-1}$, we deduce that $\bar{f}(-1)$ is p -definable. Since $f(p^0 - 1) = 1$, we obtain $f(p^e - 1) \equiv 1 \pmod{p}$ for all $e \geq 0$. \square

We show now that, for all primes $p \notin LW$, $\bar{f}(x)$ is not p -definable if x is an integer less than -1 .

Theorem 1.9. *Let k be a positive integer and $p \notin LW$. Then $\bar{f}(-k - 1)$ is not p -definable.*

Proof. We give the argument when p is odd. The argument when $p = 2$ is extremely similar. Let $x = -k - 1$. Let $e > \max\{1, \delta + \nu_p(j) : 0 < j < k\}$, where $\delta = 1$ if p is not Wieferich, and $\delta = 2$ if $p \in \{1093, 3511\}$. Then $x_{e-1} = p^e - k - 1$ and, by Theorem 1.2, $\nu_p(f(x_{e-1})) = \delta + \nu_p(k) - e$. Hence, as $e \rightarrow \infty$,

$$d_p(f(x_{e-1}), 0) = p^{e-\delta-\nu(k)} \rightarrow \infty.$$

Thus $\langle f(x_n) \rangle$ is not a Cauchy sequence. \square

The following result applies only when p is odd; the analogous statement is not true when $p = 2$. It says that -1 is the only element of $\mathbb{Z}_p - \mathbb{N}$ for which $\bar{f}(x)$ is p -definable when p is a good odd prime or 23. As noted earlier, this includes all odd primes less than 100,000,000.

Theorem 1.10. *If p is a good odd prime or $p = 23$ and $x \in \mathbb{Z}_p - \{-1, 0, 1, 2, \dots\}$, then $\bar{f}(x)$ is not p -definable.*

Proof. We will prove in Section 3 that if p is a good odd prime or 23 and $1 \leq c \leq p-1$ and $0 \leq i < p^e - 1$, then

$$(1.11) \quad \nu_p(f(c \cdot p^e + i) - f(i)) \leq 0.$$

If x is as in the theorem, then there exist infinitely many e for which (1.11) applies directly to say $\nu_p(f(x_e) - f(x_{e-1})) \leq 0$. Thus $d_p(f(x_e), f(x_{e-1})) \geq 1$ and so $\langle f(x_n) \rangle$ is not a Cauchy sequence. \square

Our other main theorem says that $\bar{f}(x)$ is 2-definable for those x whose infinitely many 1's are eventually very sparse. This is a major difference between the prime 2 and the odd primes. This theorem will be proved in Section 4.

Theorem 1.12. *Suppose $x = \sum_{i=0}^{\infty} 2^{e_i}$, $e_0 < e_1 < \dots$, and there exists a positive integer N such that $e_k > k + \sum_{i=0}^{k-1} 2^{e_i}$ for all $k \geq N$. Then $\bar{f}(x)$ is 2-definable.*

For example,

$$\bar{f}(1 + 2^3 + 2^{12} + 2^{2^{12}+13} + 2^{2^{4109}+4110} + \dots)$$

is 2-definable.

Since negative integers are the 2-adic integers with only a finite number of 0's in their binary expansion, Theorems 1.9 and 1.12 tell whether $\bar{f}(x)$ is 2-definable for x at two extremes. In Theorem 4.10 we describe 2-adic integers x with infinitely many 0's which are eventually very sparse for which $\bar{f}(x)$ is not 2-definable. It would be interesting to know more completely which 2-adic integers x have the property that $\bar{f}(x)$ is 2-definable.

2. SOME p -EXPONENTS OF $f(n)$

In this section, we prove Theorem 1.2 and a result, Proposition 2.6, which was used above and will be used again later. We begin the proof of Theorem 1.2 with a special case.

Lemma 2.1. *Let p be a prime not in LW , and $e \geq 2$. Then*

$$\nu_p(f(p^e - 2)) = \begin{cases} -(e - 2) & p \in \{1093, 3511\} \\ -(e - 1) & \text{otherwise.} \end{cases}$$

Proof. Let $\alpha_p(n)$ denote the sum of the coefficients in the base- p expansion of n . For $0 \leq i \leq p^e - 2$,

$$\begin{aligned} \nu_p \binom{p^e - 2}{i} &= \frac{1}{p-1} (\alpha_p(i) + \alpha_p(p^e - 2 - i) - \alpha_p(p^e - 2)) \\ &= \frac{1}{p-1} (\alpha_p(i) + (p-1)e - \alpha_p(i+1) - ((p-1)e - 1)) \\ &= \frac{1}{p-1} ((\alpha_p(i+1) - 1 + (p-1)\nu_p(i+1)) \\ &\quad + (p-1)e - \alpha_p(i+1) - (p-1)e + 1) \\ &= \nu_p(i+1). \end{aligned}$$

This equals $e - 1$ when $i = c \cdot p^{e-1} - 1$ for $1 \leq c \leq p - 1$, and is less than $e - 1$ for all other relevant i . The result when $p = 2$ follows since there is a single term of smallest exponent in the sum which defines $f(2^e - 2)$.

Now let p be odd. We will show that for $1 \leq c \leq p - 1$

$$(2.2) \quad \binom{p^e - 2}{c \cdot p^{e-1} - 1} / p^{e-1} \equiv (-1)^{c+1} c \pmod{p}.$$

Then we obtain

$$f(p^e - 2) = \frac{1}{p^{e-1}} \left(\sum_{c=1}^{p-1} (-1)^{c+1} \frac{1}{c} + Ap \right) + \sum_{j < e-1} \frac{1}{p^j} A_j,$$

where $A, A_j \in \mathbb{Z}_p$. By a result of Eisenstein ([1])

$$(2.3) \quad \sum_{c=1}^{p-1} (-1)^{c+1} \frac{1}{c} \equiv \frac{2^p - 2}{p} \pmod{p}.$$

If p is not a Wieferich prime, then $\nu_p(\frac{2^p - 2}{p}) = 0$ and so $\nu_p(f(p^e - 2)) = -(e - 1)$.

To prove (2.2), we note that the LHS equals

$$\frac{(p^e - 2) \cdots (p^e - cp^{e-1})}{2 \cdots (cp^{e-1} - 1)p^{e-1}} \equiv \frac{-2}{2} \cdots \frac{-(cp^{e-1} - 1)}{cp^{e-1} - 1} \cdot (-c)$$

mod p . There are $cp^{e-1} - 2$ of the fractions equal to -1 , which when multiplied together and by $-c$ give the desired $(-1)^{c+1}c$.

If $p = 1093$ or 3511 , we consider $p^{e-1}f(p^e - 2) \pmod{p^2}$. This equals

$$2p^{e-1} \sum_{i=1}^{(p^2-1)/2} \binom{p^e - 2}{i \cdot p^{e-2} - 1}^{-1}.$$

Similarly to the above argument for (2.2), we can show that

$$\frac{p^{e-1}}{\binom{p^{e-2}}{i \cdot p^{e-2}-1}} \equiv (-1)^{i-\lfloor \frac{i}{p} \rfloor + 1} \frac{p}{i \binom{p-1}{\lfloor i/p \rfloor}} \pmod{p^2}.$$

Maple computes

$$\sum_{i=1}^{(p^2-1)/2} (-1)^{i-\lfloor \frac{i}{p} \rfloor + 1} \frac{p}{i \binom{p-1}{\lfloor i/p \rfloor}} \pmod{p^2}$$

to equal $487 \cdot 1093$ when $p = 1093$ and $51 \cdot 3511$ if $p = 3511$. Thus $\nu_p(p^{e-1} f(p^e - 2)) = 1$ when $p \in \{1093, 3511\}$. \square

Proof of Theorem 1.2. The recursive formula

$$(2.4) \quad f(n) = \frac{n+1}{2n} f(n-1) + 1$$

was proved in [2]. We invert it to obtain

$$(2.5) \quad f(n-1) = (f(n) - 1) \frac{2n}{n+1}.$$

Our proof of Theorem 1.2 is by induction on k , with Lemma 2.1 being the case $k = 1$. Assume the theorem has been proved for $k - 1$. Let

$$\bar{n} = \begin{cases} n & p = 2 \\ 1 & p \text{ odd, not Wieferich} \\ 2 & p \in \{1093, 3511\}. \end{cases}$$

Then, with $\nu_p(u) = \nu_p(u') = 0$,

$$f(p^e - k) - 1 = \frac{u \cdot p^{\nu_p(k-1) + \bar{k} - 1}}{p^e} - 1 = u' \cdot p^{\nu_p(k-1) + \bar{k} - 1 - e},$$

since $e > \nu_p(k-1) + \bar{k} - 1$. Now, using (2.5),

$$f(p^e - k - 1) = u' \cdot p^{\nu_p(k-1) + \bar{k} - 1 - e} \cdot \frac{2(p^e - k)}{p^e - k + 1} = u'' \cdot p^{\nu_p(k) + \bar{k} - e},$$

with $\nu_p(u'') = 0$, since $\nu_p(p^e - k) = \nu_p(k)$ and $\nu_p(p^e - k + 1) = \nu_p(k - 1)$. \square

The following proposition was used in the proof of 1.8 and will be used in the proof of 3.2.

Proposition 2.6. *For any odd prime p and $e \geq 1$ and $1 \leq c \leq p - 1$,*

$$f(cp^e - 1) - f(cp^{e-1} - 1) \equiv c(1 - 2^{p-1})p^{e-1} f(cp^{e-1} - 1) \pmod{p^{e+1}}.$$

Proof. For any $1 \leq j \leq (p-1)/2$, we have

$$\begin{aligned} & \binom{cp^e - 1}{ip + 2j - 1}^{-1} + \binom{cp^e - 1}{ip + 2j}^{-1} = \binom{cp^e}{ip + 2j} / \left(\binom{cp^e - 1}{ip + 2j - 1} \binom{cp^e - 1}{ip + 2j} \right) \\ & = \frac{cp^e}{ip + 2j} / \binom{cp^e - 1}{ip + 2j} \equiv \frac{cp^e}{2j} / \left(\binom{cp^{e-1} - 1}{i} \binom{p-1}{2j} \right) \pmod{p^{e+1}}. \end{aligned}$$

The case $c = 1$, $i = 0$ says

$$\binom{p^e - 1}{2j - 1}^{-1} + \binom{p^e - 1}{2j}^{-1} \equiv \frac{p^e}{2j} / \binom{p-1}{2j} \pmod{p^{e+1}}.$$

Combining these, we obtain

$$\binom{cp^e - 1}{ip + 2j - 1}^{-1} + \binom{cp^e - 1}{ip + 2j}^{-1} \equiv c \binom{cp^{e-1} - 1}{i}^{-1} \left(\binom{p^e - 1}{2j - 1}^{-1} + \binom{p^e - 1}{2j}^{-1} \right) \pmod{p^{e+1}}.$$

Summing this over j gives

$$\begin{aligned} \sum_{k=1}^{p-1} \binom{cp^e - 1}{ip + k}^{-1} & \equiv c \binom{cp^{e-1} - 1}{i}^{-1} \sum_{k=1}^{p-1} \binom{p^e - 1}{k}^{-1} \\ & \equiv c(1 - 2^{p-1})p^{e-1} \binom{cp^{e-1} - 1}{i}^{-1} \pmod{p^{e+1}}, \end{aligned}$$

using Lemma 2.9 at the last step. Now sum over i to obtain

$$(2.7) \quad \sum_{k \neq 0 \pmod{p}} \binom{cp^e - 1}{k}^{-1} \equiv c(1 - 2^{p-1})p^{e-1} f(cp^{e-1} - 1) \pmod{p^{e+1}}.$$

Summing $\binom{cp^{e-1} - 1}{pj}^{-1} - \binom{cp^{e-1} - 1}{j}^{-1}$ over j gives

$$\sum_{k \equiv 0 \pmod{p}} \binom{cp^e - 1}{k}^{-1} - f(cp^{e-1} - 1) \equiv 0 \pmod{p^{e+2}}$$

by Lemma 2.8. Add this to (2.7) to obtain the claim of the proposition. \square

The above proof required the following lemmas.

Lemma 2.8. *If $1 < u < p$ and $j < up^{e-1}$, and $b = ap + u$ with $a \geq 0$, then*

$$\nu_p \left(\binom{bp^e - 1}{pj}^{-1} - \binom{bp^{e-1} - 1}{j}^{-1} \right) \geq e + 2.$$

Proof. Note that

$$\begin{aligned} & \binom{bp^e - 1}{pj}^{-1} - \binom{bp^{e-1} - 1}{j}^{-1} \\ &= \frac{(pj)(pj-1)\cdots 1}{(bp^e - pj)(bp^e - (pj-1))\cdots (bp^e - 1)} - \frac{j(j-1)\cdots 1}{(bp^{e-1} - j)(bp^{e-1} - (j-1))\cdots (bp^{e-1} - 1)}. \end{aligned}$$

When the factors in the numerator and denominator of the second quotient are multiplied by p , they just give the p -divisible factors of the first quotient. Thus the expression equals $\left(\prod_{\substack{i \\ bp^{e-1} \nmid i}} \frac{i}{bp^e - i} - 1\right) \cdot \binom{bp^{e-1} - 1}{j}^{-1}$, where the product is taken over $0 < i < pj$ with $i \not\equiv 0 \pmod p$. Since $\nu_p\left(\binom{bp^{e-1} - 1}{j}\right) = 0$ and the number of values of i is even, this has the same p -exponent as $\prod i - \prod(i - bp^e)$, and this is divisible by p^{e+2} .

To see this, we will show that

$$\prod_{i=1}^{p-1} (pk + i) \equiv \prod_{i=1}^{p-1} (pk + i - bp^e) \pmod{p^{e+2}}.$$

The difference, mod p^{e+2} , is, up to unit multiples,

$$\begin{aligned} & bp^e \sigma_{p-2}(pk + 1, \dots, pk + p - 1) + (bp^e)^2 \sigma_{p-3}(pk + 1, \dots, pk + p - 1) \\ &= bp^e \sigma_{p-2}(1, \dots, p - 1) + (bp^e \cdot pk + (bp^e)^2) \sigma_{p-3}(1, \dots, p - 1). \end{aligned}$$

This is 0 since the coefficient of x (resp. x^2) in $(x+1)\cdots(x+p-1)$ is divisible by p^2 (resp. p). Indeed, the first is $(p-1)!(1 + \cdots + \frac{1}{p-1})$, and this is divisible by p^2 for $p > 3$ by [5]. Mod p , the polynomial equals $(x^2 - 1^2)(x^2 - 2^2)\cdots(x^2 - (\frac{p-1}{2})^2)$, and its coefficient of x^2 is congruent to $1 + \frac{1}{4} + \cdots + \frac{1}{((p-1)/2)^2}$. Since $i^2 \equiv (p-i)^2 \pmod p$,

this is congruent to $\frac{1}{2} \sum_{i=1}^{p-1} \frac{1}{i^2}$, and this is 0 mod p , also by [5]. \square

Lemma 2.9. *For any odd prime p ,*

$$\sum_{k=1}^{p-1} \binom{p^e - 1}{k}^{-1} \equiv p^{e-1}(1 - 2^{p-1}) \pmod{p^{e+1}}.$$

Proof. Noting that $\binom{p^e - 1}{i} \equiv (-1)^i \pmod p$, and arguing as in the proof of 2.6, we obtain

$$\sum_{k=1}^{p-1} \binom{p^e - 1}{k}^{-1} = \sum_{j=1}^{(p-1)/2} \frac{p^e}{2j} \binom{p^e - 1}{2j}^{-1} \equiv \frac{p^e}{2} \sum_{j=1}^{(p-1)/2} \frac{1}{j} \pmod{p^2}.$$

Next note that, mod p

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} = \sum_{j=1}^{(p-1)/2} \left(\frac{1}{p-(2j-1)} - \frac{1}{2j-1} \right) \equiv \sum_{j=1}^{(p-1)/2} \frac{-1}{\frac{p-1}{2} + j} \equiv \sum_{j=1}^{(p-1)/2} \frac{1}{j}.$$

Using (2.3), we obtain the claim of the lemma. \square

3. RESULTS FOR GOOD PRIMES AND 23

In this section, we prove (1.11) (and hence Theorem 1.10), deferring most details of the proof for $p = 23$ to Section 5. We will prove the following result in Section 5.

Proposition 3.1. *If p is any odd prime, $1 \leq c \leq p-1$, and $\nu_p(f(c-1)) > 0$, then*

$$f(cp-1) - f(c-1) \equiv c(1-2^{p-1})f(c-1) \pmod{p^3}.$$

The following corollary follows easily from this and Proposition 2.6.

Corollary 3.2. *If p is an odd prime, $\nu_p(f(c-1)) \leq 2$ and $e \geq 0$, then $\nu_p(f(cp^e-1)) = \nu_p(f(c-1))$.*

Proof. Since $1-2^{p-1}$ is divisible by p , the result when $\nu_p(f(c-1)) \leq 1$ follows by induction on e using Proposition 2.6. The result when $\nu_p(f(c-1)) = 2$ follows in the same way, using Proposition 3.1 for the first step. \square

Now we can prove the following result, which implies (1.11) for good primes, since $\nu_p(i+1) \leq e-1$ here.

Proposition 3.3. *If p is odd, $\nu_p(f(c-1)) \leq 1$, $0 \leq i < p^e-1$, and $1 \leq c \leq p-1$, then*

$$(3.4) \quad \nu_p(f(cp^e+i) - f(i)) = -e + \nu_p(i+1) + \nu_p(f(c-1)).$$

Proof. Let e and c be fixed, and let $\Delta(i) := f(cp^e+i) - f(i)$. The proof is by induction on i , using (2.4). From (2.4) we obtain

$$f(cp^e) - 1 = \frac{cp^e + 1}{2cp^e} f(cp^e - 1),$$

which, with Corollary 3.2, yields the claim for $i = 0$. From (2.4), we also deduce

$$(3.5) \quad \Delta(i) = \frac{cp^e + i + 1}{2(cp^e + i)} \Delta(i-1) - \frac{cp^e}{2i(cp^e + i)} f(i-1).$$

Assume the result for $i - 1$. Then the first term in the RHS of (3.5) has exponent $\nu_p(i + 1) - \nu_p(i) + (-e + \nu_p(i) + \nu_p(f(c - 1)))$, which is the desired value. The exponent of the second term in the RHS of (3.5) is

$$e - 2\nu_p(i) + \nu_p(f(i - 1)) \geq e - 2\nu_p(i) - \max_j(\nu_p(i_j^{i-1})).$$

We will know that the second term has larger exponent than the first once we have shown that, for $1 \leq i < p^e - 1$ and $j \leq i - 1$,

$$2e - 1 > 2\nu_p(i) + \nu_p(i + 1) + \nu_p(i_j^{i-1}).$$

This follows easily from the fact that $\nu_p(i_j^{i-1}) \leq e - 1 - \nu_p(i)$, since it equals the number of carries in a base- p addition whose sum is $i - 1$. \square

We will prove the following result in Section 5. The implication of this is that information about the case $e = 1$ yields similar information for all e .

Lemma 3.6. *If $1 \leq c, u \leq p - 1$ and $\nu_p(f(c - 1)) = 2$, then for all $e \geq 1$, the expression $f(cp^e + up^{e-1} - 1) - f(up^{e-1} - 1)$ is divisible by p and its residue mod p^2 is independent of e .*

Now we can prove the following analogue of Proposition 3.3.

Proposition 3.7. *Suppose $\nu_p(f(c - 1)) = 2$ with $1 \leq c \leq p - 1$, and for all u which satisfy $1 \leq u \leq p - 1$ and $\nu_p(f(u - 1)) = 0$ we have $\nu_p(f(cp + u - 1) - f(u - 1)) = 1$ and*

$$(3.8) \quad \frac{1}{p}(f(cp + u - 1) - f(u - 1)) \not\equiv \frac{c}{u}f(u - 1) \pmod{p}.$$

Then, for all $e \geq 3$ and all i satisfying $0 \leq i < p^e - 1$, (3.4) holds.

Suppose p is a prime which is not good but has $\nu_p(f(c - 1)) \leq 2$ for all $1 \leq c \leq p - 1$ and whenever $\nu_p(f(c - 1)) = 2$ the hypotheses of Proposition 3.7 are satisfied. Then (3.4) holds for $e \geq 3$ and hence so does (1.11). `Maple` easily verifies that the conditions of Proposition 3.7 are satisfied when $p = 23$ and $c = 13$. Hence (1.11), and thus also Theorem 1.10, holds when $p = 23$. Note that the proof of Theorem 1.10 only cares about large values of e in (1.11), and so our restriction to $e \geq 3$ is not a problem.

Proof of Proposition 3.7. As in the proof of Proposition 3.3, fix c and e , and let $\Delta(i) = f(cp^e + i) - f(i)$. If we assume (3.4) holds for $i - 1$, the exponent of the first

term of (3.5) equals the desired value of $\nu_p(\Delta(i))$. Now that $\nu(f(c-1)) = 2$, it can happen that the second term has the same exponent if $i = up^{e-1} - 1$ or up^{e-1} , in which case it could conceivably happen that the exponent of the combination of the two terms is larger than that of the individual terms.

In the next paragraph, we will use the hypothesis to prove directly that $\nu_p(\Delta(up^{e-1} - 1))$ and $\nu_p(\Delta(up^{e-1}))$ have the desired values when $1 \leq u \leq p-1$. The validity for $\Delta(0)$ is verified using (2.4) as in the proof of 3.3. For $0 \leq u \leq p-1$, the induction from up^{e-1} through $(u+1)p^{e-1} - 2$ works just as it did in the proof of 3.3. Thus the result is valid for all $i < p^e - 1$, as claimed.

That $\nu_p(\Delta(up^{e-1} - 1))$ has the desired value 1 is immediate from Lemma 3.6 and the hypothesis that $\nu_p(\Delta(u-1)) = 1 \pmod{p}$, (3.5) gives

$$\begin{aligned} 2up^{e-2}\Delta(up^{e-1}) &\equiv \frac{1}{p}\Delta(up^{e-1} - 1) - \frac{c}{u}f(up^{e-1} - 1) \\ &\equiv \frac{1}{p}(f(cp + u - 1) - f(u - 1)) - \frac{c}{u}f(u - 1), \end{aligned}$$

using Lemma 3.6 and Proposition 2.6 at the last step. By Proposition 3.7, the last expression has $\nu_p(-) = 0$, and so $\Delta(up^{e-1})$ has the claimed value of $-(e-2)$. \square

4. 2-PRIMARY RESULTS

In this section, we prove Theorem 1.12, which is a major difference between the situation when $p = 2$ and the odd primes. We also state and prove a theorem, 4.10, about 2-adic integers x with infinitely many, but sparse, 0's for which $\bar{f}(x)$ is not 2-definable.

Our proof of Theorem 1.12 will use the following proposition.

Proposition 4.1. *For $n \geq 1$, $\nu_2\left(\sum_{j=1}^n \frac{1}{2^{j-1}}\right) = 2\nu_2(n)$.*

Proof. The result when n is odd follows from the result for $n-1$, since we are adding a number with $\nu_2 = 0$ to one with $\nu_2 > 0$. We will use that

$$(4.2) \quad \nu_2\left(\sum_{j=m+1}^n \frac{1}{2^{j-1}}\right) = \nu_2(\sigma_{n-m-1}(2m+1, 2m+3, \dots, 2n-1)),$$

where $\sigma(-)$ denotes an elementary symmetric polynomial, and its arguments are consecutive odd integers. The following lemma about these will be useful. We will prove it after completing the proof of the proposition.

Lemma 4.3. For $e \geq 1$,

$$(4.4) \quad \sigma_{2^e-1}(-2^e+1, -(2^e-3), \dots, -1, 1, \dots, 2^e-3, 2^e-1) = 0,$$

while for $e \geq 2$,

$$(4.5) \quad \nu_2(\sigma_{2^e-2}(-2^e+1), -(2^e-3), \dots, -1, 1, \dots, 2^e-3, 2^e-1) = e-1.$$

We first prove the proposition when $n = 2^e$. Note that, mod 2^{2e+1} ,

$$\begin{aligned} & \sigma_{2^e-1}(2^e - (2^e - 1), \dots, 2^e - 1, 2^e + 1, \dots, 2^e + (2^e - 1)) \\ \equiv & \sigma_{2^e-1}(-2^e + 1, \dots, -1, 1, \dots, 2^e - 1) \\ & + 2 \cdot 2^e \sigma_{2^e-2}(-2^e + 1, \dots, -1, 1, \dots, 2^e - 1) \\ & + 3 \cdot 2^{2e} \sigma_{2^e-3}(-2^e + 1, \dots, -1, 1, \dots, 2^e - 1). \end{aligned}$$

The factors of 2 and 3 occur since when k factors are omitted, there are k ways that the first omission could have been chosen. The third term is 0 mod 2^{2e+1} since $\sigma_{2^e-3}(-2^e + 1, \dots, -1, 1, \dots, 2^e - 1)$ is the sum of $\binom{2^e}{2^e-3}$ odd numbers, and $\binom{2^e}{2^e-3}$ is even. By Lemma 4.3, the first of the three terms is 0 and the second has $\nu_2 = 2^e$, implying the proposition when $n = 2^e$.

We complete the proof by showing that validity for $n = 2^e(2a - 1)$ implies validity for $n = 2^e(2a + 1)$. This will be done by showing

$$(4.6) \quad \nu_2\left(\frac{1}{2^{e+2}a - (2^{e+1} - 1)} + \dots + \frac{1}{2^{e+2}a + (2^{e+1} - 1)}\right) > 2e.$$

This is a sum of reciprocals of consecutive odd integers. The LHS is $\nu_2(\sigma)$, where, mod 2^{2e+4} ,

$$\begin{aligned} \sigma &= \sigma_{2^{e+1}-1}(2^{e+2}a - (2^{e+1} - 1), \dots, 2^{e+2}a + (2^{e+1} - 1)) \\ &\equiv 2 \cdot 2^{e+2}a \sigma_{2^{e+1}-2}(-2^{e+1} + 1, \dots, -1, 1, \dots, 2^{e+1} - 1), \end{aligned}$$

arguing similarly to the previous paragraph. By (4.5), $\nu_2(\sigma) \geq e + 3 + e$. \square

Proof of Lemma 4.3. These are the coefficients of x and x^2 in $\prod_{j=1}^{2^e-1} (x^2 - (2j-1)^2)$.

Thus (4.4) is clear, and the LHS of (4.5) equals $\nu_2(\sigma_{2^e-1-1}(1^2, 3^2, \dots, (2^e-1)^2))$. We prove by induction on e that this equals $e - 1$. It is easily checked when $e = 2$. Since

all arguments are odd, we need

$$\nu_2\left(1 + \frac{1}{3^2} + \cdots + \frac{1}{(2^e - 1)^2}\right) = e - 1.$$

Mod 2^e , $1/j^2 \equiv 1/(2^e - j)^2$. Thus we need

$$\nu_2\left(1 + \frac{1}{3^2} + \cdots + \frac{1}{(2^{e-1} - 1)^2}\right) = e - 2,$$

and this is the induction hypothesis. \square

Our next step toward the proof of Theorem 1.12 is

Proposition 4.7. *If $e \geq 3$, then $\nu_2(f(2^e - 1)) \geq 2e$.*

In fact, we conjecture that $\nu_2(f(2^e - 1)) = 3e - 2$ for $e \geq 4$, but this seems a good bit harder to prove, and not much more useful in proving something like Theorem 1.12.

Proof of Proposition 4.7. For $1 \leq j \leq 2^{e-2} - 1$, let

$$\begin{aligned} p_{e,j} &:= \binom{2^e-1}{2j}^{-1} + \binom{2^e-1}{2j-1}^{-1} + (-1)^{j+1} \left(\binom{2^{e-1}-1}{j}^{-1} + \binom{2^{e-1}-1}{j-1}^{-1} \right) \\ &= \frac{(2j-1)!}{(2^e-1) \cdots (2^e-(2j-1))} \cdot \frac{2^e}{2^e-2j} \\ &\quad + (-1)^{j+1} \frac{(j-1)!}{(2^{e-1}-1) \cdots (2^{e-1}-(j-1))} \cdot \frac{2^{e-1}}{2^{e-1}-j} \\ &= \frac{2^{e-1}}{2^{e-1}-j} \frac{1}{\binom{2^{e-1}-1}{j-1}} \left(\frac{(-1)^j}{(1-2^e)(1-\frac{1}{3}2^e) \cdots (1-\frac{1}{2^{j-1}}2^e)} + (-1)^{j+1} \right) \\ &= u \cdot 2^{e-1-\nu_2(j)} \left(1 + 2^e \left(1 + \frac{1}{3} + \cdots + \frac{1}{2^{j-1}} \right) + 2^{2e} A - 1 \right) \\ &= u \cdot 2^{2e-1-\nu_2(j)} \left(1 + \frac{1}{3} + \cdots + \frac{1}{2^{j-1}} + 2^e A \right). \end{aligned}$$

Here $\nu_2(u) = 0$, and $A \in \mathbb{Z}_2$ since it is a combination of elementary symmetric polynomials whose arguments are fractions with odd denominators. The dots in the second (double) line range over all integers in the range, while in subsequent lines the dots range over odd integers in the range. In going from the second line to the third, we have noted that the even factors in the first fraction, after dividing by 2, give the factors of the second fraction. In going from the third line to the fourth, we have used that $\binom{2^{e-1}-1}{j-1}$ is odd. Using Proposition 4.1 and that $e > \nu_2(j)$, we obtain $\nu_2(p_{e,j}) \geq 2e - 1$.

Similarly we have

$$\begin{aligned} p_{e,\text{mid}} &:= \binom{2^e-1}{2^{e-1}-1}^{-1} - \binom{2^{e-1}-1}{2^{e-2}-1}^{-1} \\ &= \frac{1}{\binom{2^{e-1}-1}{2^{e-2}-1}} \left(\frac{1}{(1-2^e)(1-\frac{1}{3}2^e)\cdots(1-\frac{1}{2^{e-1}-1}2^e)} - 1 \right) \\ &= u'(2^e(1 + \frac{1}{3} + \cdots + \frac{1}{2^{e-1}-1}) + 2^{2e}A'), \end{aligned}$$

satisfying $\nu_2(p_{e,\text{mid}}) \geq 2e - 1$. Thus

$$\nu_2\left(\sum_{j=1}^{2^{e-2}-1} p_{e,j} + p_{e,\text{mid}}\right) \geq 2e - 1.$$

On the other hand,

$$\sum_{j=1}^{2^{e-2}-1} p_{e,j} = \sum_{i=1}^{2^{e-2}-2} \binom{2^e-1}{i}^{-1} + \binom{2^{e-1}-1}{2^{e-2}-1}^{-1} + \binom{2^{e-1}-1}{0}^{-1},$$

since most of the terms with alternating signs cancel. Hence

$$\sum_{j=1}^{2^{e-2}-1} p_{e,j} + p_{e,\text{mid}} = \sum_{i=0}^{2^{e-1}-1} \binom{2^e-1}{i}^{-1},$$

which is $\frac{1}{2}f(2^e - 1)$. Thus $\nu_2(f(2^e - 1)) \geq 2e$, as claimed. \square

Proof of Theorem 1.12. We will use Proposition 4.7 and (2.4) to prove for $i \geq 0$ and $2^e > i$,

$$(4.8) \quad \nu_2(f(2^e + i) - f(i)) \geq e - i - 1.$$

Then, with x as in Theorem 1.12, let $E_k = \sum_{i=0}^k 2^{e_i}$. The hypothesis of the theorem and (4.8) imply that for $k \geq N$, $\nu_2(f(E_k) - f(E_{k-1})) \geq k$, and so $d(f(E_k), f(E_{k-1})) \leq 2^{-k}$. Thus $\langle f(E_k) \rangle$ is a Cauchy sequence and so has a limit in \mathbb{Q}_2 . Thus $\bar{f}(x)$ is definable.

Now we prove (4.8). Let e be fixed, and $\Delta(i) = f(2^e + i) - f(i)$. Using Proposition 4.7, let $f(2^e - 1) = A \cdot 2^{2e}$ with $A \in \mathbb{Z}_2$. Using (2.4) we obtain $\Delta(0) = A(2^e + 1)2^{e-1}$ and

$$\Delta(i) = \frac{2^e + i + 1}{2(2^e + i)} \Delta(i - 1) - \frac{2^{e-1}}{(2^e + i)i} f(i - 1).$$

Applying this iteratively, we obtain

$$(4.9) \quad \Delta(i) = (2^e + i + 1) \left(A2^{e-i-1} - \sum_{j=0}^{i-1} \frac{2^{e+j-i} f(j)}{(j+1)(2^e+j+2)(2^e+j+1)} \right).$$

Thus it suffices to prove $\nu_2(f(j)) \geq -j - 1 + \nu_2((j+1)^2(j+2))$. This can be easily checked for $j \leq 3$. (It is sharp for $j = 0$ and $j = 2$.)

Since $\nu_2\binom{j}{i} \leq [\log_2(j)]$, we deduce $\nu_2(f(j)) \geq -[\log_2(j)]$. Since $j+1 \geq [\log_2(j)] + 2\nu_2(j+1) + \nu_2(j+2)$ for $j \geq 4$, as is easily proved, the desired result follows. \square

Note how a comparison of (4.8) and Proposition 3.3 points to a huge difference between the situations when p is an even or odd prime.

The following theorem describes 2-adic integers x with infinitely many 0's in their binary expansion, which are eventually very sparse, for which $\bar{f}(x)$ is not 2-definable.

Theorem 4.10. *Suppose $x = -1 - \sum_{i \geq 0} 2^{e_i}$ with $e_0 > 0$ and for all $k > 0$*

$$e_k \geq 2^{e_{k-1}} + e_{k-1} + e_0.$$

Then $\bar{f}(x)$ is not 2-definable.

Proof. Let $E_k = 2^{e_k} - 1 - \sum_{i=0}^{k-1} 2^{e_i}$. These are some of the partial sums for x . By Theorem 1.2

$$\nu_2(f(E_k)) = \sum_{i=0}^{k-1} 2^{e_i} + e_0 - e_k.$$

From this, we deduce that $\nu_2(f(E_k)) < \nu_2(f(E_{k-1}))$ and hence $\nu_2(f(E_k) - f(E_{k-1})) = \nu_2(f(E_k))$. Thus

$$d(f(E_k), f(E_{k-1})) > d(f(E_{k-1}), f(E_{k-2})).$$

These distances form an increasing sequence of 2-powers, and so $\langle f(E_k) \rangle$ cannot be a subsequence of a Cauchy sequence. \square

5. PROOFS RELEVANT TO THE CASE $\nu_p(f(c-1)) = 2$

In this section, we prove Proposition 3.1 and Lemma 3.6. The following lemmas will be useful in the proof of Proposition 3.1.

Lemma 5.1. *If $f(c-1) \equiv 0 \pmod{p}$, then $\sum_{i=1}^{c-1} i \binom{c-1}{i}^{-1} \equiv 0 \pmod{p}$.*

Proof. We will prove the stronger result that $\sum i \binom{c-1}{i}^{-1} = \frac{1}{2}(c-1)f(c-1)$. This is deduced from the following, where we use (2.4) in the last step.

$$f(c-1) + \sum i \binom{c-1}{i}^{-1} = c \sum \binom{c}{i+1}^{-1} = c(f(c) - 1) = c \frac{c+1}{2c} f(c-1).$$

□

Lemma 5.2. *If $0 \leq C \leq A < p$ and $0 \leq D \leq B < p$, then, mod p^2 ,*

(5.3)

$$\binom{Ap+B}{Cp+D}^{-1} - \binom{A}{C}^{-1} \binom{B}{D}^{-1} \equiv p \binom{A}{C}^{-1} \binom{B}{D}^{-1} \left(C \sum_{i=1}^D \frac{1}{i} + (A-C) \sum_{i=1}^{B-D} \frac{1}{i} - A \sum_{i=1}^B \frac{1}{i} \right).$$

Proof. Since, for any j , $(pj+p-1) \cdots (pj+1) \equiv (p-1)! \pmod{p^2}$, in evaluating

$$(5.4) \quad \frac{(Cp+D)!((A-C)p+(B-D))!}{(Ap+B)!} \pmod{p^2},$$

all the products which appear in between two multiples of p cancel out. We cancel out a factor of p from all multiples of p in the numerator and denominator of (5.4) and obtain that (5.4) is congruent to

$$\frac{C!(A-C)!}{A!} \cdot \frac{(Cp+D) \cdots (Cp+1)((A-C)p+(B-D)) \cdots ((A-C)p+1)}{(Ap+B) \cdots (Ap+1)}.$$

The second factor here equals $\frac{D!(B-D)!}{B!} \cdot E$, where

$$E = \frac{\left(\frac{C}{D}p+1\right) \cdots \left(\frac{C}{1}p+1\right) \left(\frac{A-C}{B-D}p+1\right) \cdots \left(\frac{A-C}{1}p+1\right)}{\left(\frac{A}{B}p+1\right) \cdots \left(\frac{A}{1}p+1\right)}.$$

The LHS of (5.3) is congruent, mod p^2 , to $\binom{A}{C}^{-1} \binom{B}{D}^{-1} (E-1)$, and this is congruent to the claimed expression. □

Proof of Proposition 3.1. We are assuming that $\sum \binom{c-1}{i}^{-1} \equiv 0 \pmod{p}$. We will prove that for $1 \leq j \leq \frac{p-1}{2}$,

$$(5.5) \quad \sum_{i=0}^{c-1} \left(\frac{1}{ip+2j} \frac{1}{\binom{cp-1}{ip+2j}} - \frac{1}{2j \binom{c-1}{i} \binom{p-1}{2j}} \right) \equiv 0 \pmod{p^2}.$$

Since

$$\binom{cp-1}{ip+2j-1}^{-1} + \binom{cp-1}{ip+2j}^{-1} = \frac{cp}{(ip+2j)\binom{cp-1}{ip+2j}},$$

when (5.5) is summed over j and multiplied by cp , we obtain

$$(5.6) \quad \sum_{k \neq 0 \pmod{p}} \binom{cp-1}{k}^{-1} \equiv c \sum_{k=1}^{p-1} \frac{1}{\binom{p-1}{k}} \sum_{i=0}^{c-1} \binom{c-1}{i}^{-1} \pmod{p^3}.$$

By Lemma 2.8

$$(5.7) \quad \binom{cp-1}{pj}^{-1} - \binom{c-1}{j}^{-1} \equiv 0 \pmod{p^3}.$$

Adding (5.6) and the sum over j of (5.7) and using Lemma 2.9 yields the claim of the proposition.

The proof of (5.5) begins by noting that the LHS equals

$$(5.8) \quad \frac{1}{cp-1} \sum_{i=0}^{c-1} \binom{cp-2}{ip+2j-1}^{-1} - \frac{1}{(p-1)\binom{p-2}{2j-1}} \sum_{i=0}^{c-1} \binom{c-1}{i}^{-1}.$$

Since we are working mod p^2 and assuming $\sum \binom{c-1}{i}^{-1} \equiv 0 \pmod{p}$, we may replace the $(p-1)$ in its coefficient by -1 . We will show

$$(5.9) \quad - \sum_{i=0}^{c-1} \binom{cp-2}{ip+2j-1}^{-1} + \binom{p-2}{2j-1}^{-1} \sum_{i=0}^{c-1} \binom{c-1}{i}^{-1} \equiv 0 \pmod{p^2}.$$

This implies that $\sum_{i=0}^{c-1} \binom{cp-2}{ip+2j-1}^{-1} \equiv 0 \pmod{p}$, and so whether its coefficient is -1 or $\frac{1}{cp-1}$ is irrelevant. Thus (5.9) implies (5.5).

To prove (5.9), we note that -1 times its LHS is a sum of terms of the form of (5.3) with $A = c-1$, $B = p-2$, $C = i$, and $D = 2j-1$. Let $S_m := \sum_{i=1}^m \frac{1}{i}$. By Lemma 5.2, the LHS of (5.9) equals -1 times

$$\begin{aligned} & p \binom{p-2}{2j-1}^{-1} \sum_{i=0}^{c-1} \binom{c-1}{i}^{-1} (iS_{2j-1} + (c-1-i)S_{p-1-2j} - (c-1)S_{p-2}) \\ &= p \binom{p-2}{2j-1}^{-1} \sum_{i=0}^{c-1} \binom{c-1}{i}^{-1} (Ki + L), \end{aligned}$$

where K and L do not depend on i . This is $0 \pmod{p^2}$, using Lemma 5.1 and the assumption that $\nu_p(f(c-1)) > 0$.

□

The remainder of the paper is devoted to the proof of Lemma 3.6. Let $D_e := f(cp^e + up^{e-1} - 1) - f(up^{e-1} - 1)$. We will show that $\nu_p(D_{e+1} - D_e) \geq 2$ for $e \geq 1$. Using (3.5), one easily shows that $\nu_p(D_1) \geq 1$. We deduce that $\nu_p(D_e) \geq 1$ for all e .

Let

$$L_e := \sum_{i < up^{e-1}} \left(\binom{cp^e + up^{e-1} - 1}{i}^{-1} - \binom{up^{e-1} - 1}{i}^{-1} \right), \text{ and } H_e := \sum_{i \geq up^{e-1}} \binom{cp^e + up^{e-1} - 1}{i}^{-1}$$

Then $D_e = L_e + H_e$. We will prove $\nu_p(L_{e+1} - L_e) \geq 2$ and $\nu_p(H_{e+1} - H_e) \geq 2$.

We have

$$(5.10) \quad L_{e+1} - L_e = \sum_{i=0}^{up^{e-1}-1} \left(\binom{cp^{e+1} + up^e - 1}{pi}^{-1} - \binom{cp^e + up^{e-1} - 1}{i}^{-1} \right)$$

$$(5.11) \quad - \left(\binom{up^e - 1}{pi}^{-1} - \binom{up^{e-1} - 1}{i}^{-1} \right)$$

$$(5.12) \quad + \sum_{\substack{k < up^e \\ k \not\equiv 0 \pmod{p}}} \left(\binom{cp^{e+1} + up^e - 1}{k}^{-1} - \binom{up^e - 1}{k}^{-1} \right).$$

Summands of (5.10) and (5.11) are divisible by p^2 by Lemma 2.8. Similarly to the beginning of the proof of 2.6, the sum (5.12) is a sum of terms of the form

$$(5.13) \quad \frac{cp^{e+1} + up^e}{2j} \binom{cp^{e+1} + up^e - 1}{2j}^{-1} - \frac{up^e}{2j} \binom{up^e - 1}{2j}^{-1}$$

with $j \not\equiv 0 \pmod{p}$. This is clearly divisible by p^2 when $e > 1$. If $e = 1$, the cp^{e+1} -part is $0 \pmod{p^2}$, and $\binom{cp^{e+1} + up^e - 1}{2j} \equiv \binom{up^e - 1}{2j} \pmod{p}$, which is good enough, since there is an additional factor of p . Thus (5.13) is $0 \pmod{p^2}$ in this case, too.

Finally,

$$(5.14) = \frac{H_{e+1} - H_e}{\sum_{i=up^{e-1}}^{cp^e+up^{e-1}-1} \binom{cp^{e+1}+up^e-1}{pi}^{-1} - \binom{cp^e+up^{e-1}-1}{i}^{-1}}$$

$$(5.15) + \sum_{\substack{k \geq up^e \\ k \neq 0 \pmod{p}}} \binom{cp^{e+1}+up^e-1}{k}^{-1}.$$

The summands of (5.14) are divisible by p^{e+1} . To see this, we use the proof of Lemma 2.8 but note that the factor $\binom{cp^e+up^{e-1}-1}{i}^{-1}$ will have $\nu_p(-) = -1$.

Similarly to the proof of 3.1, (5.15) is a sum over j of

$$\frac{cp^{e+1}+up^e}{cp^{e+1}+up^e-1} \sum_{i=up^{e-1}}^{cp^e+up^{e-1}-1} \binom{cp^{e+1}+up^e-2}{ip+2j-1}^{-1}.$$

Since the terms in the sum certainly have $\nu_p(-) \geq -1$, we are reduced to proving

$$(5.16) \quad \sum_{i=u}^{cp+u-1} \binom{cp^2+up-2}{ip+2j-1}^{-1} \equiv 0 \pmod{p}, \text{ and}$$

$$(5.17) \quad \nu_p \left(\sum_{i=up}^{cp^2+up-1} \binom{cp^3+up^2-2}{ip+2j-1}^{-1} \right) \geq 0.$$

We begin by proving (5.16).

We split the sum as

$$\sum_{a=1}^c \sum_{b=0}^{u-1} \binom{cp^2+(u-1)p+(p-2)}{ap^2+bp+2j-1}^{-1} + \sum_{a=0}^{c-1} \sum_{b=u}^{p-1} \binom{cp^2+(u-1)p+p-2}{ap^2+bp+2j-1}^{-1}.$$

The first sum, mod p , splits as $\sum_{a=1}^c \binom{c}{a}^{-1} \cdot \sum_{b=0}^{u-1} \binom{u-1}{b}^{-1} \cdot \binom{p-2}{2j-1}^{-1}$. The first factor is $f(c) - 1 = f(c-1) \frac{c+1}{2c} \equiv 0 \pmod{p^2}$, and the other factors are p -integral. With $S_m = \sum_{i=1}^m \frac{1}{i}$ and $\ell = 2j-1$, we can prove, using methods similar to those in the proof of 3.1, that the term in the second sum above is congruent, mod p to

$$\binom{cp+u-1}{ap+b}^{-1} \binom{p-2}{\ell}^{-1} (1 + p(b(S_\ell - S_{p-2-\ell}) + (u-1)(S_{p-2-\ell} - S_{p-2}))).$$

Thus the second sum becomes, mod p ,

$$(5.18) \quad \alpha \sum_{a=0}^{c-1} \sum_{b=u}^{p-1} \binom{cp+u-1}{ap+b}^{-1} + p\beta \sum_{a=0}^{c-1} \sum_{b=u}^{p-1} b \binom{cp+u-1}{ap+b}^{-1}$$

with α and β p -integral.

We can prove that when $b \geq u$, mod p^2 ,

$$p \binom{cp+u-1}{ap+b}^{-1} \equiv \frac{u}{c} \binom{c-1}{a}^{-1} \binom{b}{u} \binom{p-1}{b-u}^{-1} (1 + p(aS_b + (c-a-1)S_{u-b+p-1} - cS_{u-1})),$$

with S_m as above. Thus p times (5.18) is congruent, mod p^2 , to

$$A \sum_{a=0}^{c-1} \binom{c-1}{a}^{-1} + Bp \sum_{a=0}^{c-1} a \binom{c-1}{a}^{-1},$$

with A and B p -integral. Since $\nu_p(f(c-1)) = 2$ and using Lemma 5.1, we obtain that this expression is divisible by p^2 and hence (5.18) is divisible by p , yielding the claim.

The proof of (5.17) is very similar. In fact, the sum here is $\equiv 0 \pmod{p}$, stronger than required. If we write the sum as

$$\sum_{i_0=0}^{p-1} \sum_{t=u}^{cp+u-1} \binom{cp^3 + (u-1)p^2 + (p^2-2)}{tp^2 + (i_0p + \ell)}^{-1}$$

with $\ell = 2j - 1$ as before, then $\binom{p^2-2}{i_0p+\ell}$ behaves here as $\binom{p-2}{\ell}$ did before, and the t -sum is split in the same way as the i -sum before.

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DEPARTMENT OF MATHEMATICS, LEHIGH UNIVERSITY, BETHLEHEM, PA 18015, USA
E-mail address: `dmd1@lehigh.edu`