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1. Simplify $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$.

2. A quart of liquid contains 10% alcohol, and another 3-quart bottle full of liquid contains 30% alcohol. They are mixed together. What is the percentage of alcohol in the mixture?

3. You run the first mile at 10 miles per hour, and the second mile at 8 miles per hour. How many miles per hour is your average speed for the two-mile run?

4. What is the circumference of a circle inside of which is inscribed a triangle with side lengths 3, 4, and 5? (Note that the triangle is inside the circle.)

5. If all people eat the same amount of pizza, and a pizza 12 inches in diameter serves two people, how many inches in diameter should each of two pizzas be in order to serve three people? (Pizzas are circular and are eaten entirely.)

6. How many integers between 90 and 100 are prime?

7. If $f(x) = \frac{3x+1}{2x-1}$, then write a formula in simplified form for $f(a+2)$. There should be no parentheses in your formula.

8. Circle B passes through the center of circle A and is tangent to it. Circle C passes through the center of circle B and is tangent to it. What fraction of the area of circle A lies inside circle B but outside circle C?

9. How many integers between 1 and 99, inclusive, are not divisible by either 3 or 7? (This means that they are not divisible by 3 and also they are not divisible by 7.)

10. The number $1234567891011\cdots585960$, which consists of the first 60 positive integers written in order to form a single number with 111 digits, is modified by removing 100 of its digits. What is the smallest number which can be obtained in this way? (If the number begins with some 0-digits, you may write it either with or without the 0's. Credit will be given for both answers.)

11. Point P is inside regular octagon $ABCDEFGH$ so that triangle ABP is equilateral. How many degrees are in angle APC?

12. What 5-digit number $32a1b$ is divisible by 156? (Here a and b represent digits.)

13. Mary paid $480 to purchase a certain number of items, but the nice vendor gave her two extra. This decreased the price per item by $1. How many items did she receive (including the two extra)?

14. There is only one positive integer n for which the number obtained by removing the last three digits of n exactly equals the cube root of n. What is this integer n?

15. In how many ways can a group of 16 people be divided into eight pairs? (You may write your answer as a product of integers without multiplying out. You may also use symbols such as factorials or exponentials.)

16. Let S denote the set of positive integers ≤ 18. Thus $S = \{1, 2, \ldots, 18\}$. How many subsets of S have sum greater than 85? (You may use symbols such as exponentials or factorials in your answer.)

17. What is the remainder when 3^{2009} is divided by 21?

18. The roots of $x^2 + bx + c$ are the squares of the roots of $x^2 + dx + e$. Express b in terms of d and e.

19. A rhombus has a 60 degree angle. What is the ratio of its area to that of a circle inscribed inside it? Write your answer as a fraction.

20. What is the largest positive integer which leaves the same remainder when divided into each of 99, 141, and 204?

21. Deck A is a standard deck of cards from which the four aces have been removed. Deck B is a standard deck from which one ace, one king, one queen, and one jack have been removed. You randomly pick one of these decks and select two cards. They turn out to be a pair (i.e., both have the same rank; e.g., two 9's or two kings, etc.). What is the probability that the deck which you selected was Deck A?

22. You walk up 12 steps, going up either 1 or 2 levels with each stride. There is a snake on the 8th step, so you cannot step there. How many ways can you go up? For example, one way is 1-1-1-1-1-1-1-2-1-1-1, while another is 2-2-2-1-2-2-1.

23. Let ℓ and ℓ' be parallel lines. Four points, a, b, c, and d, on ℓ and five points, A, B, C, D, and E, on ℓ' are selected, and segments are drawn from each of a, b, c, and d to each of A, B, C, D, and E in such a way that no three segments meet in the same point. (We are not considering the intersections on ℓ and ℓ'.) How many points of intersection of these segments are there?

24. In triangle ABC, angle A is 120 degrees, $BC + AB = 21$, and $BC + AC = 20$. What is the length of BC?

25. There are three 3-digit numbers with distinct digits, all digits being nonzero, which have the property that the number equals the sum of the six 2-digit numbers which can be formed from its digits (in any order). What is the second largest of these 3-digit numbers?

26. How many solutions $x > 2/5$ does the equation $\log_x(5x - 2) = 3$ have?

27. From a rectangular solid, $3''$ by $5''$ by $7''$, are removed tunnels whose cross sections are $1''$ by $1''$ squares connecting the centers of each pair of opposite sides. The sides of these tunnels are parallel to the sides of the solid, and the centers of the tunnels are at the centers of the rectangular faces. How many square inches is the surface area of the remaining figure (including its tunnels)?

28. In triangle ABC, $BC = \sqrt{2}$, $CA = 6$, and angle $ACB = 135$ degrees. If C' is the midpoint of AB, what is the length of CC'?

29. List all integers x which satisfy: there exists an integer $y > x$ with $x^2 + y^2 = xy + 13$ and $x + y = 2xy - 17$?

30. Evaluate $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \cdots + \frac{1}{98\cdot 99\cdot 100}$ as a simple fraction.

31. What is the maximum number of points with integer coordinates which can lie on a circle with center at $(\sqrt{2}, \sqrt{3})$?

32. Polygon $ABCDEF$ has $AB = BC = CD = DE = 2$ and $EF = FA = 1$. Its interior angle at C is between 90 and 180 degrees, its interior angle at F is greater than 180 degrees, while the rest of its angles are right angles. What is its area?

33. In the polygon $ABCDEF$ of the preceding problem, what is the length of the line segment drawn from A to E?

34. What is the largest positive integer y such that there exists a positive integer x satisfying $\sqrt{x} + \sqrt{y} = \sqrt{525}$?

35. If $0 \leq \theta \leq \pi$, and $\sin(\frac{\theta}{2}) = \sqrt{1+\sin(\theta)} - \sqrt{1-\sin(\theta)}$, what are all possible values of $\tan(\theta)$?

36. Let P be the point $(2, 10)$ and $Q = (14, -5)$. What is the maximum value of the difference $PX - QX$ for all points X on the x-axis? Here PX and QX denote distance between points.

37. In triangle ABC, B' lies on AC with $AB'/B'C = 5/2$, and A' lies on CB with $CA'/A'B = 4/5$. Let P be the point of intersection of the segments BB' and AA', and let C' be the point where the extension of CP meets AB. What is the ratio CP/PC'?

38. What integer equals $\sqrt{2000 \cdot 2008 \cdot 2009 \cdot 2017 + 1296}$?

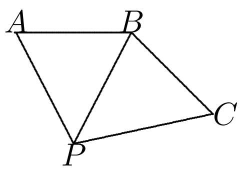
39. Suppose $ABCD$ is a quadrilateral with integer sides, right angles at A and C, and diagonal $BD = 25$. List all integer values for the distance AC.

40. What is the sum of the real solutions of $x^6 - 14x^4 - 40x^3 - 14x^2 + 1 = 0$?

Solutions to 2009 contest

The numbers in square brackets are the number of people (out of 209) who answered the question correctly, followed by the number of correct responses from the top 31 people (those with scores of ≥ 18).

1. 13/12 or 1 1/12. [200,31]It is $(6+4+3)/12$.
2. 25 or 25%. [164,30] In 4 quarts, you have $.1+3\times .3=1$ quart of alcohol.
3. 80/9 or 8 8/9. [96,30] Your total time in hours is $\frac{1}{10}+\frac{1}{8}=\frac{18}{80}$. Your speed is $2/\frac{18}{80}=80/9$.
4. 5π. [137,27] Since the triangle is a right triangle, its hypotenuse will be the diameter of the circle.
5. $6\sqrt{3}$. [82,24] Two people eat 36π sq in, so three people will need 54π sq in. If this is in two pizzas, each must have area 27π, and hence radius $\sqrt{27}=3\sqrt{3}$.
6. 1. [142,28] Only 97 is prime. Note that $91=7\cdot 13$, and 93 and 99 are divisible by 3.
7. $\frac{3a+7}{2a+3}$. [159,30] It is $\frac{3(a+2)+1}{2(a+2)-1}$.
8. 3/16. [124,31] The diameter of circle B is half that of circle A, and hence its area is 1/4 of that of the larger circle. Similarly, the area of circle C is 1/4 of that of circle B. The answer is $\frac{1}{4}-\frac{1}{4}\frac{1}{4}=\frac{3}{16}$.
9. 56. [92,26] There are 33 which are divisible by 3, and there are 14 that are divisible by 7. But there are 4 which are divisible by both. Thus there are $47-4=43$ which are divisible by at least one of 3 and 7, and hence $99-43=56$ which are divisible by neither.
10. 00000123450. [65,17] The long number has six 0's. We should use all of them, but the last one must come at the end. The other digits are the smallest digits remaining which come after the previous selected digit.
11. 112.5. [55,26] The relevant portion of the octagon is pictured below.



Angle $ABC=135$, so angle $PBC=135-60=75$. Since $BP=AB=BC$, angle $BPC=(180-75)/2=52.5$, so angle $APC=60+52.5$.

12. 32916. [139,29] Our number must be divisible by 3, 4, and 13. Consideration of divisibility by 3 and 4 implies that the number is one of 32112, 32412, 32712, 32016, 32316, 32616, or 32916. Mod 13, we have $32112\equiv 2$ and $32016\equiv 10$. Since $300\equiv 1$, the seven numbers listed above are congruent to 2, 3, 4, 10, 11, 12, 0, respectively.

Another way is to note that $206 \cdot 156 = 32136$ is the smallest multiple of 156 exceeding 32000. Keep adding 156 until you get to a number whose 10's digit is 1.

13. 32. [118,31] If x is the number of items that she received, then $\frac{480}{x-2} - \frac{480}{x} = 1$. This simplifies to $960 = x(x-2)$, yielding $x = 1 + \sqrt{961} = 32$.

14. 32768. [53,20] The integer $n = 1000x + y = x^3$ with $0 \le y < 1000$. We must have $1000x \le x^3$ and hence $1000 \le x^2$. The smallest such integer is $x = 32$. The desired n is $32^3 = 32768$. Since we must also have $x^3 < 1000x + 1000 \le 1000x + x^2$, then $x^2 < 1000 + x$, which implies $x < 33$.

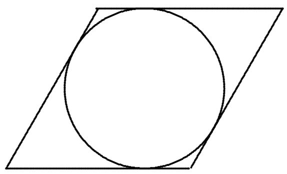
15. $15 \cdot 13 \cdot 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3$ or 2027025 or $16!/(2^8 8!)$. [28,13] Arbitrarily order the people. There are 15 ways to choose the partner of #1. Now there are 13 ways to choose the partner of the person with the lowest number of the 14 remaining people. Continue in this fashion, or argue by induction.

16. 2^{17} or 131072. [8,7] The sum of all the numbers from 1 to 18 is $18 \cdot 19/2 = 171$. Exactly half of the subsets have sum greater than 85 because the complements of these subsets are exactly those with sum ≤ 85. Since there are 2^{18} subsets altogether, the answer is 2^{17}. (You should realize that the whole set and the empty set are both considered as subsets.)

17. 12. [40,18] Working mod 21, the powers 3^n for $n = 1, \ldots, 7$ are $3, 9, 6, 18, 12, 15, 3$. Then it starts repeating with period 6. Since $2009 = 334 \cdot 6 + 5$, $3^{2009} \equiv 3^5 \equiv 12$ mod 21.

18. $b = -d^2 + 2e$ or just $-d^2 + 2e$. [25,17] If $x^2 + dx + e = (x - \alpha)(x - \beta)$, then $x^2 + bx + c = (x - \alpha^2)(x - \beta^2)$, and so $b = -(\alpha^2 + \beta^2) = -(\alpha + \beta)^2 + 2\alpha\beta = -(-d)^2 + 2e$.

19. $\frac{8\sqrt{3}}{3\pi}$. [38,22] Let the side length of the rhombus equal 1. Then its altitude equals $\sqrt{3}/2$, and hence this also equals its area, and equals the diameter of the inscribed circle, which hence has area $3\pi/16$. See diagram below.



20. 21. [42,20] If x is the number, then both $141 - 99$ and $204 - 141$ must be multiples of x. Thus x must be the greatest common divisor of 42 and 63. Hence $x = 21$.

21. 12/23. [32,19] Deck A has $12\binom{4}{2} = 72$ pairs, while Deck B has $4\binom{3}{2} + 9\binom{4}{2} = 66$ pairs. The desired probability is $72/(72 + 66)$. A more thorough solution can be given using conditional probability. The answer depends on the fact that both decks have the same number of cards (48). If this were not the case, the answer would be much more complicated.

22. 63. [29,15] You must do a 2-step from 7 to 9. Thus our desired answer equals the number of ways to do 7 steps times the number of ways to do 3 steps. The latter is 3

(viz., 1-1-1, 2-1, and 1-2), while the former is $1+\binom{6}{1}+\binom{5}{2}+\binom{4}{3}=7+10+4=21$. These latter possibilities are determined by how many 2-steps were taken among the 7. For example, if you used two 2-steps in getting to 7, then you took five strides altogether, and there are $\binom{5}{2}$ places where the two 2-steps could have been done.

23. 60. [36,18] For any ordered pair (x, y) from $\{a, b, c, d\}$ with x to the left of y and any ordered pair (X, Y) from $\{A, B, C, D, E\}$ with X to the left of Y, there will be one intersection, of the segment xY with the segment yX. Thus there are exactly $\binom{4}{2}\binom{5}{2}=6\cdot 10$ points of intersection.

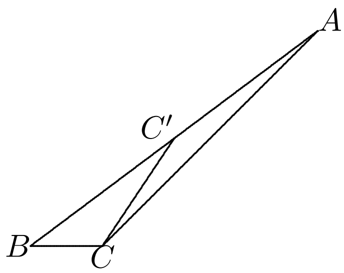
24. 13. [41,19] Let $b=AC$. Then, by the Law of Cosines, $(20-b)^2=b^2+(b+1)^2-2b(b+1)\cos(120)$, hence $400-40b+b^2=2b^2+2b+1+b(b+1)$. We obtain $0=2b^2+43b-399=(b-7)(2b+57)$. Thus $b=7$ and $BC=20-b=13$.

25. 264. [40,21] If the number equals $100a+10b+c$, then this must equal $(10a+b)+(10a+c)+(10b+a)+(10b+c)+(10c+a)+(10c+b)=22(a+b+c)$, and hence $78a=12b+21c$. We must have $c=2c'$ with $1\le c'\le 4$ and, dividing by 6, we obtain $13a=2b+7c'$. The RHS is ≤ 46, and so we must have $a=1$, 2, or 3. One now easily checks that the only possible numbers are 132, 264, and 396.

26. 2. [71,24] The equation is equivalent to $f(x)=0$ where $f(x)=x^3-5x+2$. Since $f(-100)<0$, $f(\frac{2}{5})>0$, $f(1)<0$, and $f(100)>0$, f has one root x satisfying $x<2/5$, one satisfying $2/5<x<1$, and one satisfying $x>1$.

27. 184 or 184 sq in. [37,20] On the faces, there is $2(3\cdot 5+3\cdot 7+5\cdot 7)-6=136$. The 1-by-1-by-1 cube at the very center gives no surface area. Leading into it are six tunnels of cross-sectional perimeter 4 and lengths 1, 1, 2, 2, 3, and 3, yielding an additional surface area of 48.

28. $\frac{1}{2}\sqrt{26}$. [18,13] Place B at $(0,0)$, C at $(\sqrt{2},0)$ and A at $(4\sqrt{2},3\sqrt{2})$. Then C' is at $(2\sqrt{2},\frac{3}{2}\sqrt{2})$, and so the desired length is $\sqrt{(\sqrt{2})^2+(\frac{3}{2}\sqrt{2})^2}=\sqrt{2}\sqrt{1+\frac{9}{4}}=\sqrt{26}/2$.

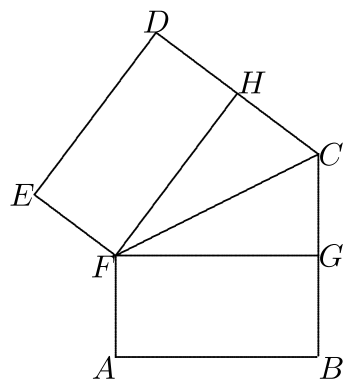


29. 3. [26,11] Let $u=x+y$ and $v=xy$. Then $u^2=3v+13$ and $u=2v-17$, which yields $2u^2-3u-77=0$ and hence $u=7$ or $-11/2$. The corresponding values of v are 12 and 23/4, respectively. If $(u,v)=(7,12)$, then $\{x,y\}=\{3,4\}$, while $(u,v)=(-11/2,23/4)$ does not give integer values for x or y.

30. 4949/19800. [7,6] Note that $\frac{1}{(n+1)n(n-1)}=\frac{1}{2n(n-1)}-\frac{1}{2(n+1)n}$. Thus the desired sum is $\frac{1}{4}-\frac{1}{2(100)(99)}$, as all intermediate terms cancel out. This equals $\frac{50\cdot 99-1}{200\cdot 99}$.

31. 1. [19,9] For any pair (m,n) of integers, there is clearly a circle passing through (m,n) centered at $(\sqrt{2},\sqrt{3})$. Suppose (m_1,n_1) and (m_2,n_2) are distinct points which lie on the circle $(x-\sqrt{2})^2+(y-\sqrt{3})^2=r^2$. Then $m_1^2-2\sqrt{2}m_1+n_1^2-2\sqrt{3}n_1=m_2^2-2\sqrt{2}m_2+n_2^2-2\sqrt{3}n_2$. Hence $(m_1-m_2)(m_1+m_2-2\sqrt{2})+(n_1-n_2)(n_1+n_2-2\sqrt{3})=0$. If m_1, m_2, n_1, and n_2 are integers, then this gives integers A, B, and C such that $A\sqrt{2}=B+C\sqrt{3}$, and at least one of A and C is nonzero. If $A=0$, we obtain that $\sqrt{3}$ is rational, which is false. Similarly we cannot have $C=0$. If both are nonzero, then squaring both sides gives the impossible conclusion that a nonzero integer times $\sqrt{3}$ equals an integer. Thus there cannot be two integer points on this circle.

32. 6. [33,15] In the figure below, rectangles $ABGF$ and $DEFH$ have area 2 each, while triangles CFH and CFG have areas 1 each.



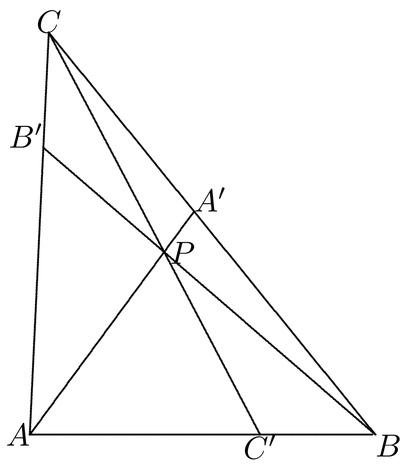
33. $.8\sqrt{5}$. [8,7] The tangent of angle GFC is $1/2$. Angle GFH is twice as large, so its tangent is $\dfrac{1}{1-\frac{1}{4}}=\frac{4}{3}$. Therefore the angle which FE makes with a horizontal to the left from F has tangent equal to $3/4$. If the figure is placed so that $F=(0,0)$, then $E=(-\frac{4}{5},\frac{3}{5})$ and $A=(0,-1)$, so the distance AE is $\sqrt{(.8)^2+(1.6)^2}=.8\sqrt{1+4}$.

34. 336. [32,14] We have $y=525+x-10\sqrt{21x}$; hence $x=21m^2$ for some positive integer m. Similarly $y=21n^2$ for some positive integer n. We obtain $m+n=5$, and hence $(m,n)=(1,4)$, $(2,3)$, $(3,2)$, or $(4,1)$. The largest y is $21\cdot 4^2$.

35. 0 and $-4/3$. [3,3] Squaring yields $\sin^2(\frac{\theta}{2})=2-2\sqrt{1-\sin^2\theta}=2-2|\cos\theta|$. Hence $1-\cos\theta=4(1-|\cos\theta|)$. If $\cos\theta\geq 0$, we obtain $3\cos\theta=3$, so $\cos\theta=1$ and hence $\tan(\theta)=0$. If $\cos\theta<0$, then $\cos\theta=-3/5$ and so $\tan\theta=-4/3$.

36. 13. [5,1] Note that $QX=Q'X$, where $Q'=(14,5)$. For any X, $PX\leq PQ'+Q'X$, hence $PX-Q'X\leq PQ'=13$. Equality will be attained when P, Q', and X lie in a straight line, which occurs when X is the point of intersection of the extension of PQ' with the x-axis.

37. 6/5 or 1.2. [2,0] Refer to the drawing below.



One slick way to solve this uses what is called Mass Point Geometry. If you have not heard of this, you can learn about it by googling the term. If you put a mass 5 at C, 2 at A, and 4 at B, then the center of mass of $\{A, C\}$ is at B', and it is as if there was mass 7 there. The center of mass of $\{A, B, C\}$ will lie on BB'. Similarly the center of mass of $\{B, C\}$ lies at A', since the products of mass times distance from A' are the same for B and C. Thus the center of mass of $\{A, B, C\}$ also lies on AA', and hence is at P. This P must also be the center of mass of CC', with mass 6 at C'. Hence the ratio CP/PC' is 6/5. The problem can be worked in other ways. One combines Ceva's theorem and Menelaus's theorem. Another uses vectors. Append directions to all the segments in the diagram, and write vector equations for each triangle. Insert the given information: for example, if you let $v = AB'$, then $B'C = \frac{2}{5}v$. You can find the desired ratio by solving the six equations. Another method uses equations involving areas of subtriangles.

38. 4034036. [8,6] Let $A = 2000$ and $B = A^2 + 17A + 36$. The expression inside the square root is $A(A+17)(A+8)(A+9)+1296 = (A^2+17A)(A^2+17A+72)+1296 = (B-36)(B+36)+1296 = B^2$. Thus the answer is $B = 2000^2 + 34000 + 36$.

39. 20, 24, 25. (any order) [5,4] The sets $\{AB, AD\}$ and $\{BC, CD\}$ can each be either of $\{7, 24\}$ or $\{15, 20\}$. BD is the diameter of a circle passing through all four points, so the Theorem of Ptolemy applies to say that $25AC = AB \cdot CD + AD \cdot BC$. The possibilities for the RHS are $7 \cdot 7 + 24 \cdot 24$, $7 \cdot 24 + 24 \cdot 7$, $7 \cdot 15 + 24 \cdot 20$, $7 \cdot 20 + 24 \cdot 15$, $15 \cdot 15 + 20 \cdot 20$, and $15 \cdot 20 + 20 \cdot 15$. These sums are 625, 336, 585, 500, 625, and 600, respectively. The integer ratios when these are divided by 25 are as claimed.

40. 5. [1,1] Let $y = x + \frac{1}{x}$, and note $y^3 = x^3 + \frac{1}{x^3} + 3y$. After dividing the given equation by x^3, it becomes $y^3 - 17y - 40 = 0$ or $(y-5)(y^2+5y+8) = 0$, which has $y = 5$ as its only real root. Thus $x + \frac{1}{x} = 5$, so $x^2 - 5x + 1 = 0$, which has real roots whose sum is 5.