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A COMPLETE PROOF OF THE POINCARÉ AND GEOMETRIZATION CONJECTURES – APPLICATION OF THE HAMILTON-PERELMAN THEORY OF THE RICCI FLOW*

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Abstract. In this paper, we give a complete proof of the Poincaré and the geometrization conjectures. This work depends on the accumulative works of many geometric analysts in the past thirty years. This proof should be considered as the crowning achievement of the Hamilton-Perelman theory of Ricci flow.

Key words. Ricci flow, Ricci flow with surgery, Hamilton-Perelman theory, Poincaré Conjecture, geometrization of 3-manifolds

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Introduction. In this paper, we shall present the Hamilton-Perelman theory of Ricci flow. Based on it, we shall give the first written account of a complete proof of the Poincaré conjecture and the geometrization conjecture of Thurston. While the complete work is an accumulated efforts of many geometric analysts, the major contributors are unquestionably Hamilton and Perelman.

An important problem in differential geometry is to find a canonical metric on a given manifold. In turn, the existence of a canonical metric often has profound topological implications. A good example is the classical uniformization theorem in two dimensions which, on one hand, provides a complete topological classification for compact surfaces, and on the other hand shows that every compact surface has a canonical geometric structure: a metric of constant curvature.

How to formulate and generalize this two-dimensional result to three and higher dimensional manifolds has been one of the most important and challenging topics in modern mathematics. In 1977, W. Thurston [122], based on ideas about Riemann surfaces, Haken's work and Mostow's rigidity theorem, etc, formulated a geometrization conjecture for three-manifolds which, roughly speaking, states that every compact orientable three-manifold has a canonical decomposition into pieces, each of which admits a canonical geometric structure. In particular, Thurston's conjecture contains, as a special case, the Poincaré conjecture: A closed three-manifold with trivial fundamental group is necessarily homeomorphic to the 3-sphere $\mathbb{S}^3$. In the past thirty years, many mathematicians have contributed to the understanding of this conjecture of Thurston. While Thurston's theory is based on beautiful combination of techniques from geometry and topology, there has been a powerful development of geometric analysis in the past thirty years, lead by S.-T. Yau, R. Schoen, C. Taubes, K. Uhlenbeck, and S. Donaldson, on the construction of canonical geometric structures based on nonlinear PDEs (see, e.g., Yau's survey papers [129, 130]). Such canonical geometric structures include Kähler-Einstein metrics, constant scalar curvature metrics, and self-dual metrics, among others. However, the most important contribution for geometric analysis on three-manifolds is due to Hamilton.

In 1982, Hamilton [58] introduced the Ricci flow

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

to study compact three-manifolds with positive Ricci curvature. The Ricci flow, which evolves a Riemannian metric by its Ricci curvature, is a natural analogue of the heat equation for metrics. As a consequence, the curvature tensors evolve by a system of diffusion equations which tends to distribute the curvature uniformly over the manifold. Hence, one expects that the initial metric should be improved and evolve into a canonical metric, thereby leading to a better understanding of the topology of the underlying manifold. In the celebrated paper [58], Hamilton showed that on a compact three-manifold with an initial metric having positive Ricci curvature, the Ricci flow converges, after rescaling to keep constant volume, to a metric of positive constant sectional curvature, proving the manifold is diffeomorphic to the three-sphere $\mathbb{S}^3$ or a quotient of the three-sphere $\mathbb{S}^3$ by a linear group of isometries. Shortly after, Yau suggested that the Ricci flow should be the best way to prove the structure theorem for general three-manifolds. In the past two decades, Hamilton proved many important and remarkable theorems for the Ricci flow, and laid the foundation for the program to approach the Poincaré conjecture and Thurston's geometrization conjecture via the Ricci flow.

The basic idea of Hamilton's program can be briefly described as follows. For any given compact three-manifold, one endows it with an arbitrary (but can be suitably normalized by scaling) initial Riemannian metric on the manifold and then studies the behavior of the solution to the Ricci flow. If the Ricci flow develops singularities, then one tries to find out the structures of singularities so that one can perform (geometric) surgery by cutting off the singularities, and then continue the Ricci flow after the surgery. If the Ricci flow develops singularities again, one repeats the process of performing surgery and continuing the Ricci flow. If one can prove there are only a finite number of surgeries during any finite time interval and if the long-time behavior of solutions of the Ricci flow with surgery is well understood, then one would recognize the topological structure of the initial manifold.

Thus Hamilton's program, when carried out successfully, will give a proof of the Poincaré conjecture and Thurston's geometrization conjecture. However, there were obstacles, most notably the verification of the so called "Little Loop Lemma" conjectured by Hamilton [63] (see also [17]) which is a certain local injectivity radius estimate, and the verification of the discreteness of surgery times. In the fall of 2002 and the spring of 2003, Perelman [103, 104] brought in fresh new ideas to figure out important steps to overcome the main obstacles that remained in the program of Hamilton. (Indeed, in page 3 of [103], Perelman said "the implementation of Hamilton program would imply the geometrization conjecture for closed three-manifolds" and "In this paper we carry out some details of Hamilton program".) Perelman's breakthrough on the Ricci flow excited the entire mathematics community. His work has since been examined to see whether the proof of the Poincaré conjecture and geometrization program, based on the combination of Hamilton's fundamental ideas and Perelman's new ideas, holds together. The present paper grew out of such an effort.

Now we describe the three main parts of Hamilton's program in more detail.

(i) Determine the structures of singularities

Given any compact three-manifold M with an arbitrary Riemannian metric, one evolves the metric by the Ricci flow. Then, as Hamilton showed in [58], the solution $g(t)$ to the Ricci flow exists for a short time and is unique (also see Theorem 1.2.1). In fact, Hamilton [58] showed that the solution $g(t)$ will exist on a maximal time interval $[0, T)$, where either $T = \infty$, or $0 < T < \infty$ and the curvature becomes unbounded as t tends to T. We call such a solution $g(t)$ a maximal solution of the Ricci flow. If $T < \infty$ and the curvature becomes unbounded as t tends to T, we say the maximal solution develops singularities as t tends to T and T is the singular time.

In the early 1990s, Hamilton systematically developed methods to understand the structure of singularities. In [61], based on suggestion by Yau, he proved the fundamental Li-Yau [82] type differential Harnack estimate (the Li-Yau-Hamilton estimate) for the Ricci flow with nonnegative curvature operator in all dimensions. With the help of Shi's interior derivative estimate [114], he [62] established a compactness theorem for smooth solutions to the Ricci flow with uniformly bounded curvatures and uniformly bounded injectivity radii at the marked points. By imposing an injectivity radius condition, he rescaled the solution to show that each singularity is asymptotic to one of the three types of singularity models [63]. In [63] he discovered (also independently by Ivey [73]) an amazing curvature pinching estimate for the Ricci flow on three-manifolds. This pinching estimate implies that any three-dimensional singularity model must have nonnegative curvature. Thus in dimension three, one only needs to obtain a complete classification for nonnegatively curved singularity models.

For Type I singularities in dimension three, Hamilton [63] established an isoperimetric ratio estimate to verify the injectivity radius condition and obtained spherical or necklike structures for any Type I singularity model. Based on the Li-Yau-Hamilton estimate, he showed that any Type II singularity model with nonnegative curvature is either a steady Ricci soliton with positive sectional curvature or the product of the so called cigar soliton with the real line [66]. (Characterization for nonnegatively curved Type III models was obtained in [30].) Furthermore, he developed a dimension reduction argument to understand the geometry of steady Ricci solitons [63]. In the three-dimensional case, he showed that each steady Ricci soliton with positive curvature has some necklike structure. Hence Hamilton had basically obtained a canonical neighborhood structure at points where the curvature is comparable to the maximal curvature for solutions to the three-dimensional Ricci flow.

However two obstacles remained: (a) the verification of the imposed injectivity radius condition in general; and (b) the possibility of forming a singularity modelled on the product of the cigar soliton with a real line which could not be removed by surgery. The recent spectacular work of Perelman [103] removed these obstacles by establishing a local injectivity radius estimate, which is valid for the Ricci flow on compact manifolds in all dimensions. More precisely, Perelman proved two versions of "no local collapsing" property (Theorem 3.3.3 and Theorem 3.3.2), one with an entropy functional he introduced in [103], which is monotone under the Ricci flow, and the other with a space-time distance function obtained by path integral, analogous to what Li-Yau did in [82], which gives rise to a monotone volume-type (called reduced volume by Perelman) estimate. By combining Perelman's no local collapsing theorem I′ (Theorem 3.3.3) with the injectivity radius estimate of Cheng-Li-Yau (Theorem 4.2.2), one immediately obtains the desired injectivity radius estimate, or the Little Loop Lemma (Theorem 4.2.4) conjectured by Hamilton.

Furthermore, Perelman [103] developed a refined rescaling argument (by considering local limits and weak limits in Alexandrov spaces) for singularities of the Ricci flow on three-manifolds to obtain a uniform and global version of the canonical neighborhood structure theorem. We would like to point out that our proof of the singularity structure theorem (Theorem 7.1.1) is different from that of Perelman in two aspects: (1) we avoid using his crucial estimate in Claim 2 in Section 12.1 of [103]; (2) we give a new approach to extend the limit backward in time to an ancient solution. These differences are due to the difficulties in understanding Perelman's arguments at these points.

(ii) Geometric surgeries and the discreteness of surgery times

After obtaining the canonical neighborhoods (consisting of spherical, necklike and caplike regions) for the singularities, one would like to perform geometric surgery and then continue the Ricci flow. In [64], Hamilton initiated such a surgery procedure for the Ricci flow on four-manifolds with positive isotropic curvature and presented a concrete method for performing the geometric surgery. His surgery procedures can be roughly described as follows: cutting the neck-like regions, gluing back caps, and removing the spherical regions. As will be seen in Section 7.3 of this paper, Hamilton's geometric surgery method also works for the Ricci flow on compact orientable three-manifolds.

Now an important challenge is to prevent surgery times from accumulating and make sure one performs only a finite number of surgeries on each finite time interval. The problem is that, when one performs the surgeries with a given accuracy at each surgery time, it is possible that the errors may add up to a certain amount which could cause the surgery times to accumulate. To prevent this from happening, as time goes on, successive surgeries must be performed with increasing accuracy. In [104], Perelman introduced some brilliant ideas which allow one to find "fine" necks, glue "fine" caps, and use rescaling to prove that the surgery times are discrete.

When using the rescaling argument for surgically modified solutions of the Ricci flow, one encounters the difficulty of how to apply Hamilton's compactness theorem (Theorem 4.1.5), which works only for smooth solutions. The idea to overcome this difficulty consists of two parts. The first part, due to Perelman [104], is to choose the cutoff radius in neck-like regions small enough to push the surgical regions far away in space. The second part, due to the authors and Chen-Zhu [34], is to show that the surgically modified solutions are smooth on some uniform (small) time intervals (on compact subsets) so that Hamilton's compactness theorem can still be applied. To do so, we establish three time-extension results (see Step 2 in the proof of Proposition 7.4.1.). Perhaps, this second part is more crucial. Without it, Shi's interior derivative estimate (Theorem 1.4.2) may not applicable, and hence one cannot be certain that Hamilton's compactness theorem holds when only having the uniform C^0 bound on curvatures. We remark that in our proof of this second part, as can be seen in the proof of Proposition 7.4.1, we require a deep comprehension of the prolongation of the gluing "fine" caps for which we will use the recent uniqueness theorem of Bing-Long Chen and the second author [33] for solutions of the Ricci flow on noncompact manifolds.

Once surgeries are known to be discrete in time, one can complete the classification, started by Schoen-Yau [109, 110], for compact orientable three-manifolds with positive scalar curvature. More importantly, for simply connected three-manifolds, if one can show that solutions to the Ricci flow with surgery become extinct in finite time, then the Poincaré conjecture would follow. Such a finite extinction time result was proposed by Perelman [105], and a proof also appears in Colding-Minicozzi [42]. Thus, the combination of Theorem 7.4.3 (i) and the finite extinction time result provides a complete proof to the Poincaré conjecture.

(iii) The long-time behavior of surgically modified solutions.

To approach the structure theorem for general three-manifolds, one still needs to analyze the long-time behavior of surgically modified solutions to the Ricci flow. In [65], Hamilton studied the long time behavior of the Ricci flow on compact three-manifolds for a special class of (smooth) solutions, the so called nonsingular solutions. These are the solutions that, after rescaling to keep constant volume, have (uniformly) bounded curvature for all time. Hamilton [65] proved that any three-dimensional nonsingular solution either collapses or subsequently converges to a metric of constant curvature on the compact manifold or, at large time, admits a thick-thin decomposition where the thick part consists of a finite number of hyperbolic pieces and the thin part collapses. Moreover, by adapting Schoen-Yau's minimal surface arguments in [110] and using a result of Meeks-Yau [86], Hamilton showed that the boundary of hyperbolic pieces are incompressible tori. Consequently, when combined with the collapsing results of Cheeger-Gromov [24, 25], this shows that any nonsingular solution to the Ricci flow is geometrizable in the sense of Thurston [122]. Even though the nonsingular assumption seems very restrictive and there are few conditions known so far which can guarantee a solution to be nonsingular, nevertheless the ideas and arguments of Hamilton's work [65] are extremely important.

In [104], Perelman modified Hamilton's arguments to analyze the long-time behavior of arbitrary smooth solutions to the Ricci flow and solutions with surgery to the Ricci flow in dimension three. Perelman also argued that the proof of Thurston's geometrization conjecture could be based on a thick-thin decomposition, but he could only show the thin part will only have a (local) lower bound on the sectional curvature. For the thick part, based on the Li-Yau-Hamilton estimate, Perelman [104] established a crucial elliptic type estimate, which allowed him to conclude that the thick part consists of hyperbolic pieces. For the thin part, he announced in [104] a new collapsing result which states that if a three-manifold collapses with (local) lower bound on the sectional curvature, then it is a graph manifold. Assuming this new collapsing result, Perelman [104] claimed that the solutions to the Ricci flow with surgery have the same long-time behavior as nonsingular solutions in Hamilton's work, a conclusion which would imply a proof of Thurston's geometrization conjecture. Although the proof of this new collapsing result promised by Perelman in [104] is still not available in literature, Shioya-Yamaguchi [118] has published a proof of the collapsing result in the special case when the manifold is closed. In the last section of this paper (see Theorem 7.7.1), we will provide a proof of Thurston's geometrization conjecture by only using Shioya-Yamaguchi's collapsing result. In particular, this gives another proof of the Poincaré conjecture.

We would like to point out that Perelman [104] did not quite give an explicit statement of the thick-thin decomposition for surgical solutions. When we were trying to write down an explicit statement, we needed to add a restriction on the relation between the accuracy parameter ε and the collapsing parameter w. Nevertheless, we are still able to obtain a weaker version of the thick-thin decomposition (Theorem 7.6.3) that is sufficient to deduce the geometrization result.

In this paper, we shall give complete and detailed proofs of what we outlined above, especially of Perelman's work in his second paper [104] in which many key ideas of the proofs are sketched or outlined but complete details of the proofs are often missing. As we pointed out before, we have to substitute several key arguments of Perelman by new approaches based on our study, because we were unable to comprehend these original arguments of Perelman which are essential to the completion of the geometrization program.

Our paper is aimed at both graduate students and researchers who want to learn Hamilton's Ricci flow and to understand the Hamilton-Perelman theory and its application to the geometrization of three-manifolds. For this purpose, we have made the paper to be essentially self-contained so that the proof of the geometrization is accessible to those who are familiar with basics of Riemannian geometry and elliptic and parabolic partial differential equations. The reader may find some original papers, particularly those of Hamilton's on the Ricci flow, before the appearance of Perelman's preprints in the book "Collected Papers on Ricci Flow" [17]. For introductory materials to the Hamilton-Perelman theory of Ricci flow, we also refer the reader to the recent book by B. Chow and D. Knopf [39] and the forthcoming book by B. Chow, P. Lu and L. Ni [41]. We remark that there have also appeared several sets of notes on Perelman's work, including the one written by B. Kleiner and J. Lott [78], which cover part of the materials that are needed for the geometrization program. There also have appeared several survey articles by Cao-Chow [16], Milnor [91], Anderson [4] and Morgan [95] for the geometrization of three-manifolds via the Ricci flow.

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