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THE CONNECTIVE MORAVA K -THEORY OF THE EILENBERG-MACLANE SPACE $K(\mathbf{Z}_p, 2)$

DONALD M. DAVIS, DOUGLAS C. RAVENEL, AND W. STEPHEN WILSON

ABSTRACT. We develop tools for computing the connective n -th Morava K -theory of spaces. Starting with a Universal Coefficient Theorem that computes the cohomology version from the homology version, we show that every step in the process of computing one is mirrored in the other and that this can be used to make computations. As our example, we compute the connective n -th Morava K -theory of the second mod p Eilenberg-MacLane space.

1. INTRODUCTION

Being able to compute is central to much of algebraic topology. Computing generalized (co)homology theories of basic spaces usually runs from difficult to impossible. One exception has been the extraordinary K -theories of Jack Morava, $K(n)_*(X)$. They have a Kunneth isomorphism that makes them more tractable to computations than most.

There is a connective version of Morava K -theories, $k(n)_*(X)$, and in this paper we make some progress towards computing with this. In particular, we develop some tools that can be applied to this problem in general, and then we apply them to compute the n -th connective Morava K -theory of the second mod p Eilenberg-MacLane space, $K_2 = K(\mathbf{Z}_p, 2)$, where \mathbf{Z}_p is the integers modulo the prime p .

The origin of this problem is now securely in the past. It was known from Anderson-Hodgkin, [AH68], that $K(1)_*(K_2)$ was trivial. The third author searched, periodically over the decades, for the differentials in the Atiyah-Hirzebruch spectral sequence (AHSS) that would reduce the already small $H_*(K_2; k(1)_*)$ to zero (at $p = 2$). The differentials in the AHSS are the same as those in the Adams spectral sequence (ASS), so this paper finally gives the third author great satisfaction. The project grew into this paper.

The main computation of the paper is to compute both $k(n)_*(K_2)$ and $k(n)^*(K_2)$ as $k(n)_*$ (and $k(n)^*$) modules.

One of our main tools is obtained by combining results of Robinson and Lazarev for computational purposes.

Theorem 1.1. *For X of finite type with $K(n)_*(X)$ finitely generated over $K(n)_*$, there is a Universal Coefficient spectral sequence (UCT), that collapses:*

$$\mathrm{Ext}_{k(n)_*}^{s,t}(k(n)_*(X), k(n)_*) \Rightarrow k(n)^{s+t}(X)$$

Robinson, in [Rob87], created the UCT, and in [Rob89], he, and also Lazarev in [Laz01], show that $k(n)$ satisfies the UCT criteria. We show the UCT collapses.

From this result, we prove the next important tool.

Theorem 1.2 (The Pairing). *For X of finite type with $K(n)_*(X)$ finitely generated over $K(n)_*$, there is a differential $d^r(q) = v^r m$ in the ASS for $k(n)^*(X)$ if and only if there is a corresponding $d_r(m') = v^r q'$ in the ASS for $k(n)_*(X)$, with $|q| = |q'|$ and $|m| = |m'|$.*

It is the interaction between $k(n)$ cohomology and homology from these two results that allows us to do our computation. Theorem 1.1 gives a duality of sorts between $k(n)_*(X)$ and $k(n)^*(X)$, but Theorem 1.2 goes even further and says that there is a duality every step of the way in the computation. In our case, we have that K_2 is an H-space so both Adams spectral sequences are multiplicative. Although this does not give us a Hopf algebra, there is enough similarity in the structure that we can make good use of it.

The plan of the paper is to state the results of the main computation in the next section. We set up some notation in Section 3. In Section 4 we compute the E_2 term of the ASS for $k(n)^*(K_2)$. We give some necessary definitions and numbers in Section 5. In Section 6, we prove the two theorems in the introduction and establish some other preliminaries we need. All the hard work is done in Section 7 where the differentials are computed. The results for $k(n)_*(K_2)$ are all collected in Section 8 and the final section is devoted to describing the results at $p = 2$.

2. STATEMENT OF RESULTS

In this section we define only what we need to efficiently state the results of our main computation of $k(n)^*(K_2)$. Many details will be properly developed later.

We need to establish some basic notation. All our cohomology and homology groups will be mod p . The connective n -th Morava K -theory spectrum, $k(n)$, has $k(n)^* = \mathbf{Z}_p[v]$ with $|v| = -2(p^n - 1)$. This v is usually called v_n but we will suppress the n .

We let $P[x]$, $E[x]$, and $\Gamma[x]$ be the polynomial, exterior, and divided power algebras on x over \mathbf{Z}_p . In addition, we need the truncated polynomial algebra, $TP_k[x] = P[x]/(x^k)$, and its dual, $\Gamma_k[x]$. The algebra structure of $\Gamma[x]$ is $\otimes_{k \geq 0} TP_p[\gamma_{p^k}(x)]$. For $p = 2$, this degenerates to an exterior algebra.

For an algebra A , we let \overline{A} denote the augmentation ideal of A , which, in our case, is always just the positive degree elements.

To compute with the ASS, we need (for p an odd prime)

$$(2.1) \quad \begin{aligned} H^*K_2 &= P[t_2] \otimes_{i>0} P[z_i] \otimes_{i \geq 0} E[u_i] \\ |z_i| &= 2(p^i + 1) \quad |u_i| = 2p^i + 1 \end{aligned}$$

Let $y_j = t_2^{p^j}$. In particular, $t_2 = y_0$ and $t_2^p = y_0^p = y_1$. In general, $y_j^p = y_{j+1}$ with $|y_j| = 2p^j$.

We define w_{n+i} , for $0 \leq i \leq n$ (with $z_0 = 0$)

$$(2.2) \quad w_{n+i} = u_{n+i} + u_{n-i}(z_i)^{p^n - p^{n-i}}$$

and inductively

$$(2.3) \quad w_{n+j+(n+1)} = y_j^{p-1} w_{n+j} z_{n+j+1}^{p^n - 1}$$

We can now state the main computational result of the paper.

Theorem 2.4. *There are explicit numbers, $r(j)$ and $r'(j)$ (see Section 5), such that the odd primary $k(n)^*(K_2)$, as a $k(n)^*$ -module, is the sum of the following:*

$$P[v] \otimes_{0 < i < n} TP_{p^i}[z_{n-i}]$$

$$\bigoplus_{0 < j} TP_{r(j)}[v] \otimes P[y_{j+1}] \otimes TP_{p-1}[y_j] \otimes \overline{E}[w_{n+j}] \otimes_{0 < i \leq n} E[w_{n+j+i}] \otimes_{0 \leq s} TP_{p^n}[z_{n+j+s+1}]$$

$$\bigoplus_{0 \leq j} TP_{r'(j)}[v] \otimes P[y_{j+1}] \otimes_{0 < i \leq n} E[w_{n+j+i}] \otimes \overline{TP}_{p^n}[z_{n+j+1}] \otimes_{0 < s} TP_{p^n}[z_{n+j+s+1}]$$

plus a computable family of \mathbf{Z}_p 's annihilated by v .

Remark 2.5. If we invert v , it kills all but the first line and it becomes $K(n)^*(K_2)$, [RW80][dual to Theorem 11.1]. This $P[v]$ -free part is all that appears in negative degrees, where it is finite in each degree. In addition, every positive degree is also finite.

3. OUR ASS SPECTRAL SEQUENCE NOTATION

The $k(n)$ under consideration here is the the connective version of Morava's n -th extraordinary K -theory. We have $k(n)^* = P[v]$ with $|v| = -2(p^n - 1)$ and $k(n)_* = P[v]$ with $|v| = 2(p^n - 1)$. It has cohomology, $H^*(k(n)) = \mathcal{A}/\mathcal{A}(Q_n)$ where \mathcal{A} is the mod p Steenrod algebra and Q_n is the n -th Milnor primitive. We need the algebra $E[Q_n]$. The ASS for $k(n)^*(K_2)$ has $E_2^{s,t} = \text{Ext}_{\mathcal{A}}^{s,t}(H^*(k(n)), H^*K_2)$. The change of rings theorem turns this into $\text{Ext}_{E[Q_n]}^{s,t}(\mathbf{Z}_p, H^*K_2)$. This converges to $k(n)^{-(t-s)}(K_2)$. We use the usual grading for the ASS so that $E_r^{s,t}$ is at the $(t - s, s)$ coordinates but then we give the negative x-axis positive degrees, rewriting $E_r^{s,t}$ as $G_{s-t,s}^r$ in position $(t - s, s)$. We use d^r for our cohomology differentials.

We also need the ASS for $k(n)_*(K_2)$, and need to have distinct notation to clearly separate it from the cohomology notation. It has $E_2^{s,t} = \text{Ext}_{\mathcal{A}}^{s,t}(H^*(k(n) \wedge K_2), \mathbf{Z}_p)$. The change of rings theorem turns this into $\text{Ext}_{E[Q_n]}^{s,t}(H^*K_2, \mathbf{Z}_p)$. This converges to $k(n)_{t-s}(K_2)$. We use the usual grading for the ASS so that $E_r^{s,t}$ is at the $(t - s, s)$ coordinates. Here we don't need the negative grading, but to distinguish this from the cohomology ASS, we write $E_r^{s,t}$ as $G_r^{t-s,s}$ in position $(t - s, s)$. Here we use d_r for the differential so we can keep track of which is which.

We need the $E[Q_n]$ -structure of H^*K_2 . Any $E[Q_n]$ -module M is the sum of a trivial module and a free module. As a result, it is easy to compute $\text{Ext}_{E[Q_n]}^{s,t}(\mathbf{Z}_p, M)$ and $\text{Ext}_{E[Q_n]}^{s,t}(M, \mathbf{Z}_p)$. We have $|Q_n| = 2p^n - 1$. Then $\text{Ext}_{E[Q_n]}^{s,t}(\mathbf{Z}_p, \mathbf{Z}_p) = P[v]$, with $v \in \text{Ext}_{E[Q_n]}^{1, 2p^n - 1}$, but the convention for the sign of the degree of v depends on whether we are computing cohomology ($|v| = -2(p^n - 1)$) or homology ($|v| = 2(p^n - 1)$).

We have $\text{Ext}_{E[Q_n]}^{s,t}(\mathbf{Z}_p, E[Q_n]) = \Sigma^{2p^n-1}\mathbf{Z}_p$ in $G_{2p^n-1,0}^2$ and $\text{Ext}_{E[Q_n]}^{s,t}(E[Q_n], \mathbf{Z}_p) = \mathbf{Z}_p$ in $G_2^{0,0}$.

4. THE Q_n HOMOLOGY OF H^*K_2 AND THE ASS E_2 TERM

Following Tamanoi, [Tam99][Theorem 5.2], we have, at odd primes, $z_i = Q_i Q_0 t_2$ ($z_0 = 0$) and $u_i = Q_i t_2$, giving us H^*K_2 in equation (2.1). Continuing to follow Tamanoi:

$$\begin{aligned} Q_n z_i &= 0 \\ Q_n u_n &= 0 \\ Q_n u_i &= (Q_{n-1} u_{i-1})^p \quad 0 < i \neq n \\ &= z_{n-i}^{p^i} \quad 0 \leq i < n \\ &= z_{i-n}^{p^n} \quad 0 < n < i \end{aligned}$$

The w_{n+i} of equation (2.2) are modified u_{n+i} so that $Q_n(w_{n+i}) = 0$.

Rewrite H^*K_2 as the associated graded object of a filtration:

$$\begin{aligned} &\left(E[u_i]_{0 \leq i < n} \otimes P[(z_{n-i})^{p^i}] \right) \otimes \left(E[u_{2n+i}]_{0 < i} \otimes P[(z_{n+i})^{p^n}] \right) \\ &\otimes \left(TP_p[y_0] \otimes E[w_n] \right)_{0 < i < n} \otimes TP_{p^i}[z_{n-i}] \\ &\otimes P[y_1]_{0 < i \leq n} \otimes E[w_{n+i}]_{n < s} \otimes TP_{p^n}[z_{n+s}] \end{aligned}$$

Computing the Q_n homology from this, the first line gives zero. The first part of the second line gives $E[y_0^{p-1}w_n]$. We rename $y_0^{p-1}w_n = w_{n+1/2}$. The second part of the second line is what will give us $K(n)^*(K_2)$.

The Q_n homology is:

$$(4.1) \quad \otimes_{0 < i < n} TP_{p^i}[z_{n-i}] \otimes P[y_1] \otimes E[w_{n+1/2}]_{0 < i \leq n} \otimes E[w_{n+i}]_{0 < s} \otimes TP_{p^n}[z_{n+s}]$$

Theorem 4.2. *We have elements $v \in G_{-2(p^n-1),1}^2$, $y_1 \in G_{2p,0}^2$, $w_{n+i} \in G_{2p^{n+i}+1,0}^2$, $w_{n+1/2} \in G_{2(p^n-1)+2p+1,0}^2$ and $z_j \in G_{2(p^j+1),0}^2$. The E_2 term of the odd primary Adams spectral sequence for $k(n)^*(K_2)$ is*

$$P[v]_{0 < i < n} \otimes TP_{p^i}[z_{n-i}] \otimes P[y_1] \otimes E[w_{n+1/2}]_{0 < i \leq n} \otimes E[w_{n+i}]_{0 < s} \otimes TP_{p^n}[z_{n+s}]$$

plus a computable family of filtration-0 \mathbf{Z}_2 's annihilated by v coming from the $E[Q_n]$ -free part of H^*K_2 .

Proof. The Q_n homology of H^*K_2 gives us the trivial $E[Q_n]$ -module part. The rest is free over $E[Q_n]$. We have already computed Ext for both kinds of modules. The result follows. \square

5. NUMBERS AND DEFINITIONS

Although we are only interested in odd primes until the last section, all of the formulas and numbers in the section work for $p = 2$ except for one small deviation.

In this section we give some definitions and compute some numbers we need. We already have elements y_j , z_j and w_{n+i} . (The odd prime definitions for these elements work for $p = 2$ with one minor exception.) This is in preparation for our differentials

$$(5.1) \quad d^{r(j)}(y_j) = v^{r(j)}w_{n+j} \quad d^{r'(j)}(w_{n+j+1/2}) = v^{r'(j)}z_{n+j+1}$$

The second gives rise to the element $w_{n+j+(n+1)}$ previously defined by equation (2.3) and the first defines

$$(5.2) \quad w_{n+j+1/2} = y_j^{p-1}w_{n+j}$$

as we shall see in Section 7.

We collect some easily read off numbers

Lemma 5.3.

$$\begin{aligned} |y_j| &= 2p^j & j \geq 0 \\ |z_j| &= 2(p^j + 1) & j > 0 \\ |w_{n+j+1/2}| &= 2p^j(p-1) + |w_{n+j}| \\ |w_{n+j+(n+1)}| &= 2p^j(p-1) + |w_{n+j}| + 2(p^n - 1)(p^{n+j+1} + 1) \\ q(k) &= 1 + p^{n+1} + p^{2(n+1)} + \dots + p^{(k-1)(n+1)} = q(k-1) + p^{(k-1)(n+1)} \\ q(0) &= 0 \end{aligned}$$

Our prospective differentials, equations (5.1), obey the equations

$$(5.4) \quad \begin{aligned} |y_j| + 1 + 2r(j)(p^n - 1) &= |w_{n+j}| \\ |w_{n+j+1/2}| + 1 + 2r'(j)(p^n - 1) &= |z_{n+j+1}| \end{aligned}$$

Lemma 5.5. *We write $j = i + k(n + 1)$ $0 \leq j$ $0 \leq i < n + 1$*

$$\begin{aligned} |w_{n+j}| &= 2(p^n - 1) \left(p^{i+1}(p^n - 1)q(k) + k + p^i \right) + 2p^j + 1 \\ |w_{n+j+1/2}| &= 2(p^n - 1) \left(p^{i+1}(p^n - 1)q(k) + k + p^i \right) + 2p^{j+1} + 1 \\ r(j) &= p^{i+1}(p^n - 1)q(k) + k + p^i \\ r'(j) &= p^{j+1} - p^{i+1}(p^n - 1)q(k) - k - p^i \\ p^{j+1} &= r(j) + r'(j) \\ r(j+1) &> r'(j) > r(j) \quad p \text{ odd} \\ r(j+1) &> r'(j) \geq r(j) \quad p = 2 \\ p^j &\geq r(j) > p^{j-1} \\ p^{j+1} - p^{j-1} &> r'(j) \geq p^{j+1} - p^j \\ r(j + (n + 1)) &= r(j) + p^{j+1}(p^n - 1) + 1 \\ r'(j + (n + 1)) &= r'(j) + p^{j+(n+1)}(p - 1) - 1 \end{aligned}$$

Proof. The one exception for $p = 2$ comes when $j \leq n + 1$ and we have $r(j) = r'(j) = 2^j$.

The proof of the first formula is the most complicated. It is done by induction on k . We do it last. The same technique works for the last two formulas.

The formula for $|w_{n+j+1/2}|$ follows from the definition, equation (5.2), and the formula for w_{n+j} . The formulas for $r(j)$ and $r'(j)$ follow immediately from the first two formulas and equations (5.4).

That $p^{j+1} = r(j) + r'(j)$ is now obvious. The inequalities $r(j+1) > r'(j) > r(j)$ and $p^j \geq r(j)$ are left to the reader. We show $r(j) > p^{j-1}$ to illustrate.

$$r(j) = p^{i+1}(p^n - 1)q(k) + k + p^i = p^{i+1}(p^n - 1)q(k - 1) + p^j - p^{j-n} + k + p^i.$$

Since $p^j - p^{j-n} + p^i > p^{j-1}$, we are done. This is even easier if $k = 0$.

We do the formula for $|w_{n+j}|$ in detail. Our induction is on k . It is easy to start. When $k = 0$, we have $|w_{n+j}| = |u_{n+j}| = 2p^{n+j} + 1$. We assume our formula for $|w_{n+i+(k-1)(n+1)}|$ and use the inductive formula in Lemma 5.3. So $|w_{n+j}| =$

$$2(p^n - 1) \left(p^{i+1}(p^n - 1)q(k-1) + k - 1 + p^i \right) + 2p^{i+(k-1)(n+1)} + 1 \\ + 2p^{i+(k-1)(n+1)}(p-1) + 2(p^n - 1)(p^j + 1)$$

Replace the $q(k-1)$ and incorporate the last term into the first.

$$2(p^n - 1) \left(p^{i+1}(p^n - 1)(q(k) - p^{(k-1)(n+1)}) + k - 1 + p^i + p^j + 1 \right) \\ + 2p^{i+(k-1)(n+1)} + 1 + 2p^{i+(k-1)(n+1)}(p-1)$$

There is some cancellation in both rows giving

$$2(p^n - 1) \left(p^{i+1}(p^n - 1)q(k) + k + p^i + p^{i+1+(k-1)(n+1)} \right) \\ + 2p^{i+1+(k-1)(n+1)} + 1$$

Take out the last term in the top row.

$$2(p^n - 1) \left(p^{i+1}(p^n - 1)q(k) + k + p^i \right) \\ + 2(p^n - 1)p^{i+1+(k-1)(n+1)} + 2p^{i+1+(k-1)(n+1)} + 1$$

The last row reduces giving the desired result.

$$2(p^n - 1) \left(p^{i+1}(p^n - 1)q(k) + k + p^i \right) + 2p^j + 1$$

□

6. PRELIMINARIES BEFORE THE PROOF

Lemma 6.1 (*Divisibility Criteria*). *If $d^r(q) = v^r m$, then $|q| + 1 + 2r(p^n - 1) = |m|$, i.e. $|m| - 1 - |q| \equiv 0 \pmod{|v|}$.*

Proof of Theorem 1.1. In [Laz01][Corollary 11.8] and [Rob89][Theorem 2.3], the odd primary $k(n)$ is shown to be A_∞ . In private communication, Lazarev says that his argument for $k(n)$ works just as well for $p = 2$.

In [Rob87][p. 257], Robinson produces a Universal Coefficient Theorem (UCT) for A_∞ spectra. In our case this gives the spectral sequence of Theorem 1.1. For spaces of finite type with $K(n)_*(X)$ finitely generated, $k(n)_*(X)$ is the sum of a free

module (of finite dimension) over $k(n)_*$ and a sum of torsion modules, $TP_k[v]$. The above Ext is easy to compute and everything is in Ext^0 and Ext^1 . More precisely:

$$\text{Ext}_{k(n)_*}^{0,*}(k(n)_*, k(n)_*) = k(n)^*$$

$$\text{Ext}_{k(n)_*}^{1,*}(TP_k[v], k(n)_*) = TP_k[v]$$

with generator in $\text{Ext}_{k(n)_*}^{1,|v^k|}$ and $v \in \text{Ext}_{k(n)_*}^{0,-2(p^n-1)}$

The entire E_2 term is in Ext^0 and Ext^1 . This is peculiar to $k(n)$. As a result, the spectral sequence collapses. \square

Proof of Theorem 1.2. If we have $d^r(q) = v^r m$ in the ASS for $k(n)^*(X)$, it means we have (a cohomology) $TP_r[v]$ with generator in the degree of m . From the UCT, to get this, we must have a (homology) $TP_r[v]$ with generator in the degree of q . To get this in the ASS for $k(n)_*(X)$, we must have a differential $d_r(m') = v^r q'$ with the mentioned degrees. Reverse the argument to get the other direction. \square

Before we state the next result, we need

$$(6.2) \quad H_*K_2 = \Gamma[l_2^*] \otimes_{i>0} \Gamma[z_i^*] \otimes_{i \geq 0} E[u_i^*]$$

Here we have $y_j^* = \gamma_{pj}(l_2^*)$ dual to y_j in cohomology.

In Theorem 8.1, we compute the E_2 term for the ASS for $k(n)_*(K_2)$. In particular, $\Gamma[y_1^*]$ is there.

Lemma 6.3. *The z_j are all permanent cycles in the ASS for $k(n)^*(K_2)$ and there is a non-zero differential $d^r(y_j)$ for some $r \leq p^j$. In the ASS for $k(n)_*(K_2)$, $v^r y_j^*$ is hit by a differential for some $r \leq p^j$.*

Proof. Tamanoi, in [Tam97], (actually done much earlier in his 1983 masters thesis in Japan), and then again later in [RWY98], the image of the map $BP^*(K(\mathbf{Z}_p, i)) \rightarrow H^*K(\mathbf{Z}_p, i)$ is computed. In particular, the answer for $i = 2$ contains the z_j , where $j > 0$. This map factors through $k(n)^*(K_2)$ so we conclude that the z_j cannot support a differential.

Setting $b_i \in k(n)_{2i}\mathbb{C}\mathbb{P}^\infty$, we consider the composition

$$\mathbb{C}\mathbb{P}^\infty \xrightarrow{p} \mathbb{C}\mathbb{P}^\infty \longrightarrow K_2$$

Define $b(s) = \sum_i b_i s^i$ and $b_{(j)} = b_{p^j}$. Note that $b_{(j)}$ maps to $y_j^* \in k(n)_*(K_2)$. We follow [RW77][Theorem 3.8(ii)] and use the fact that for $k(n)$, $[p](s) = vs^{p^n}$. The composition above takes $b(s)$ to zero, but the first map takes $b(s) \rightarrow b(vs^{p^n})$. In particular, we see that $v^{p^j} b_{(j)}$ maps to zero, giving $v^{p^j} y_j^* = 0 \in k(n)_*(K_2)$.

Since we must have a $d_r(m) = v^r y_j^*$ with $r \leq p^j$, from Theorem 1.2, we must have a corresponding $d^r(q') = v^r m'$ with $|q'| = 2p^j = |y_j^*|$. By the time we get to the point of worrying about y_j , it will be the only element in degree $|q'|$ and so must be q' . \square

7. THE SPECTRAL SEQUENCE

Theorem 7.1. *In the odd primary Adams spectral sequence for $k(n)^*(K_2)$, the differentials are:*

$$\text{For } 0 < j \quad d^{r(j)}(y_j) = v^{r(j)} w_{n+j} \quad d^{r'(j-1)}(w_{n+j-1+1/2}) = v^{r'(j-1)} z_{n+j}.$$

Ignoring the permanent v -free terms and the previously created v -torsion

$$E_{r'(j-1)+1} = P[v, y_j] \otimes_{0 \leq i \leq n} E[w_{n+j+i}] \otimes_{0 \leq s} TP_{p^n}[z_{n+j+s+1}]$$

$$E_{r(j)+1} = P[v, y_{j+1}] \otimes E[w_{n+j+1/2}] \otimes_{0 < i \leq n} E[w_{n+j+i}] \otimes_{0 \leq s} TP_{p^n}[z_{n+j+s+1}]$$

Proof of Theorem 2.4. The action of $d^{r(j)}$ takes place in $P[v, y_j] \otimes E[w_{n+j}]$. We can break this up into $P[v, y_{j+1}] \otimes TP_p[y_j] \otimes E[w_{n+j}]$. It creates v -torsion $TP_{r(j)}[v] \otimes P[y_{j+1}] \otimes TP_{p-1}[y_j] \otimes \bar{E}[w_{n+j}]$.

The action of $d^{r'(j)}$ takes place in $P[v] \otimes E[w_{n+j+1/2}] \otimes TP_{p^n}[z_{n+j+1}]$. It creates v -torsion $TP_{r'(j)}[v] \otimes \bar{TP}_{p^n}[z_{n+j+1}]$.

This is the correct description of $k(n)^*(K_2)$ as a $k(n)^*$ -module. However, to be precise, we might have to alter the generator names as follows. If, for example, some $P[v]/(v^{r(j)})$ generator in the theorem had $v^{r(j)} \neq 0$, it would have to jump to a higher filtration, but there it would be $v^r x$ where x has a higher degree than our generator. We could then adjust our generator by adding $v^j x$ for some j . This would push our element to an even higher filtration, but this process has to stop

in a finite number of steps. Despite the normal limitations of the ASS, in our case we can discover the real $k(n)^*$ -module structure. \square

Proofs for Theorems 7.1 and 8.2. Our proof starts with showing the asserted differentials must happen. Then we have to show that there are no additional differentials. This is where the full power of *The Pairing* comes in.

We assume, inductively, we have $E_{r'(j-1)+1}$. We must have a $d^r(y_j) = v^r q$ where q is odd degree and $r \leq p^j$ by Lemma 6.3. There are few odd degree elements in this range. We will show that if $q = w_{n+j+1}$, we would have $r > p^j$. This eliminates all $q = w_{n+j+i}$, $i > 1$, because their degree is even higher. We want to show

$$|w_{n+j+1}| - 1 - |y_j| > 2p^j(p^n - 1)$$

Replace w_{n+j+1} using equation (5.4) to get

$$|y_{j+1}| + 1 + 2r(j+1)(p^n - 1) - 1 - |y_j|.$$

It is enough to have $r(j+1) > p^j$, but this is in Lemma 5.5.

The only remaining elements of odd degree are $y_j^s w_{n+j}$. However, since we know w_{n+j} meets the *Divisibility Criteria*, for this to meet it with $s > 0$, we must have s at least $p^n - 1$. Then the differential would be $p^j + r(j)$, and this is greater than p^j so can't happen. We conclude that we *must* have $d^{r(j)}y_j = v^{r(j)}w_{n+j}$ as desired.

The action of this differential takes place in $P[v, y_j] \otimes E[w_{n+j}]$ which can be broken up as $P[v, y_{j+1}] \otimes TP_p[y_j] \otimes E[w_{n+j}]$. The remaining v -torsion free part is $P[v, y_{j+1}] \otimes E[w_{n+j+1/2}]$, giving us $E_{r(j)+1}$.

We already know the $P[v]$ -free elements that survive. Since z_{n+j+1} is not one of them but is a permanent cycle, we know that some $v^r z_{n+j+1}$ must be hit by a differential coming from an odd degree element.

Lemma 7.2. *If $d^r(w_{n+j+s}) = v^r z_{n+j+1}$, $s > 0$, then $r \leq r'(j-1)$ for $n > 1$. In the $n = 1$ case, $d^r(w_{n+j+s}) = v^r z_{n+j+1}$, $s > 0$, cannot exist.*

Proof. For $n > 1$, it is enough to study the $s = 1$ case. We would have

$$|w_{n+j+1}| + 1 + 2r(p^n - 1) = |z_{n+j+1}|$$

Replace $|w_{n+j+1}|$ using equation (5.4)

$$|y_{j+1}| + 1 + 2r(j+1)(p^n - 1) + 1 + 2r(p^n - 1) = |z_{n+j+1}|$$

Plugging in the numbers for y_{j+1} and z_{n+j+1} and rearranging, we get

$$2r(j+1)(p^n - 1) + 2r(p^n - 1) = 2p^{n+j+1} - 2p^{j+1} = 2p^{j+1}(p^n - 1)$$

So, $r = p^{j+1} - r(j+1)$.

We need to show this is $\leq r'(j-1)$. Using the formulas from Lemma 5.5 for $r(j+n+1)$ and $r'(j+n+1)$, we have $p^{j+1} - r(j+1) = p^{j+1-n} - r(j-n) - 1$ and $r'(j-1) = r'(j-n-2) + p^{j-1}(p-1) - 1$. When $n > 1$, the p^j clearly dominates and we get our inequality. The induction is easy to start. The inequality is false for $n = 1$. This implies the $s > 1$ case.

We are left with showing that the differential cannot exist when $n = 1$. Our real interest is in showing what can and can't hit $v^r z_{n+j+1}$. The numbers make it easy to rule out w_{n+j+2} . The remaining odd degree elements are w_{n+j+1} and $w_{n+j+1/2}$. Both meet the *Divisibility Criteria*, the first with $p^{j+1} - r(j+1)$ and the second with $r'(j)$. All we need now is to rule out w_{n+j+1} in this $n = 1$ case.

We know what must happen with $d^{r(j)}(y_j)$, so we can ask what happens with y_{j+1} since $d^{r(j)}$ is zero on it. We know that the differential on y_{j+1} must have $r \leq p^{j+1}$. It is an easy check to see that if $d^r(y_{j+1}) = v^r w_{n+j+2}$, then $r = p^{j+1} + r(j+2)$, which is too big. Our only choices are w_{n+j+1} and $w_{n+j+1/2}$. If we can show that $w_{n+j+1/2}$ doesn't work, then it would have to hit w_{n+j+1} and so this would not be available to hit z_{n+j+1} . This is all a bit contorted just because $n = 1$ is special sometimes. Doing our usual kinds of numerical computations, we see that if $d^r(y_{j+1}) = v^r w_{n+j+1/2}$ for $n = 1$, then $r = r(j)$. But $d^{r(j)}$ is zero on y_{j+1} . \square

Lemma 7.2 rules out all w_{n+j+i} with $i > 0$. Now the only odd degree elements left are the $y_{j+1}^s w_{n+j+1/2}$. We know $w_{n+j+1/2}$ would work with differential $r'(j)$ because of the *Divisibility Criteria*. The *Divisibility Criteria* requires s to be a multiple of $p^n - 1$. The lowest non-zero s is $s = p^n - 1$ and this would give a differential of length $r'(j) - p^{j+1}$, but $p^{j+1} > r'(j)$ so this cannot happen. We *must* have $d^{r'(j)}(w_{n+j+1/2}) = v^{r'(j)} z_{n+j+1}$.

The action of $d^{r'(j)}$ takes place in $P[v] \otimes E[w_{n+j+1/2}] \otimes TP_{p^n}[z_{n+j+1}]$ and results in the $P[v]$ -free part being $E[w_{n+j+(n+1)}]$, giving us $E_{r'(j)+1}$.

Having computed these differentials, we can use *The Pairing* of Theorem 1.2 to get the dual differentials for the ASS for $k(n)_*(K_2)$ in Theorem 8.2. We first show $d_{r(j)}(w_{n+j}^*) = v^{r(j)} y_j^*$. We know, from Lemma 6.3 that some differential must hit

some $v^r y_j^*$ with $r \leq p^j$. From *The Pairing*, we know that some element, q , in the degree of y_j^* must have $d_{r(j)}(m) = v^{r(j)}q$. However, in $E_{r'(j-1)}$, we see that y_j^* is the only element there is in that degree and w_{n+j}^* is the only odd degree generator in the correct degree. The only option is the expected result. Again, the pairing gives us a $d_{r'(j)}$ in degrees corresponding to z_{n+j+1}^* and $w_{n+j+1/2}^*$. There are no other options, so $d_{r'(j)}$ is as advertised.

Having computed these two must-have differentials, it gives us the description for

$$(7.3) \quad E_{r'(j)+1} = P[v, y_{j+1}] \otimes_{0 \leq i \leq n} E[w_{n+j+i+1}] \otimes_{0 \leq s} TP_p^n [z_{n+j+s+2}]$$

We are not done. We must show that there is no action of a d^r on this for $r'(j-1) < r \leq r'(j)$.

There are no differentials on the z 's because they are permanent cycles. This leaves only the w 's and y_{j+1} . The lowest odd degree element is w_{n+j+1} and we know that such a differential on y_{j+1} would be $r(j+1) > r'(j)$, so we cannot have a differential on y_{j+1} .

All that is left is to show there is no r in this range with $d^r(w_{n+j+i+1}) \neq 0$. Let r be the smallest r in the range $r'(j-1) < r \leq r'(j)$ with $d^r(w_{n+j+i+1}) = v^r m \neq 0$, for the $w_{n+j+i+1}$ of smallest degree. We know from Theorem 1.2, *The Pairing*, that there is an m' , with $|m'| = |m|$, in the homology ASS with $d_r(m') = v^r q' \neq 0$. If m' is decomposable, then there must be an element, m'' , with lower degree than m' with $d_r(m'') = v^r q'' \neq 0$. For example, if $m' = ab$, then $d_r(m') = d_r(a)b \pm ad_r(b)$ and either $d_r(a)$ or $d_r(b)$ is non-zero. In either case, we get our m'' with degree less than $|m'|$. Again, by *The Pairing*, there is a q in the cohomology ASS with $|q| = |q''| < |w_{n+j+i+1}|$ with $d^r(q) \neq 0$. This contradicts our choice of $w_{n+j+i+1}$. We conclude that if there is such an r , m' is indecomposable. Theorem 1.2, *The Pairing*, is pretty vague about what the corresponding elements are. All it really gives us are degrees. However, we know where all the even degree (and since we started with the odd degree $w_{n+j+i+1}$, we are looking for an even degree element) elements are in our homology $E_{r'(j)+1}$. These elements are the y_k^* , $k > j$, and the $\gamma_{p^k}(z_{n+j+s+2}^*)$. We have similar looking elements in the cohomology $E_{r'(j)+1}$. They are not known to be "dual" in any sense, but they are in the right degrees. All we will use about these cohomology elements is their degree. If we can show that there are no differentials that hit elements in these degrees, we are done.

We have three main ways to show a differential cannot exist. (1) We can use the *Divisibility Criteria*, (2) we can show that a prospective d^r has $r > r'(j)$, or (3) we can show that $r \leq r'(j-1)$.

The first we have to check to see if there is some s with $d^r(w_{n+j+i+1}) = v^r y_s$. For this, we must have

$$|w_{n+j+i+1}| + 1 + 2r(p^n - 1) = |y_s|$$

but we can replace the first term using equation (5.4)

$$|y_{j+i+1}| + 1 + 2r(j+i+1)(p^n - 1) + 1 + 2r(p^n - 1) = |y_s|$$

so

$$|y_s| - |y_{j+i+1}| - 2 = 2p^s - 2p^{j+i+1} - 2$$

is both positive and divisible by $2(p^n - 1)$. Let s' and $(j+i+1)'$ be s and $(j+i+1) \bmod n$. Then, $\bmod 2(p^n - 1)$, this is $2p^{s'} - 2p^{(j+i+1)'} - 2$. This cannot be zero $\bmod 2(p^n - 1)$ so the *Divisibility Criteria* tells us we cannot have this differential.

The elements $z_{n+j+s+2}^{p^k}$ below are in degrees that correspond to the degrees of the remaining even degree generators in the homology version. We have to show, using only their degrees, that there is no differential $d^r(w_{n+j+i+1}) = v^r z_{n+j+s+2}^{p^k}$ with $0 \leq i < n$, $0 \leq s$, $0 \leq k < n$, $0 \leq j$, and $r'(j-1) < r \leq r'(j)$. We don't have to worry about the w with $i = n$ because it was created by $d^{r'(j)}$. (This is a degree reason.)

We will assume that $j > 0$. Our proof works for $j = 0$ but there are parts that become degenerate and even easier.

We replace $|w_{n+j+i+1}|$ with $|y_{j+i+1}| + 1 + 2r(j+i+1)(p^n - 1)$. so we have

$$|y_{j+i+1}| + 1 + 2r(j+i+1)(p^n - 1) + 1 + 2r(p^n - 1) = |z_{n+j+s+2}^{p^k}|$$

Turning this into numbers and rearranging,

$$\begin{aligned} 2(p^n - 1)(r(j+i+1) + r) &= |z_{n+j+s+2}^{p^k}| - |y_{j+i+1}| - 2 \\ &= p^k 2(p^{n+j+s+2} + 1) - 2p^{j+i+1} - 2 = 2(p^{n+j+s+k+2} + p^k) - 2p^{j+i+1} - 2 \end{aligned}$$

It is r we are interested in, so rewrite

$$(7.4) \quad 2r(p^n - 1) = 2p^{n+j+s+k+2} + 2p^k - 2p^{j+i+1} - 2 - 2r(j+i+1)(p^n - 1)$$

For starters, we want to show that if $s = i - k$, we have $r'(j) < r$. This eliminates all $p^{k'}$ powers where $k' > k$ as well. Also, when $k = 0$, it eliminates $s = i$, and consequently, all $s > i$ and all k , for $s \geq i$.

We prefer to work with $r(-)$, so replace $r'(j) = p^{j+1} - r(j)$ and $s = i - k$ in equation (7.4). We want

$$(p^{j+1} - r(j))2(p^n - 1) < 2p^{n+j+i+2} + 2p^k - 2p^{j+i+1} - 2 - 2r(j+i+1)(p^n - 1)$$

rearrange

$$p^{j+1}2(p^n - 1) + 2r(j+i+1)(p^n - 1) + 2p^{j+i+1} + 2 < 2p^{n+j+i+2} + 2p^k + 2r(j)(p^n - 1)$$

We can use $r(j+i+1) \leq p^{j+i+1}$ to show

$$p^{j+1}2(p^n - 1) + 2p^{j+i+1}(p^n - 1) + 2p^{j+i+1} + 2 < 2p^{n+j+i+2} + 2p^k + r(j)2(p^n - 1)$$

expand and rearrange

$$2p^{n+j+i+1} + 2p^{n+j+1} + 2 < 2p^{n+j+i+2} + 2p^k + r(j)2(p^n - 1) + 2p^{j+1}$$

This is pretty clearly always true, so we are done with this.

We move to the case with $s = i - k$, but now, instead of taking the p^k power of the z , we just do p^{k-1} . We show that this differential cannot exist because of the *Divisibility Criteria*, not because it is too short. We show that the differential would be too short if we use p^{k-2} , i.e. that $r \leq r'(j-1)$. If we do that, then all the lower powers work as well. In particular, the $s < i - k$ will also work.

Using equation (7.4) but with p^{k-1} instead of p^k ,

$$\begin{aligned} 2r(p^n - 1) &= 2p^{n+j+i+1} + 2p^{k-1} - 2p^{j+i+1} - 2 - 2r(j+i+1)(p^n - 1) \\ &= 2(p^n - 1)(p^{j+i+1} - r(j+i+1)) + 2p^{k-1} - 2 \end{aligned}$$

We should be able to solve for r , but we can't unless $k = 1$, but this was already solved back with Lemma 7.2. With $0 < k - 1 < n - 1$, this can never be divisible by $2(p^n - 1)$. So, we can eliminate the $s = i - k$ and p^{k-1} case. All we did here was show this couldn't exist. We didn't show that it was too short if it did exist. So, this doesn't solve the problem for $p^{k'}$ powers with $k' < k - 1$.

If we move on to the p^{k-2} case, we can show it is too short. That will imply all the lower powers as well. For this we go back to equation (7.4) but now with p^{k-2} instead of p^k .

$$\begin{aligned} 2r(p^n - 1) &= 2p^{n+j+i} + 2p^{k-2} - 2p^{j+i+1} - 2 - 2r(j+i+1)(p^n - 1) \\ &= 2p^{k-2} - 2 - 2r(j+i+1)(p^n - 1) \end{aligned}$$

We want this to be less than $2r'(j-1)(p^n - 1)$, or

$$2p^{k-2} - 2 - 2r(j+i+1)(p^n - 1) \leq 2r'(j-1)(p^n - 1)$$

rearrange

$$2p^{k-2} \leq 2r'(j-1)(p^n-1) + 2 + 2r(j+i+1)(p^n-1)$$

Recall that $n > k$, so if $r(j+i+1) > 0$, this is true.

This concludes all of the cases we needed to check. There are no more differentials than those already produced.

Of course if there are no more differentials for the ASS for $k(n)^*(K_2)$, then *The Pairing* says there are no more for $k(n)_*(K_2)$. \square

8. THE DUAL OF EVERYTHING

We have already given H_*K_2 in equation (6.2). We need the Q_n homology of H_*K_2 , but it is just dual to equation (4.1). It gives us the E_2 term of the ASS for $k(n)_*(K_2)$, so we'll skip writing it down.

We give the E_2 term of the ASS for $k(n)_*(K_2)$, describe all the differentials, and give the final result as a $k(n)_*$ -module. The proofs are dual to the proofs for $k(n)^*(K_2)$ and are actually, of necessity, carried out simultaneously with those.

Theorem 8.1. *We have elements $v \in G_2^{2(p^n-1),1}$, $y_j^* \in G_2^{2p^j,0}$, $w_{n+i}^* \in G_2^{2p^{n+i}+1,0}$, and $z_j^* \in G_2^{2(p^j+1),0}$. The E_2 term of the odd primary Adams spectral sequence for $k(n)_*(K_2)$ is*

$$P[v] \otimes_{0 < i < n} \Gamma_{p^i}[z_{n-i}^*] \otimes \Gamma[y_1^*] \otimes E[w_{n+1/2}^*] \otimes_{0 < i \leq n} E[w_{n+i}^*] \otimes_{0 < s} \Gamma_{p^n}[z_{n+s}^*]$$

plus a computable family of filtration-0 \mathbf{Z}_2 's annihilated by v coming from the $E[Q_n]$ -free part of H^*K_2 .

Theorem 8.2. *In the odd primary Adams spectral sequence for $k(n)_*(K_2)$, the differentials are:*

$$\text{For } 0 < j, \quad d_{r(j)}(w_{n+j}^*) = v^{r(j)}y_j^* \quad d_{r'(j-1)}(z_{n+j}^*) = v^{r'(j-1)}w_{n+j-1+1/2}^*$$

Ignoring the permanent v -free terms and the previously created v -torsion

$$E_{r'(j-1)+1} = P[v] \otimes \Gamma[y_j^*] \otimes_{0 \leq i \leq n} E[w_{n+j+i}^*] \otimes_{0 \leq s} \Gamma_{p^n}[z_{n+j+s+1}^*]$$

$$E_{r(j)+1} = P[v] \otimes \Gamma[y_{j+1}^*] \otimes E[w_{n+j+1/2}^*] \otimes_{0 < i \leq n} E[w_{n+j+i}^*] \otimes_{0 \leq s} \Gamma_{p^n}[z_{n+j+s+1}^*]$$

Theorem 8.3. *The odd primary $k(n)_*(K_2)$, as a $k(n)_*$ -module, is the sum of the following:*

$$P[v] \otimes_{0 < i < n} \Gamma_{p^i}[z_{n-i}^*]$$

$$\bigoplus_{0 < j} TP_{r(j)}[v] \otimes \Gamma[y_{j+1}^*] \otimes \overline{TP}_p[y_j^*] \otimes_{0 < i \leq n} E[w_{n+j+i}^*] \otimes_{0 \leq s} \Gamma_{p^n}[z_{n+j+s+1}^*]$$

$$\bigoplus_{0 \leq j} TP_{r'(j)}[v] \otimes \Gamma[y_{j+1}^*] \otimes \overline{E}[w_{n+j+1/2}^*] \otimes_{0 < i \leq n} E[w_{n+j+i}^*] \otimes \Gamma_{p^{n-1}}[z_{n+j+1}^*] \otimes_{0 < s} \Gamma_{p^n}[z_{n+j+s+1}^*]$$

plus a computable family of \mathbf{Z}_p 's annihilated by v coming from the $E[Q_n]$ -free part of H^*K_2 .

9. MODIFICATIONS FOR $p = 2$

All we do in this section is to lay out the results for $k(n)^*(K_2)$ for $p = 2$. We skip the homology version and proofs. We do this with a twinge of guilt. The very first case done was the $p = 2, n = 1$ case, and there, the generally useful *Divisibility Criteria* is worthless. Consequently, there are lots of little *ad hoc* arguments that must be done in that case.

For $p = 2$, $H^*K_2 = P[\iota_2] \otimes_{i \geq 0} P[u_i]$, with $u_i = Q_i \iota_2$. We let $u_i^2 = z_{i+1} = Q_{i+1} Q_0 \iota_2$ ($z_0 = 0$). In an attempt to be as similar as possible with notation, we have $|u_i| = 2 \times 2^i + 1$ and $|z_i| = 2(2^i + 1), i > 0$. We have $|Q_n| = 2 \times 2^n - 1$. We also have $y_0 = \iota_2$ and $y_j = \iota_2^{2^j}$ in degree 2×2^j .

$$\begin{aligned} Q_n y_0 &= u_n \\ Q_n u_i &= (u_{n-i-1})^{2^{i+1}} = (z_{n-i})^{2^i} & 0 \leq i < n \\ Q_n u_n &= 0 \\ Q_n u_i &= (u_{i-n-1})^{2^{n+1}} = (z_{i-n})^{2^n} & 0 < n < i \end{aligned}$$

The old formulas used for odd primes mostly work here

$$\begin{aligned} w_{n+i} &= u_{n+i} + u_{n-i} (z_i)^{2^n - 2^{n-i}} & 0 \leq i \leq n & \quad w_{n+j+1/2} = y_j w_{n+j} \\ w_{n+(n+1)} &= u_{2n+1} + y_0 w_n z_{n+1}^{2^n - 1} & w_{n+j+(n+1)} &= y_j w_{n+j} z_{n+j+1}^{2^n - 1} \quad j > 0 \end{aligned}$$

In the ASS for odd primes, we had $d^1(y_0) = vw_n$ so that both y_0 and w_n were part of the $E[Q_n]$ -free part of E_2 . In the $p = 2$ case we have $d^1(y_0w_n) = vw_n^2 = vz_{n+1}$ so z_{n+1} , and all its powers, are also in the $E[Q_n]$ -free part. In general, we have $d^1(y_0w_n^k) = vw_n^{k+1}$. Because we would normally have $w_{n+(n+1)} = y_0w_nz_{n+1}^{2^n-1} = y_0w_n^{2^{n+1}-1}$, giving $d^1(w_{n+(n+1)}) = vz_{n+1}^{2^n}$, we would not have w_{2n+1} . Making the adjustment above allows us to keep a new version of w_{2n+1} with $d^1(w_{2n+1}) = Q_n(w_{2n+1}) = 0$.

Rewriting H^*K_2 at $p = 2$

$$\begin{aligned} & \left(E[y_0] \otimes P[w_n] \right) \otimes \left(E[u_i]_{0 \leq i < n} \otimes P[(z_{n-i})^{2^i}] \right) \otimes \left(E[u_{n+(n+1)+i}]_{0 < i} \otimes P[(z_{n+1+i})^{2^n}] \right) \\ & \quad \otimes_{0 < i < n} TP_{2^i}[z_{n-i}] \otimes P[y_1] \otimes_{0 \leq i \leq n} TP_{2^{n+1}}[w_{n+i+1}] \otimes_{0 < s} TP_{2^n}[z_{n+(n+1)+s+1}] \end{aligned}$$

The Q_n homology is

$$\otimes_{0 < i < n} TP_{2^i}[z_{n-i}] \otimes P[y_1] \otimes_{0 \leq i \leq n} TP_{2^{n+1}}[w_{n+i+1}] \otimes_{0 < s} TP_{2^n}[z_{n+(n+1)+s+1}]$$

Theorem 9.1. *We have elements $v \in G_{-2(2^n-1),1}^2$, $y_1 \in G_{4,0}^2$, $w_{n+i} \in G_{2^{n+i+1}+1,0}^2$ and $z_j \in G_{2^{j+1}+2,0}^2$. The E_2 term of the $p = 2$ Adams spectral sequence for $k(n)^*(K_2)$ is*

$$P[v]_{0 < i < n} \otimes TP_{2^i}[z_{n-i}] \otimes P[y_1] \otimes_{0 \leq i \leq n} TP_{2^{n+1}}[w_{n+i+1}] \otimes_{0 < s} TP_{2^n}[z_{n+(n+1)+s+1}]$$

plus a computable family of filtration-0 \mathbf{Z}_2 's annihilated by v coming from the $E[Q_n]$ -free part of H^*K_2 .

For convenience we reset $z_{n+i+1} = w_{n+i}^2$ for $0 < i \leq n + 1$.

Proposition 9.2. *In the $p = 2$ Adams spectral sequence for $k(n)^*(K_2)$, the differentials are:*

For $0 < j \leq n + 1$, $r(j) = 2^j = r'(j)$. Although $w_{n+j+1/2} = y_jw_{n+j}$, for $j \leq n + 1$, this is not a generator.

$$d^{2^j}(y_j) = v^{2^j}w_{n+j} \quad \text{and} \quad d^{2^j}(y_jw_{n+j}) = v^{2^j}w_{n+j}^2 = v^{2^j}z_{n+j+1}$$

$$\text{For } j > n + 1 \quad d^{r(j)}(y_j) = v^{r(j)}w_{n+j} \quad d^{r'(j)}(w_{n+j+1/2}) = v^{r'(j)}z_{n+j+1}$$

and, ignoring the permanent free terms and the previously created v -torsion

For $0 < j \leq n + 1$, $r(j) = 2^j = r'(j)$

$$E_{2^{j+1}} = P[v, y_{j+1}] \otimes_{j \leq i \leq n} TP_{2^{n+1}}[w_{n+i+1}] \otimes_{0 \leq i < j} E[w_{2^{n+2}+i}] \otimes_{0 \leq s} TP_{2^n}[z_{2^{n+2}+s}]$$

For $n + 2 < j$,

$$E_{r'(j-1)+1} = P[v, y_j] \otimes_{0 \leq i \leq n} E[w_{n+j+i}] \otimes_{0 \leq s} TP_{2^n}[z_{n+j+s+1}]$$

For $n + 1 < j$,

$$E_{r(j)+1} = P[v, y_{j+1}] \otimes E[w_{n+j+1/2}] \otimes_{0 < i \leq n} E[w_{n+j+i}] \otimes_{0 \leq s} TP_{2^n}[z_{n+j+s+1}]$$

We could rewrite $TP_{2^{n+1}}[w_{n+i+1}]$ as $E[w_{n+i+1}] \otimes TP_{2^n}[z_{n+i+2}]$ for $0 \leq i \leq n$. If we did that, we could write proposition 9.2 without the exceptional cases. Since our interest is in the $k(n)^*$ -module structure and not so much in the multiplicative structure, we do this for our final result.

Theorem 9.3. *The 2-primary $k(n)^*(K_2)$ as a $k(n)^*$ -module is the sum of the following:*

$$\begin{aligned} & P[v] \otimes_{0 < i < n} TP_{2^{n-i}}[z_{i+1}] \\ & \oplus_{0 < j} TP_{r(j)}[v] \otimes P[y_{j+1}] \otimes \overline{E}[w_{n+j}] \otimes_{0 < i \leq n} E[w_{n+j+i}] \otimes_{0 \leq s} TP_{2^n}[z_{n+j+s+1}] \\ & \oplus_{0 < j} TP_{r'(j)}[v] \otimes P[y_{j+1}] \otimes_{0 < i \leq n} E[w_{n+j+i}] \otimes \overline{TP}_{2^n}[z_{n+j+1}] \otimes_{0 \leq s} TP_{2^n}[z_{n+j+s+1}] \end{aligned}$$

plus a computable family of \mathbf{Z}_2 's annihilated by v coming from the $E[Q_n]$ -free part of H^*K_2 .

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DEPARTMENT OF MATHEMATICS, LEHIGH UNIVERSITY, BETHLEHEM, PA 18015, USA

Email address: dmd1@lehigh.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER, NY 14627, USA

Email address: dravenel@ur.rochester.edu

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 01220, USA

Email address: wwilson3@jhu.edu