2020 LEHIGH UNIVERSITY HIGH SCHOOL MATH CONTEST, annotated by the number of the 82 students who scored at least 19 answered them correctly.

1. [82] Simplify $1/(\frac{1}{5} - \frac{1}{6})$.
2. [81] What is the area of a triangle with side lengths 16, 30, and 34?
3. [80] How many liters of 20% vinegar solution should be added to 4 liters of 50% vinegar solution to make a 30% vinegar solution?
4. [78] Alex can paint a room in 3 hours. Working together efficiently without overlapping, Alex and Chris can paint it in 72 minutes. How many minutes would it take Chris to paint it alone?
5. [69] Professor Davis prefers to give the Saturday Lehigh Contest on February 29 whenever possible. In what year will be the next time (after 2020) that February 29 falls on a Saturday?
6. [72] If a glasses-wearing student enters the library, the fraction of students wearing glasses will be 1/4. If, instead, a glasses-wearing student leaves the library, the fraction wearing glasses will be 1/5. How many students are in the library (before either hypothetical change)?
7. [77] Suppose $p > 0$ and $x^2 + px + 1 = 0$ has solutions which differ by 1. What is the value of p?
8. [78] Points A and B lie on a circle of radius 15 with $AB = 24$. Point C lies outside the circle on ray AB with $BC = 28$. What is the distance from C to the center of the circle?
9. [79] What is the smallest positive integer for which the product of its digits equals 16,000?
10. [69] What is the minimum value of x^2+y^2 for all (x, y) satisfying $(x - 5)^2 + (y - 12)^2 = 196$?

11. [74] What is the area of a trapezoid $ABCD$ with parallel sides $AD = 2$ and $BC = 8$, while $AB = 6$ and $DC = 4$?

12. [63] Compared to 20 years ago, the number of boys taking this contest increased by 20%, the number of girls increased by 30%, and the total number of students increased by 22%. What fraction of the students taking the contest this year are girls? (Fictitious numbers.)

13. [46] Let r_1, r_2, r_3, and r_4 denote the (possibly complex) roots of the polynomial $p(x) = 20x^4 + 13x^3 + 11x^2 + 16$. What is the numerical value of $(r_1 + 1)(r_2 + 1)(r_3 + 1)(r_4 + 1)$?

14. [78] List all positive integers n for which $\lfloor n^2/5 \rfloor$ is a prime number.

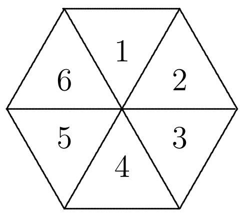
15. [76] Point E is the midpoint of AD in square $ABCD$. Point F is the intersection of BE and AC. What fraction of the area of the square is contained in quadrilateral $CDEF$?

16. [77] What is the sum of the infinite series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{6} + \tfrac{1}{8} + \tfrac{1}{9} + \tfrac{1}{12} + \cdots$$

consisting of the reciprocals of all positive integers which have no prime factors greater than 3?

17. [39] In how many ways can the hexagonal window below be painted so that each triangle is either Red, Green, Blue, or Yellow, and adjacent triangles have different colors?



18. [30] How many ordered pairs (a, b) of integers (not necessarily positive) satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{24}$?

19. [51] How many integers from 10000 to 99999 have distinct digits such that the last digit equals the sum of the first four?

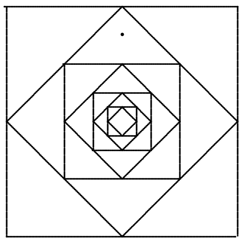
20. [49] Among families with exactly two children, at least one of which is a boy born on Sunday, what fraction have two boys? Assume genders are equally likely, days of birth are equally likely, and these are independent.

21. [41] What is the smallest integer N satisfying $(m+\frac{1}{2021})^2 < N < (m+\frac{1}{2020})^2$ for some positive integer m?

22. [71] How many positive divisors does $56^4 - 20^4$ have?

23. [38] If $\log_4(x+2y) + \log_4(x-2y) = 1$, what is the smallest possible value of $x - y$?

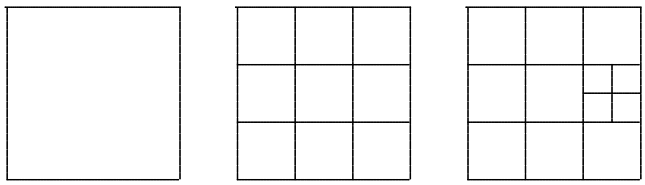
24. [78] A random point is selected in the square pictured below, with the nested squares continuing *ad infinitum*. What is the average of the number of squares in which the point lies? For example, the indicated point lies in 2 squares.



25. [63.5] A 3-by-3 grid is filled randomly with the integers from 1 to 9, with each of these integers included once. What is the probability that each row sum is odd and each column sum is odd?

26. [19] A point P inside a square $ABCD$ has $AP = 5$, $BP = 2\sqrt{2}$, and $CP = 3$. What is the area of the square?

27. [40] Start with a square. Perform any nonnegative number of steps, where a step means dividing an existing subsquare into n^2 congruent smaller subsquares for some $n \geq 2$ by horizontal and vertical lines. Any sequence of steps leaves a certain number of squares, counting only squares which are not further subdivided. What is the sum of all positive integers which cannot be obtained as the number of squares in this way? For example, the diagrams below show how 1, 9, and 12 can be obtained. Note that 2 and 3 are the first two integers that cannot be obtained.



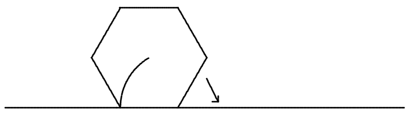
28. [10] What is the largest integer with distinct digits which are all odd such that its product with some positive integer has distinct digits which are all even?

29. [14.5] A tightrope walker stands in the center of a rope of length 16 meters. Every minute she walks forward one meter with probability 2/3 and backward with probability 1/3. What is the probability that she reaches the forward end before getting to the backward end?

30. [44] A function f satisfies $f(n) = n - 2$ if $n > 2200$, and $f(n) = f(f(n + 5))$ if $n \leq 2200$. What is $f(2020)$?

31. [37] A triangle has side lengths 3, 4, and 5. Three spheres are tangent to the plane of the triangle, one at each vertex of the triangle, and are tangent to one another. What is the radius of the largest of these three spheres?

32. [40.5] A regular hexagon with sidelength 1 “rolls” along a line. What is the length of the path that the vertex P travels as the hexagon “rolls” through one full cycle? (There will be six stages to this cycle.)



33. [20] Let a_n denote the nth smallest positive integer for which the sum of the digits equals 7. For example, $a_1 = 7$, $a_2 = 16$, and $a_3 = 25$. If $a_n = 1006$, what is the integer a_{9n}?

34. [17] The weekday on which a person is born is random. A large number of people start naming, one at a time, the weekday on which they were born, until all seven weekdays have been mentioned. What is the expected value of the number of people who will have named a day when, for the first time, all seven days have been mentioned? Assume that each day is equally likely at each step.

35. [3] Suppose $p(x)$ is a polynomial with integer coefficients and $q(x) = \frac{p(x)}{x(1-x)}$. If, for all $x \neq 0$ or 1, $q(x) = q(\frac{1}{1-x})$, $p(2) = 5$ and $p(3) = 5$, then what is the numerical value of $p(4)$?

36. [4] Let x and y be positive real numbers and θ an angle which is not an integer multiple of $\pi/2$. Suppose

$$\frac{\sin\theta}{x} = \frac{\cos\theta}{y} \quad \text{and} \quad \frac{\cos^4\theta}{x^4} + \frac{\sin^4\theta}{y^4} = \frac{7\sin 2\theta}{x^3y + xy^3}.$$

What is the value of $\frac{x}{y} + \frac{y}{x}$?

37. [2] What are the three largest integers a for which there exist positive integers b and c satisfying $\frac{a}{a+1} = \frac{b}{b+2} + \frac{c}{c+3}$?

38. [4] What is the number of ordered triples (a_1, a_2, a_3) of positive integers such that $a_1 + a_2 + a_3 = 2020$, a_1 is not divisible by 2, a_2 is not divisible by 3, and a_3 is not divisible by 4?

39. [3] Let C_1 be a circle of radius 1, and P_1 a square circumscribed around C_1. For each $n \geq 2$, let C_n be the circle circumscribed around P_{n-1}, and let P_n be a regular 2^{n+1}-gon circumscribed around C_n. What is the numerical value of $\lim_{n\to\infty} r_n$, where r_n is the radius of C_n?

40. [2] Triangle ABC has $AB = AC = 3$ and $BC = 1$. Point D lies on AB so that the inradius of triangles ACD and BCD both equal the same number r. What is the value of r?

SOLUTIONS

1. 30. It is $1/\frac{1}{30}$.
2. 240. You should recognize this as twice an 8-15-17 Pythagorean triple. So its area is $\frac{1}{2} \cdot 16 \cdot 30$.
3. 8. $0.2x + 0.5 \cdot 4 = 0.3(x + 4)$, so $0.1x = 0.8$.
4. 120. Alex can paint 1/3 of the room in an hour, and so can paint 2/5 in 6/5 hours (72 minutes). Thus Chris can paint 3/5 in 6/5 hours, and hence could do the whole thing in 2 hours.
5. 2048. There are $4 \cdot 365 + 1 = 1461$ days between leap days. This is 208 weeks plus 5 days. Thus the weekday of February 29 advances by 5 every four years. Since 5 and 7 are relatively prime, seven 4-year periods will be required until it returns to a Saturday.
6. 31. Let G be the number of glasses-wearers out of T total students. Then $\frac{G-1}{T-1} = \frac{1}{5}$ and $\frac{G+1}{T+1} = \frac{1}{4}$. The solution is $G = 7$, $T = 31$.
7. $\sqrt{5}$. If $x^2 + px + 1 = 0$ and $(x + 1)^2 + p(x + 1) + 1 = 0$, then $2x + 1 + p = 0$. Therefore $(\frac{-p-1}{2})^2 + p(\frac{-p-1}{2}) + 1 = 0$, which reduces to $-\frac{p^2}{4} + \frac{5}{4} = 0$. Or, using the discriminant, solve $\sqrt{p^2 - 4} = 1$.
8. 41. Let M be the midpoint of AB, and O the center of the circle. Then the right triangle OMB implies that $OM = 9$, and then the right triangle OMC implies that $OC = 41$.
9. 255588. Since $16000 = 2^7 \cdot 5^3$, we must have three 5s. To minimize the number of digits, we should choose two factors of 8 and one of 2.
10. 1. A point at radius 14 from $(5, 12)$ on a radial line through the origin will lie at distance 1 from the origin, since $(5, 12)$ lies at distance 13 from the origin. Let $C = (5, 12)$, $O = (0, 0)$, and P lie on the given circle. Then $CP \leq CO + OP$, so $OP \geq 14 - 13$.

11. $40\sqrt{2}/3$. If h is the altitude and x the projection of DC onto BC, then $6^2 - (6-x)^2 = h^2 = 4^2 - x^2$. This yields $x = \frac{4}{3}$ and $h = \frac{8}{3}\sqrt{2}$. The area is $h \cdot \frac{2+8}{2}$.

12. 13/61. If B and G denote the number of boys (resp. girls) taking the contest 20 years ago, the current breakdown is $\frac{6B}{5} + \frac{13G}{10} = \frac{122(B+G)}{100}$. This reduces to $B = 4G$. The current fraction who are girls is $\frac{13}{10}/\frac{122 \cdot 5}{100} = 13/61$.

13. 1.7. These are the roots of $q(x) = p(x-1)$. The product of the roots of $q(x)$ equals its constant term, $q(0)$, divided by its coefficient of x^4, which is still 20. Since $q(0) = p(-1) = 20 - 13 + 11 + 16 = 34$, the answer is 34/20.

14. 4, 5, 6. For $d = 0, 1, 2, 3, 4$, $\lfloor (5k+d)^2/5 \rfloor = 5k^2$, $(5k+2)k$, $(5k+4)k$, $(5k+1)(k+1)$, and $(5k+3)(k+1)$, respectively. The only primes occur for $(d,k) = (0,1)$, $(1,1)$, and $(4,0)$.

15. 5/12. Let s denote the side length of the square. Triangles AEF and CBF are similar, and so the area of CBF is $\frac{1}{2}s \cdot \frac{2}{3}s = \frac{1}{3}s^2$. The area of ABE is $\frac{1}{4}s^2$. Thus the desired fraction is $1 - \frac{1}{3} - \frac{1}{4}$.

16. 3. It equals

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{2^i 3^j} = \left(\sum_{i=0}^{\infty} \frac{1}{2^i}\right)\left(\sum_{j=0}^{\infty} \frac{1}{3^j}\right) = \frac{1}{1-\frac{1}{2}} \cdot \frac{1}{1-\frac{1}{3}} = 2 \cdot \tfrac{3}{2} = 3.$$

17. 732. There are $4 \cdot 3^4 = 324$ ways to color all except #4. Of these, $4 \cdot 3^2 = 36$ have 1, 3, and 5 the same color, while $4 \cdot 3 \cdot 2^2 = 48$ have 3 and 5 the same color, but different from 1. These 84 colorings with 3 and 5 the same have 3 choices for #4, so 252 of these. The other 240 ways of coloring all except #4 have 2 choices for #4, so 480 of them.

18. 41. The equation can be manipulated to $(a-24)(b-24) = 24^2 = 576$. Note that $576 = 2^6 \cdot 3^2$ has $7 \cdot 3 = 21$ positive divisors, and so 42 positive or negative divisors x. For all of

these except $x = -24$, letting $a = x + 24$ and $b = \frac{576}{x} + 24$ gives a solution to the required equation.

19. 126. Since $1 + 2 + 3 + 4 = 10$, one of the first four digits must be 0, and it cannot be in the first position. The possibilities for the other three digits are, in some order, 123, 124, 125, 126, 134, 135, and 234. For each of these, there are 3 choices for the first digit, 3 for the second, and then 2 for the third. So the answer is $7 \cdot 18$.

20. 13/27. We list the possible outcomes such as ((B,Sun),(G,Wed)), listing the younger child first. There are 27 possibilities which include (B,Sun) in at least one component. Of these, 13 have B in both components.

21. 1020101. If $N = m^2 + k$, then $\frac{2m}{2021} + \frac{1}{2021^2} < k < \frac{2m}{2020} + \frac{1}{2020^2}$. Then $m = 1010$ and $k = 1$ works.

22. 216. It factors as $2^8(14^2+5^2)(14-5)(14+5) = 2^8 3^2 \cdot 13 \cdot 17 \cdot 19$, so the number of divisors is $9 \cdot 3 \cdot 2^3$.

23. $\sqrt{3}$. The equation implies $x^2 - 4y^2 = 4$ and $x \pm 2y > 0$, so $x > 0$. The minimum value of $x - y$ can be found using calculus. Here is a solution without calculus. Let $u = x - y > 0$ since $x = \sqrt{4 + 4y^2}$. Then $(y+u)^2 - 4y^2 - 4 = 0$, so $3y^2 - 2uy + 4 - u^2 = 0$. Since this has solutions, $(-2u)^2 - 12(4 - u^2) \geq 0$ so $u \geq \sqrt{3}$. One can check that $x = \frac{4}{3}\sqrt{3}$ and $y = \frac{1}{3}\sqrt{3}$ works.

24. 2. Since the four outer triangles at each stage comprise half of the remaining area, the answer is $1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 3 \cdot \frac{1}{2^3} + \cdots$. This equals 2, as can be seen from

$$\begin{aligned} & \left(\tfrac{1}{2} + \tfrac{1}{4} + \cdots\right) + \left(\tfrac{1}{4} + \tfrac{1}{8} + \cdots\right) + \left(\tfrac{1}{8} + \tfrac{1}{16} + \cdots\right) + \cdots \\ = \; & \left(\tfrac{1}{2} + \tfrac{1}{4} + \cdots\right)\left(1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots\right) = 1 \cdot 2 = 2. \end{aligned}$$

25. 1/14. The five odd integers will have to be placed so that one row has three of them and one column has three of them. There are nine ways to choose the (row,column) pair, each of which

has three odd entries. Once this choice has been made, there are $5! \cdot 4!$ ways to place the integers. So the answer is $9 \cdot 5! \cdot 4!/9!$. (You can also think about placing five 1's and four 0's.)

26. 29. Let x denote the perpendicular distance from P to AB, y the perpendicular distance from P to BC, and s the side length of the square. Then $25 = (s-y)^2 + x^2$, $8 = x^2 + y^2$, and $9 = (s-x)^2 + y^2$. The first two equations give $y = (s^2 - 17)/2s$ and the second two give $x = (s^2 - 1)/2s$. Substituting these into the middle equation reduces to $2(s^4 - 34s^2 + 145) = 0$, which implies $s^2 = 5$ or 29. But 5 is too small.

27. 49. The numbers which cannot be constructed are 2, 3, 5, 6, 8, 11, and 14. You can add 3 or 8 squares at any step (also 15 and other $m^2 - 1$, but these are not necessary). We want the numbers not expressible as $1 + 3k + 8\ell$ for nonnegative integers k and ℓ. By considering mod 3 congruence, you can easily see that our list is correct.

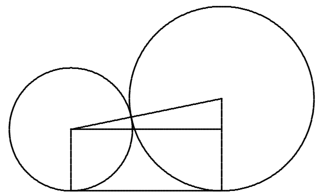
28. 7531. Its product with 8 is 60248. Let x be a number with distinct odd digits and y a number such that $x \cdot y$ has distinct even digits. Clearly y must be even and $y \neq 10$. If $y = 2$, 4, 6, or 8, and x contains a 9, then the digit in the product preceding the $9 \cdot y$ digit would be odd, due to carrying. Thus $x \leq 7531$ if $y < 12$. If $x > 7531$ and $y \geq 12$, then $x \cdot y > 86420$, which is the largest number with distinct even digits.

29. 256/257. After 2 minutes, she is 2 meters forward with probability $(2/3)^2$ and 2 meters backward with probability $(1/3)^2$. Ignore any period of no net movement. Now group periods into 2-minute intervals. The probabilities of moving forward 4 or backward 4 are $(2/3)^4$ and $(1/3)^4$, respectively. Similarly, we find that she is $2^8 = 256$ times as likely to move 8 steps forward as to move 8 steps backward.

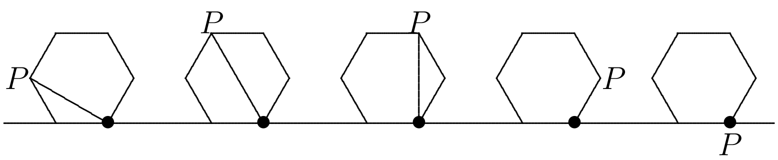
Alternate solution: A path which reaches the forward end first is a sequence containing $(k+8)$ F's and k B's, ending with an F. Any such sequence has probability $(\frac{2}{3})^{k+8}(\frac{1}{3})^k$. There is a corresponding sequence obtained by interchanging F's and B's which reaches the back end first and has probability $(\frac{1}{3})^{k+8}(\frac{2}{3})^k$. The ratio of the probabilities is $2^8 : 1$. Thus the probability of reaching the front end first is 256/257.

30. 2201. Let $\overline{m}$ denote the remainder when m is divided by 3. We claim $f(n) = 2199 + \overline{n+1}$ for $n \le 2203$. This is true for $2201 \le n \le 2203$, then for $2198 \le n \le 2200$ since $f(n) = f(f(n+5)) = f(n+3) = n+1$, then for $n \le 2197$ since $f(n) = f(f(n+5)) = f(2199 + \overline{n+6}) = 2199 + \overline{n+1}$.

31. 10/3. Let a, b, and c be the side lengths of the triangle. For side a, with spheres of radius $r_2 > r_1$ tangent to its end points, the diagram below shows that $(r_1 + r_2)^2 = (r_2 - r_1)^2 + a^2$, so $a^2 = 4r_1r_2$. Doing this for the other sides, and multiplying yields $a^2b^2c^2 = 64r_1^2r_2^2r_3^2$, so $abc = 8r_1r_2r_3$. Thus $bc = 2ar_3$, so $r_3 = bc/2a$. In our case, the largest radius is $4 \cdot 5/2 \cdot 3$.



32. $\frac{2\pi}{3}(2+\sqrt{3})$. The path consists of six arcs with central angle $\pi/3$, with respective radii 1, $\sqrt{3}$, 2, $\sqrt{3}$, 1, and 0. The diagram shows the position of P, the new pivot point, and the radius of rotation of P after each of the first five rotations.



33. 100024. The number of integers with $\le d$ digits and digital sum 7 equals the number of nonnegative integer solutions of $x_1 + \cdots + x_d = 7$, which equals $\binom{d+6}{d-1}$. This formula, quite well known, can be proved by a "stars-and-bars" argument. There

are 36 such integers with ≤ 3 digits. Thus $1006 = a_{37}$. We need the integer a_{333}. The number of our integers with ≤ 5 digits is $\binom{11}{4} = 330$. So we want the third such number with six digits.

34. 18.15. The answer is $1 + \frac{7}{6} + \frac{7}{5} + \frac{7}{4} + \frac{7}{3} + \frac{7}{2} + \frac{7}{1}$, where the ith summand, $\frac{7}{7-(i-1)}$, is the expected number of tries for the ith new name to occur after $i - 1$ days have been named. This is true because on each try there is probability $p = \frac{7-(i-1)}{7}$ of a new day being named, and the expected number of trials until the first such event is $1/p$.

35. -5. The relations on p and q imply that $p(x) = -(1-x)^3 p(\frac{1}{1-x})$. This implies that the degree of p is at most 3, since otherwise the right hand side would not be a polynomial, as it would contain a term $\frac{1}{1-x}$. Let $p(x) = ax^3 + bx^2 + cx + d$. The relation implies that $a = d$ and $b = -c - 3d$. The given values for $p(2)$ and $p(3)$ imply that $5 = -3a - 2c$ and $5 = a - 6c$, hence $a = c = -1$. We then obtain $p(x) = -x^3 + 4x^2 - x - 1$ and $p(4) = -5$.

36. $\sqrt{6}$. Let k satisfy $x = k \sin\theta$ and $y = k\cos\theta$. We obtain

$$\frac{\cos^4\theta}{\sin^4\theta} + \frac{\sin^4\theta}{\cos^4\theta} = \frac{14\sin\theta\,\cos\theta}{\sin\theta\,\cos\theta(\sin^2\theta + \cos^2\theta)} = 14.$$

Let $S = \frac{x}{y} + \frac{y}{x}$. Then $(S^2 - 2)^2 - 2 = \frac{\sin^4\theta}{\cos^4\theta} + \frac{\cos^4\theta}{\sin^4\theta} = 14$, so $S = \sqrt{6}$.

37. 11, 23, and 27. Since

$$1 > \frac{a}{a+1} = \frac{b}{b+2} + \frac{c}{c+3} = \frac{bc + 3b + 2c + bc}{bc + 3b + 2c + 6},$$

we have $bc < 6$. There are 10 possible pairs (b, c), and $(4, 1)$, $(1, 5)$, and $(5, 1)$ yield $\frac{11}{12}$, $\frac{23}{24}$, and $\frac{27}{28}$, the largest obtainable fractions of the desired form.

38. 510050. The generating function with exponents giving the possible values of a_1 is

$$x + x^3 + x^5 + \cdots = \tfrac{1}{1-x} - \tfrac{1}{1-x^2} = \tfrac{x(1-x)}{(1-x)(1-x^2)}.$$

Similarly, the generating function for a_2 is $\frac{1}{1-x} - \frac{1}{1-x^3} = \frac{x(1-x^2)}{(1-x)(1-x^3)}$, and for a_3 it is $\frac{x(1-x^3)}{(1-x)(1-x^4)}$. We want the coefficient of x^{2020} in their product, $\frac{x^3}{(1-x)^2(1-x^4)}$, so the coefficient of x^{2017} in

$$(1 + 2x + 3x^2 + 4x^3 + \cdots)(1 + x^4 + x^8 + x^{12} + \cdots).$$

This coefficient is $2 + 6 + \cdots + 2018 = 505 \cdot 1010$.

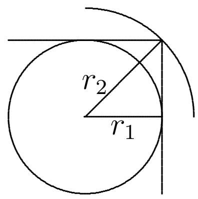
39. $\pi/2$. The diagram below shows that $r_2 = r_1/\cos(\pi/4)$. Similarly $r_n = r_{n-1}/\cos(\pi/2^n)$, and so $r_n = 1/\prod_{k=2}^{n} \cos(\pi/2^k)$. Next note that, for any n and any x,

$$\begin{aligned} \sin(x) &= 2\sin(\tfrac{x}{2})\cos(\tfrac{x}{2}) = 2^2 \sin(\tfrac{x}{4})\cos(\tfrac{x}{4})\cos(\tfrac{x}{2}) \\ &= \cdots = 2^n \sin(x/2^n) \prod_{k=1}^{n} \cos(x/2^k). \end{aligned}$$

Thus

$$\frac{\sin x}{x} = \lim_{n\to\infty} \left(\frac{\sin(x/2^n)}{x/2^n} \prod_{k=1}^{n} \cos(x/2^k) \right) = \prod_{k=1}^{\infty} \cos(x/2^k),$$

since $\lim\limits_{h\to 0}(\sin(h)/h) = 1$. Let $x = \pi/2$ and find that $\frac{2}{\pi} = \prod\limits_{k=1}^{\infty} \cos(\pi/2^{k+1})$. Hence $\lim\limits_{n\to\infty} r_n = 1/\frac{2}{\pi}$.



40. $(\sqrt{35} - \sqrt{5})/12$. Let $x = BD$ and $y = CD$. Since the product of inradius by semiperimeter equals area, we obtain that the ratio of areas $\frac{ACD}{BCD}$ equals $\frac{6-x+y}{y+1+x}$. On the other hand, it equals $\frac{AD}{BD} = \frac{3-x}{x}$. Equating these ratios reduces to $y = \frac{4x-3}{3-2x}$. Applying the Law of Cosines to triangle BCD yields $y^2 = x^2 + 1 - 2x \cdot \frac{1}{6}$. Substituting the value of y into this reduces to the equation $12x^4 - 40x^3 + 3x^2 + 27x = 0$. This factors as $x(x-3)(12x^2 - 4x - 9) = 0$ with $x = \frac{1+2\sqrt{7}}{6}$ the root of interest. We obtain $y =$

$\sqrt{7}/2$. The area of triangle ABC is $\frac{1}{2}\sqrt{3^2 - (\frac{1}{2})^2} = \sqrt{35}/4$. On the other hand, it equals r times the sum, $\frac{6-x+y}{2} + \frac{y+1+x}{2}$, of the semiperimeters. This becomes $(7+2y)r = \sqrt{35}/2$. Substituting our value of y reduces to the claimed value of r.