2018 LEHIGH UNIVERSITY HIGH SCHOOL MATH CONTEST

1. A right triangle has hypotenuse 29 and one leg 21. What is the length of the other leg?
2. Don is 2/3 of the way through his run. After running another 1/2 mile, he is 3/4 of the way through his run. What is the total number of miles in his run?
3. Write $8^{1/3} \cdot 81^{-1/4}$ in simplest form.
4. If $\frac{2}{x} = \frac{y}{3} = \frac{x}{y}$, then what is the value of x?
5. The changes in the population of a city in four consecutive censuses are, respectively, 20% decrease, 20% decrease, 20% increase, and 20% increase. Rounded to the nearest percent, what is the overall change after the four censuses? Include the sign of the change, positive, negative, or zero in your answer.
6. What is the sum of the solutions of $|x + 3| = 3|x - 2|$?
7. A small car radiator has 6-liter capacity. If it is filled with liquid which is 40% antifreeze, how many liters of the liquid must be replaced with pure antifreeze in order that the radiator becomes filled with a mixture which is 60% antifreeze?
8. Suppose the earth is a perfect sphere 25000 miles in circumference, and suppose it is possible to erect a telephone line on poles around the equator. Assuming that the telephone wire would then form a circle concentric with the equator, what is the height of the poles, rounded to the nearest foot, if the length of the wire is 100 feet longer than the circumference of the earth?
9. If all people eat the same amount, and a circular pizza 12 inches in diameter serves two people, how many inches in diameter must each of three equal pizzas be in order to serve four people?

10. The angle of elevation to the top of a building is 45 degrees. If you move 20 feet farther away, the angle of elevation to the top of the building becomes 30 degrees. What is the height of the building (in feet)?

11. Suppose f is a function such that for every real number x, $f(x) + f(1 - x) = 10$ and $f(1 + x) = 4 + f(x)$. What is the value of $f(100) + f(-100)$?

12. What is the sum of the digits of the smallest positive integer which is divisible by 99 and has all of its digits equal to 2?

13. A point is located in the interior of a rectangle so that its distances from two opposite corners are 5 and 14, while its distance from a third corner is 10. What is its distance from the fourth corner?

14. How many positive integers are divisors of
$$29^4 + 4 \cdot 29^3 + 6 \cdot 29^2 + 4 \cdot 29 + 1?$$

15. A semicircle and an equilateral triangle are constructed on the same side of the same base of length 2. What is the area of the region inside the triangle but outside the semicircle?

16. If $x > y > 0$ and $2 \log_{10}(x - y) = \log_{10} x + \log_{10} y$, what is the value of x/y?

17. What is the value of b if a, b, and c satisfy the following equations?
$$\begin{aligned} & 1 + a + b + c \\ = \; & 16 + 8a + 4b + 2c \\ = \; & 81 + 27a + 9b + 3c \\ = \; & 256 + 64a + 16b + 4c \end{aligned}$$

18. What is the largest 2-digit prime factor of $\binom{200}{100}$?

19. How many 6-digit numbers whose leftmost digit is 1 have exactly two pairs of identical digits (and no digit occurs three or more times)?

20. A point inside a circle of radius $\sqrt{50}$ lies 2 units directly below a point on the circle, and 6 units directly to the right of a point on the circle. What is the distance from the center of the circle to this point?

21. List all real values of x which satisfy

$$\frac{6}{\sqrt{x-8}-9}+\frac{1}{\sqrt{x-8}-4}+\frac{7}{\sqrt{x-8}+4}+\frac{12}{\sqrt{x-8}+9}=0.$$

22. What is the fifth smallest positive integer such that if the leftmost digit is removed, the resulting number is one-fifth of the original?

23. The solution set of the inequality $\frac{4x^2}{(1-\sqrt{1+2x})^2}<2x+9$ is of the form $-a \leq x < b$, $x \neq 0$, with $a, b > 0$. What is the ordered pair $(-a, b)$?

24. Two noncongruent isosceles triangles with integer sides have equal perimeters and equal areas and ratio of bases 9:8. What is the common perimeter of the smallest pair of such triangles?

25. What is the smallest integer $n > 1$ such that 3^n ends with 003?

26. If $(x^2-5x+7)(4x-5)(2x+1) = Ax^4+Bx^3+Cx^2+Dx+E$, what is the value of $80A+28B+8C+4D$?

27. In triangle ABC, D lies on BC such that AD bisects angle A. The point P lies on AC such that BP passes through the midpoint of AD. If $AB = 3$ and $AC = 1$, what is the length of AP?

28. A sequence is formed by taking products of corresponding terms of two arithmetic sequences. Its first three terms are 10, 37, and 86. What is its 11th term?

29. Let $f(x) = \frac{1}{3(1-x)}$ for $|x| < 1$. The domain of f is the interval $(-1, 1)$. Let f^n denote the n-fold iterate of f. Thus $f^1 = f$ and $f^n = f \circ f^{n-1}$. What is the largest value of n for which f^n has nonempty domain? Note that the domain $\mathrm{dom}(g \circ h)$ of a composite is the set of x in $\mathrm{dom}(h)$ such that $h(x)$ is in $\mathrm{dom}(g)$.

30. A sequence satisfies $a_1 = -14$, $a_2 = 14$, and for $k \geq 3$, $a_k = a_{k-2} + \frac{4}{a_{k-1}}$. What is the smallest value of k for which $a_k = 0$?

31. The eight people in three families sit randomly around a round table. Two families have three members, and the other has two. What is the probability that everyone sits next to at least one person from a family other than his/her own?

32. Triangle ABC has sides AB, AC, and CB of lengths 5, 3, and 7, respectively. Side CB is extended through B to point D. Point F is the intersection of the bisector of external angle ABD with the extension of line AC. What is the length of segment AF?

33. How many positive integers ≤ 2018 have strictly more 1's than 0's in their binary expansion?

34. Triangle ABC has $AB = 8$, $AC = 9$, and $BC = 7$. Point O is the center of the circle inscribed in this triangle. Segment DE passes through O and is parallel to BC, with D and E lying on AB and AC, respectively. What is the length of DE?

35. Point P lies on the positive x-axis outside the circle $x^2 + y^2 = 1$. Points A and B lie on this circle, one in the first quadrant and one in the fourth quadrant, so that PA and PB are tangent to the circle, and angles APO and BPO are both 45 degrees, where O is the origin. What is the total area of the infinite sequence of circles tangent to one another and to the two segments, and centered on the positive x-axis such that the first of these circles is tangent to the given circle? Do not include the given circle in your area sum.

36. A slot machine has ten pictures, which occur randomly. One of them is a snowman. Kenny and Audrey each have a machine,

and each plays until he/she gets a snowman. What is the probability that their number of plays is within one of each other?

37. If $A_1, A_2, \ldots, A_n$ are consecutive vertices of a regular n-gon such that $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$, what is the value of n?
38. Find all sets of three prime numbers $\{p, q, r\}$ such that $p+q = r$ and $(r-p)(q-p) - 279p$ is a perfect square.
39. Find the maximum value of $(ab + bc + cd)/(a^2 + b^2 + c^2 + d^2)$ for positive real numbers a, b, c, and d.
40. What is the largest value of $\sin x$ which satisfies

$$\sin x + \sin 2x + \cos x = 0?$$

SOLUTIONS, annotated with the number of people answering correctly, out of the 56 people who scored at least 20.

1. 20. [55] $29^2 - 21^2 = (29-21)(29+21) = 8 \cdot 50 = 400$. The answer is $\sqrt{400}$.
2. 6. [56] $\frac{3}{4} - \frac{2}{3} = \frac{1}{12}$. The answer is $12 \cdot \frac{1}{2}$.
3. 2/3. [56] $\sqrt[3]{8} = 2$ and $\sqrt[4]{81} = 3$.
4. $\sqrt[3]{12}$. [53] $x^2 = 2y$ and $y = 6/x$. Thus $x^3 = 12$.
5. -8% or 8% decrease. [52.5] The ratio of the final population to the original is $.8^2 \cdot 1.2^2 = (1-.04)^2 = 1 - .08 + .0016$.
6. $\frac{21}{4}$ or 5.25. [54] Either $x+3 = 3(x-2)$, in which case $x = 4.5$, or $x+3 = -3(x-2)$, in which case $x = 3/4$.
7. 2. [46] If x is the desired number of liters, then $x + .4(6-x) = 3.6$, so $.6x = 1.2$.
8. 16. [48] The increase in the circumference equals 2π times the increase in the radius, which is the height of the poles. Thus the height of the poles is $\frac{100}{2\pi}$ feet, which is approximately 16.
9. $4\sqrt{6}$. [49] $\frac{12^2}{2} = 3d^2/4$, so $d^2 = 288/3 = 96 = 4^2 \cdot 6$.
10. $10(\sqrt{3}+1)$. [56] If x is the height of the building, then the 30 degree angle will be part of a triangle with base $x+20$ and height x. Thus $\frac{x+20}{x} = \sqrt{3}$, and so $x = \frac{20}{\sqrt{3}-1} = 10(\sqrt{3}+1)$.
11. 6. [54] Replace x by $-x$ in the second property, and get $f(1-x) = 4 + f(-x)$. Substitute this in the first property and get $f(x) + 4 + f(-x) = 10$ for all x.
12. 36. [55] The number must be divisible by 9, so its number of digits must be divisible by 9. It must be divisible by 11, so it must have an even number of digits. Thus the number has 18 2's.
13. 11. [51] If A, B, C, and D are the distances from the four edges, moving around the rectangle, then $A^2 + B^2 = 5^2$, $B^2 + C^2 = 10^2$, and $C^2 + D^2 = 14^2$. Our desired distance is $\sqrt{A^2 + D^2}$.

Subtracting the middle equation from the sum of the first and third yields $A^2 + D^2 = 25 + 196 - 100 = 11^2$.

14. 125. [56] The number is $(29+1)^4 = 2^4 \cdot 3^4 \cdot 5^4$. The number of divisors of this is $(4+1)^3$.

15. $\frac{1}{2}\sqrt{3} - \frac{\pi}{6}$. [49] From the center of the base, a rhombus is formed with other vertices the interior points of intersection of the semicircle and the triangle, and the vertex of the triangle. Its area is that of two unit equilateral triangles, $\frac{1}{2}\sqrt{3}$. The desired area equals that minus 1/3 of the area of the semicircle.

16. [50] $\frac{1}{2}(3+\sqrt{5})$. $(x-y)^2 = xy$, so $x^2 - 3xy + y^2 = 0$. If $r = \frac{x}{y}$, then $r^2 - 3r + 1 = 0$ so $r = \frac{1}{2}(3 \pm \sqrt{5})$. Since $x/y > 1$, we choose the + sign.

17. 35. [38] Let $f(x) = x^4 + ax^3 + bx^2 + cx$. This satisfies $f(1) = f(2) = f(3) = f(4)$. If K denotes this common value, then
$$x^4 + ax^3 + bx^2 + cx - K = (x-1)(x-2)(x-3)(x-4),$$
so $b = 1\cdot 2 + 1\cdot 3 + 1\cdot 4 + 2\cdot 3 + 2\cdot 4 + 3\cdot 4 = 35$.

18. 61. [51] The number is $\frac{200\cdot 199\cdots 101}{100!}$. Primes such as 97 down through 67 will occur once in the numerator and once in the denominator. 61 will occur twice in the numerator and once in the denominator.

19. 22680. [27] There are $\binom{9}{2}\binom{5}{2}\binom{3}{2}7$ ways in which the 1 is not paired. The first factor is "which two numbers are paired." The second is the positions of the larger pair of identical digits. The $\binom{3}{2}$ is the position of the smaller pair, and the 7 is which was the non-paired non-1 digit. There are $5 \cdot 9 \cdot \binom{4}{2} \cdot 8 \cdot 7$ ways in which 1 is one of the paired digits. The $\binom{4}{2}$ is the position of the other paired digit, and the $8 \cdot 7$ is what digits went in the remaining spaces. The total is $9 \cdot 8 \cdot 7 \cdot 5(3+6)$.

20. $\sqrt{26}$. [25.5] If the circle is $x^2 + y^2 = 50$ and the coordinates of the desired point are (a, b), then $a^2 + (b+2)^2 = 50$ and $(a-6)^2 + b^2 = 50$. Thus $a^2 + b^2 + 4b = 46$ and $a^2 + b^2 - 12a = 14$,

hence $4b + 12a = 32$. Thus $a^2 + (8 - 3a)^2 - 12a = 14$, or $10a^2 - 60a + 50 = 0$. This gives (a, b) as $(1, 5)$ or $(5, -7)$. The second point lies outside the circle.

21. 17, 44. Let $y = \sqrt{x - 8}$. [34] The equation becomes $\frac{18y-54}{y^2-81} + \frac{8y-24}{y^2-16} = 0$, or $(y - 3)(\frac{18}{y^2-81} + \frac{8}{y^2-16}) = 0$. Thus $y = 3$ or $26y^2 - (8 \cdot 81 + 16 \cdot 18) = 0$. The second equation yields $y^2 = 36$, so $y = 6$. Finally $x = 3^2 + 8$ or $6^2 + 8$.

22. 1250. [46] If d is the first digit, with $k+1$ digits overall, and m is what remains when the first digit is removed, then $d \cdot 10^k + m = 5m$. If $k = 1$, then $5d = 2m$, yielding the number 25. If $k \geq 2$, then $25d \cdot 10^{k-2} = m$. We must have $d \leq 3$, so there is no carrying. The first five numbers are 25, 125, 250, 375, and 1250.

23. $(-\frac{1}{2}, \frac{45}{8})$. [22] Let $y = \sqrt{1 + 2x}$ so $y^2 = 1 + 2x$. The inequality becomes $(y^2 - 1)^2 < (1 - y)^2(y^2 + 8)$, hence $(y + 1)^2 < y^2 + 8$, so $y < \frac{7}{2}$. Thus $1 + 2x < \frac{49}{4}$ so $x < \frac{45}{8}$. We need $x \geq -\frac{1}{2}$ in order that $\sqrt{1 + 2x}$ is defined. The left hand side of the inequality is not defined when $x = 0$.

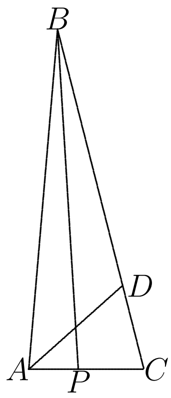
24. 868. [10] Let the sides of one be a, a, $16c$, and the sides of the other be b, b, $18c$. Then $a + 8c = b + 9c$, so $c = a - b$. Also $8c\sqrt{a^2 - (8c)^2} = 9c\sqrt{b^2 - (9c)^2}$. Since $a + 8c = b + 9c$, this implies $8\sqrt{a - 8c} = 9\sqrt{b - 9c}$, so $64a - 512c = 81b - 729c$. Inserting $c = a - b$ leads to $281a = 298b$. The smallest value will occur when $a = 298$, $b = 281$, and $c = 17$.

25. 101. [24] Since $3^{n-1} \equiv 1 \bmod 1000$, we must have $n - 1 = 4k$. Mod 1000, $0 \equiv 3^{4k} - 1 = (80 + 1)^k - 1 \equiv 80k + \binom{k}{2} \cdot 80^2$. This implies that k is a multiple of 5, and then that k is a multiple of 25.

26. -68. [42] If $p(x)$ is the given polynomial, the desired expression equals $p(3) - p(-1) = 1 \cdot 7 \cdot 7 - 13(-9)(-1) = 49 - 117$.

27. 3/7. [36] $BD = 3 \cdot DC$ by the Angle Bisector Theorem. Assign mass points so that $m(B) = 1$. Then $m(C) = 3$ and $m(D) = 4$.

Hence $m(A) = 4$ and $\frac{AP}{PC} = \frac{3}{4}$.



28. 1270. [39] Such a sequence will be quadratic; i.e., $x_n = a + bn + cn^2$. Thus

$$\begin{aligned} a + b + c &= 10 \\ a + 2b + 4c &= 37 \\ a + 3b + 9c &= 86 \end{aligned}$$

Solving these by row reduction shows that $x_n = 5 - 6n + 11n^2$, so $x_{11} = 5 - 66 + 11^3$. Alternatively, solve $aa' = 10$, $(a+d)(a'+d') = 37$, and $(a+2d)(a'+2d') = 86$, and then find $(a+10d)(a'+10d')$.

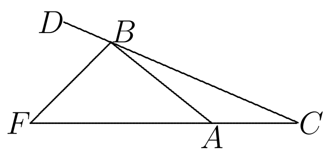
29. 5. [15] First note that $f^{-1}(x) = 1 - \frac{1}{3x}$ provided $x > \frac{1}{6}$. Now

$$\begin{aligned} \mathrm{dom}(f^2) &= f^{-1}(-1,1) = f^{-1}(\tfrac{1}{6},1) = (-1,\tfrac{2}{3}) \\ \mathrm{dom}(f^3) &= f^{-1}(\mathrm{dom}(f^2)) = f^{-1}(-1,\tfrac{2}{3}) = f^{-1}(\tfrac{1}{6},\tfrac{2}{3}) = (-1,\tfrac{1}{2}) \\ \mathrm{dom}(f^4) &= f^{-1}(\mathrm{dom}(f^3)) = f^{-1}(-1,\tfrac{1}{2}) = f^{-1}(\tfrac{1}{6},\tfrac{1}{2}) = (-1,\tfrac{1}{3}) \\ \mathrm{dom}(f^5) &= f^{-1}(\mathrm{dom}(f^4)) = f^{-1}(-1,\tfrac{1}{3}) = f^{-1}(\tfrac{1}{6},\tfrac{1}{3}) = (-1,0) \\ \mathrm{dom}(f^6) &= f^{-1}(\mathrm{dom}(f^5)) = f^{-1}(-1,0) = f^{-1}(\tfrac{1}{6},0) = \emptyset. \end{aligned}$$

30. 51. [16] Note that $a_k a_{k-1} = a_{k-1}a_{k-2} + 4$, so if $p_k = a_k a_{k-1}$, then $p_k = p_{k-1} + 4$. Thus $p_k = -196 + 4(k-2)$. This equals 0 when $k = 51$. Thus $a_{50}a_{49} \neq 0$ but $a_{51}a_{50} = 0$.

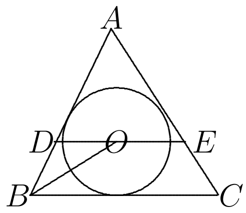
31. 53/70. [12] Number the seats and treat family members as indistinguishable. There are $\frac{8!}{3!3!2!} = 560$ ways to seat the people. If families A and B are the ones with three members, there are $8 \cdot \frac{5!}{3!2!} = 80$ ways in which family A sits together (so that one of their members is not sitting next to anyone from another family) and also 80 ways in which family B sits together. There are $8 \cdot 3$ ways in which both families A and B sit together. Thus there are $160 - 24 = 136$ ways in which someone is surrounded by family members. The answer is $(560 - 136)/560$.

32. 15/2 or 7.5. [9] Let θ denote $\angle FBA$, and let x denote the desired length AF. Then $\frac{\sin F}{5} = \frac{\sin\theta}{x}$. Note that $\angle FBC = 180 - \angle FBD = 180 - \theta$. Hence $\frac{\sin F}{7} = \frac{\sin(180-\theta)}{x+3}$. Since $\sin(180-\theta) = \sin\theta$, we obtain $\frac{x}{5} = \frac{x+3}{7}$, so $x = 15/2$.



33. 1170. [6] We will count the numbers through 2047 having this property and then subtract 29, since all numbers from 2019 to 2047 begin with at least six 1's. Half of the numbers with an even number of bits have this property, since they have an odd number of bits after their initial 1. Of the 2^{2k} numbers with $2k$ bits following the initial 1, the number with this property is $\frac{1}{2}(2^{2k} + \binom{2k}{k})$ since those with k 1's will be included, and half of the others will. Thus the number ≤ 2047 is $\frac{1}{2}(2047 + \binom{0}{0} + \binom{2}{1} + \binom{4}{2} + \binom{6}{3} + \binom{8}{4} + \binom{10}{5}) = \frac{1}{2}(2047+1+2+6+20+70+252) = 1199$.

34. 119/24. [12] BO bisects angle B, and since also $\angle OBC = \angle DOB$, triangle DOB is isosceles with $DB = DO$. Similarly $OE = EC$. Thus the perimeter of ADE equals $8+9$. By similar triangles DE/BC equals the ratio of the perimeters which is $(8+9)/(8+9+7)$.



35. $\frac{\pi}{8}(3\sqrt{2}-4)$. [7] The upper segment to which all the circles are tangent is $x+y=\sqrt{2}$. If r is the radius of the first new circle, with center at $(1+r,0)$, the point of tangency of this circle with the line $x+y=\sqrt{2}$ will be at the point $(1+r+r\frac{\sqrt{2}}{2}, r\frac{\sqrt{2}}{2})$. Thus $1+r+r\sqrt{2}=\sqrt{2}$, so $r=\frac{\sqrt{2}-1}{\sqrt{2}+1}=3-2\sqrt{2}$. The ratios of radii of successive circles will have this same value of r. Thus the total area is

$$\pi r^2/(1-r^2)=\pi(\tfrac{1}{2}(\tfrac{1}{1-r}+\tfrac{1}{1+r})-1)=\pi(\tfrac{1}{2}(\tfrac{1}{2\sqrt{2}-2}+\tfrac{1}{4-2\sqrt{2}})-1)$$
$$=\pi(\tfrac{1}{4}(\sqrt{2}+1+\tfrac{2+\sqrt{2}}{2})-1)=\tfrac{\pi}{8}(3\sqrt{2}-4).$$

36. 14/95. [11] Let p denote the desired probability. The probability that they both get a snowman on the first try is 1/100. The probability that exactly one of them gets a snowman on the first roll is 18/100. If that happens the probability that the other person gets a snowman on the second roll is 1/10. Thus the probability that their number of plays is within one of each other and at least one got a snowman on the first play is $\frac{1}{100}+\frac{18}{1000}$. If neither of them got a snowman on the first play, it is as if they are starting over, and this happens with probability 81/100. Thus $p=\frac{1}{100}+\frac{18}{1000}+\frac{81}{100}p$, so $p=\frac{100}{19}\cdot\frac{28}{1000}=\frac{28}{190}$.

37. 7. [6] If 2θ subtends one side, then $\frac{1}{\sin\theta}=\frac{1}{\sin(2\theta)}+\frac{1}{\sin(3\theta)}$, hence $\sin(2\theta)\sin(3\theta)=\sin(\theta)(\sin(2\theta)+\sin(3\theta))$. This becomes $2\cos\theta\sin(3\theta)=\sin(2\theta)+\sin(3\theta)$. Using a product-to-sum formula, the LHS equals $\sin(4\theta)+\sin(2\theta)$, so we obtain $\sin(4\theta)=$

$\sin(3\theta)$. From this, you can probably guess $\theta = \pi/7$, or deduce it from $4\theta = \pi - 3\theta$.

38. [8] $\{2, 29, 31\}$ and $\{2, 281, 283\}$. We must have $p = 2$. Then $q(q-2) - 558 = x^2$ so $(q-1)^2 - x^2 = 559 = 13 \cdot 43$. Thus $(q-1-x, q-1+x) = (13, 43)$ or $(1, 559)$. Thus $q - 1 = 28$ or 280. In both cases, q and $q+2$ are primes.

39. [3] $\frac{1}{4}(\sqrt{5}+1)$. For any number t, $ab \le \frac{t}{2}a^2 + \frac{1}{2t}b^2$, $bc \le \frac{1}{2}b^2 + \frac{1}{2}c^2$, and $cd \le \frac{1}{2t}c^2 + \frac{t}{2}d^2$ by AM-GM. Choose $t = \frac{1}{2}(1+\sqrt{5})$ so that $\frac{t}{2} = \frac{1}{2t} + \frac{1}{2}$. Then $ab + bc + cd \le \frac{t}{2}(a^2 + b^2 + c^2 + d^2)$, so the desired max is $t/2$. Equality occurs if $ta = b = c = td$.

40. [1] $\frac{1}{2}\sqrt{2 + \sqrt{2(\sqrt{5}-1)}}$. $\sin^2(2x) = (\sin x + \cos x)^2 = 1 + \sin(2x)$. Thus $\sin(2x) = \frac{1}{2}(1 \pm \sqrt{5})$. Using the + sign gives a value greater than 1, so we have $2\sin x \cos x = \frac{1}{2}(1 - \sqrt{5})$. We obtain $\sin^2 x(1 - \sin^2 x) = \frac{1}{16}(6 - 2\sqrt{5})$. The quadratic formula gives $\sin^2 x = \frac{1}{2}\left(1 \pm \sqrt{1 - (3-\sqrt{5})/2}\right) = \frac{1}{2} \pm \frac{1}{4}\sqrt{2(\sqrt{5}-1)}$. Choosing the + sign makes $\sin x$ as claimed. This solution has $\sin x \approx .945$ and $\cos x \approx -.327$. The other solution reverses these. It is obtained by choosing the minus sign in the final square root.