1. By how much does $\frac{1}{3}$ of $\frac{5}{2}$ exceed $\frac{1}{2}$ of $\frac{1}{3}$?

2. What fraction of the area of a circle of radius 5 lies between radius 3 and radius 4?

3. A ticket fee was \$10, but then it was reduced. The number of customers increased by 50%, but the amount of money received only increased by 20%. How many dollars was the reduced ticket price?

4. If $6x + 7y = 2007$ and $7x + 6y = 7002$, then what is the value of $x + y$?

5. What is the sum of the digits of $10^{55} - 55$?

6. There are three consecutive positive integers such that the square of the second minus twelve times the first is three less than twice the third. What is the smallest of the three integers?

7. How many perfect cubes greater than 1 are divisors of 9^9?

8. The harmonic mean of two numbers is defined to be the reciprocal of the average of the reciprocals of the numbers. Find x such that the harmonic mean of 4 and x is 6.

9. What is the smallest possible value of $|x-1|+|y-x|+|y-2007|$ for any real numbers x and y?

10. What is the value of $\log_2(\log_2(\log_2(16)))$?

11. Two fair 6-sided dice, colored red and white, are tossed. What is the probability that the number on the red die is at least as large as that on the white die?

12. What is the smallest positive number which is equal to the cube of one positive integer and also is equal to the fourth power of a different positive integer?

13. What value of c occurs in a solution of the system of equations $a+b+c = 14$, $ab = 14$, and $c^2 = a^2 + b^2$?

14. If 11 positive integers $a_1 \leq a_2 \leq \cdots \leq a_{11}$ have the property that no three of them are the sides of a triangle, what is the smallest possible value of a_{11}/a_1?

15. A cube and sphere have the same surface area. What is the ratio of the volume of the sphere to that of the cube?

16. A rhombus $ABCD$ with sides of length 5 lies in the first quadrant with A at the origin. If AD has slope 1/2 and AB has slope 2, then what are the coordinates of C?

17. For how many primes p is $p^2 + 3p - 1$ also prime?

18. A large container, labeled R, is partially filled with 4 quarts of red paint. Another large container, labeled W, is partially filled with 5 quarts of white paint. A small empty can is completely filled with red paint taken from R, and the contents of the can then emptied into W. After thorough mixing of the contents of W, the can is completely filled with some of this mixture from W, and the contents of the can then emptied into R. The ratio of red paint to white in R is now 3:1. What is the size of the can, in quarts?

19. What is the largest number A such that the graphs of $x^2 = y^2$ and $(x - A)^2 + y^2 = 1$ intersect?

20. A number is written with one 1, followed by three 3's, then five 5's, then seven 7's, then nine 9's, then eleven 11's, etc. Thus it begins 13335555577..., and just as it gets to the 13's it has ...11131313.... What is its 999th digit?

21. What is the radius of a circle inscribed in an isosceles right triangle whose legs have length 1?

22. In what base b is the equation $53 \times 15 = 732$ valid? Here all three numbers are base-b numbers, and b must be a positive integer.

23. A strip of rubber is initially 80 cm long, and after every minute it is instantaneously and uniformly stretched by 40 cm. An ant moves at a rate of 42 cm per minute, starting at one end. Each time the strip is stretched, the ant is moved to a position on the modified strip proportional to its position on the strip before the stretching took place. How many minutes does it take the ant to cross the strip?

24. The first term of an infinite geometric series is 10, and the sum of the series lies between 9 and 11, inclusive. What is the range of values for the common ratio r between terms of this series?

25. If the roots of $x^2 - bx + c = 0$ are $\sin(\pi/9)$ and $\cos(\pi/9)$, then express b in terms of c. (Don't write "$b =$"; just write the expression involving c.)

26. Let R denote the set of points (x, y) satisfying $x^2 - 4|x| + y^2 + 3 \leq 0$. What is the area of R?

27. The sum of the 3-digit numbers $35x$ and $4y7$ is divisible by 36. Find all possible ordered pairs (x, y).

28. What is the ratio of the area of a regular 10-gon to that of a regular 20-gon inscribed in the same circle? Express your answer using a single trig function with its angle in degrees.

29. Let $k > 0$ and let A lie on the curve $y = k\sqrt{x}$ so that the vertical and horizontal segments from A to the x- and y-axes are sides of a square with the origin at the vertex opposite A. Let B be the vertex of this square lying below A on the x-axis. Let C be the point on the same curve between the origin and A such that the vertical segment from C to the x-axis and the horizontal segment from C to AB is a square with B at the vertex opposite C. What is the ratio of the side length of the second square to that of the first?

30. What is the smallest multiple of 999 which does not have any 9's among its digits? (If the number is $n = 999 \cdot d$, write n, not d.)

31. Let $P(n)$ denote the product of the digits of n, and let $S(n)$ denote the sum of the digits of n. How many positive integers n satisfy $n = P(n) + S(n)$?

32. For each point in the plane, consider the sum of the squares of its distances from the four points $(-3,-1)$, $(-1,0)$, $(1,2)$, and $(1,3)$. What is the smallest number achieved as such a sum?

33. What is the remainder when $x^3 + x^6 + x^9 + x^{27}$ is divided by $x^2 - 1$?

34. Find k so that the solutions of $x^3 - 3x^2 + kx + 8 = 0$ are in arithmetic progression.

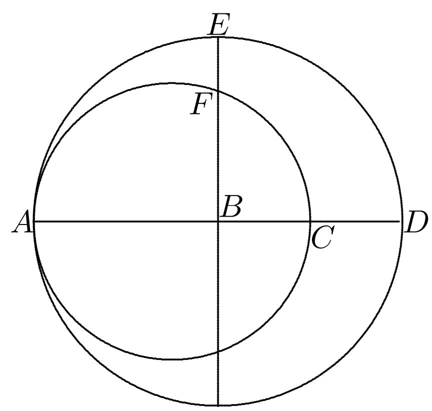
35. In triangle ABC, D lies on BC so that $AC = 3$, $AD = 3$, $BD = 8$, and $CD = 1$. What is the length of AB?

36. What is the ninth digit from the right in 101^{20}?

37. Two circles of radius 1 overlap so that the center of each lies on the circumference of the other. What is the area of their union?

38. The first two positive integers n for which $1 + 2 + \cdots + n$ is a perfect square are 1 and 8. What are the next two?

39. In the diagram below, which is not drawn to scale, the circles are tangent at A, the center of the larger circle is at B, $CD = 42$, and $EF = 24$. What are the radii of the circles?



40. What is the side length of an equilateral triangle ABC for which there is a point P inside it such that $AP = 6$, $BP = 8$, and $CP = 10$?

Solutions to 2007 contest

Numbers in brackets are the number of people who answered the question correctly, first out of the 28 people who scored at least 20, and then out of the other 228 people.

1. 2/3. [28, 193] The difference is $\frac{5}{6} - \frac{1}{6}$.
2. 7/25. [27, 138.5] The fraction is $(4^2 - 3^2)/5^2 = 7/25$.
3. 8. [24, 136] If the new fee is x and the original number of customers was N, then $1.2 \cdot 10N = 1.5Nx$. Thus $x = 12/1.5 = 8$.
4. 693. [28, 105] Add the equations to get $13(x+y) = 9009$. Thus $x+y = 9009/13 = 693$.
5. 486. [22, 81] The number has 53 9's followed by the digits 45. Thus the answer is $54 \cdot 9$.
6. 12. [28, 141] If x is our desired number, then $(x+1)^2 - 12x = 2(x+2) - 3$. Thus $x^2 + 2x + 1 - 12x = 2x + 1$, and hence $x = 0$ or 12.
7. 6. [25, 46] They are 3^3, 3^6, 3^9, 3^{12}, 3^{15}, and 3^{18}.
8. 12. [24, 140] The harmonic mean of x and y equals $1/(\frac{1}{2}(\frac{1}{x} + \frac{1}{y})) = \frac{2xy}{x+y}$. We have $6 = 8x/(x+4)$. We obtain $2x = 24$.
9. 2006. [25, 130] This value will be achieved whenever $1 \le x \le y \le 2007$. The sum must be ≥ 2006 by the general fact that $|a-b| + |b-c| \ge |a-c|$.
10. 1. [28, 138] $\log_2(16) = 4$. $\log_2(4) = 2$. $\log_2(2) = 1$.
11. 7/12. [25.5, 101.5] There is 1/6 chance that they are equal, and then $\frac{1}{2}(1 - \frac{1}{6})$ that the red die is greater. The desired answer is $\frac{1}{6} + \frac{5}{12}$.
12. 4096. [26, 66] This is 2^{12} which equals 16^3 and 8^4. Note that 1 is ruled out since it is not the cube and fourth power of *different* integers. To see that there is nothing smaller, one can easily check that none of $2^4, \ldots, 7^4$ is a cube.
13. 6. [25, 37] We have $c^2 = (a+b)^2 - 28 = (14-c)^2 - 28$. We obtain $28c = 196 - 28$, hence $c = 6$.
14. 89. [24, 21] We may set $a_1 = 1$. To make the numbers as small as possible, we should choose $a_2 = a_1$ and $a_n = a_{n-2} + a_{n-1}$ for $n \ge 3$. Thus we obtain the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, and 89.
15. $\sqrt{6/\pi}$. [15.5, 23] If s is the side of the cube and r the radius of the sphere, then $6s^2 = 4\pi r^2$ and so $(r/s)^2 = 6/(4\pi)$. The ratio which we desire is $\frac{4}{3}\pi(r/s)^3 = \frac{4}{3}\pi(6/4\pi)^{3/2} = \sqrt{6/\pi}$.
16. $(3\sqrt{5}, 3\sqrt{5})$. [23, 49] The vector AC is the sum of AD and AB, which are, respectively, $\sqrt{5}\langle 2, 1\rangle$ and $\sqrt{5}\langle 1, 2\rangle$.
17. 1. [22, 58] For $p = 3$, we obtain the prime 17. Every prime $p \ne 3$ is of the form $3k \pm 1$ for some k and hence has $p^2 - 1$ a multiple of 3. Thus $p^2 + 3p - 1$ is also a multiple of 3.

18. 5/4 or 1.25. [15, 19] Let x denote the size of the can. After the first transfer, R has $4-x$ red, while W had 5 white and x red. After the second transfer, R has $4-x+x\frac{x}{5+x}$ red and $x\frac{5}{5+x}$ white. Thus $(4-x)(5+x)+x^2 = 3\cdot 5x$. This simplifies to $20 = 16x$.

19. $\sqrt{2}$. [21, 16] Method 1: $x^2 - 2Ax + A^2 + x^2 = 1$. Thus $2x^2 - 2Ax + A^2 - 1 = 0$. This has a solution when its discriminant, $(-2A)^2 - 8(A^2-1)$, is nonnegative. Thus $8 - 4A^2 \geq 0$. Hence $|A| \leq \sqrt{2}$. Method 2: The circle of radius 1 centered at $(A, 0)$ will be tangent to the line $y = x$. The radius to the point of tangency will have slope -1. This radius, the tangent line from $(0, 0)$, and the line from $(0, 0)$ to $(A, 0)$ form a 45-45-90 triangle with legs of length 1. Thus $A^2 = 1^2 + 1^2$.

20. 5. [9, 48] The sum of the odd integers through $2m-1$ is m^2. Going through the 9's, there will have been 25 digits. At the end of the 43's, we will have used $2\cdot 22^2 - 25 = 943$ digits. The reason for the 25 subtracted comes from thinking of the digits through the 9's as if they had been paired, but then subtracting off for the fact that they weren't. The 45's will take us beyond digit 999; the 4's will occur at even-numbered digits and the 5's at odd-numbered.

21. $1 - \frac{\sqrt{2}}{2}$. [21, 14] Method 1: Place the triangle with vertices at (0,0), (1,0), and (0,1). The center of the inscribed circle must be at (r, r) such that $r^2 = (\frac{1}{2} - r)^2 + (\frac{1}{2} - r)^2$. These are two forms for r^2. The equation reduces to $0 = r^2 - 2r + \frac{1}{2}$. The quadratic formula gives $r = 1 \pm \frac{1}{2}\sqrt{2}$. Method 2: The altitude to the hypotenuse equals $\sqrt{2}/2$ and also equals $r + r\sqrt{2}$.

22. 13. [26, 26] We must have $(5b+3)(b+5) = 7b^2 + 3b + 2$. Thus $2b^2 - 25b - 13 = 0$. Find $b = 13$ or $-1/2$ either by factoring or by the quadratic formula.

23. $2\frac{10}{21}$. [19.5, 33.5] After one minute, it will be 42/80. After the stretching, it will be 63/120. After the second minute, it will be 105/120. After the second stretching, it will be 140/160. It requires another 20/42 minutes to get to the end.

24. $-\frac{1}{9} \leq r \leq \frac{1}{11}$. [18, 26.5] We have $9 \leq \frac{10}{1-r} \leq 11$. We obtain $11r \leq 1$ and $9r \geq -1$.

25. $\sqrt{1+2c}$. [12.5, 4.5] We have $c = \sin(\pi/9)\cos(\pi/9)$ and $b = \sin(\pi/9) + \cos(\pi/9)$. Then $b^2 = 1 + 2\sin(\pi/9)\cos(\pi/9)$, since $\sin^2(-) + \cos^2(-) = 1$. Hence $b^2 = 1 + 2c$. Finally note $b > 0$ so we choose the positive square root.

26. 2π. [11, 2] For $x \geq 0$, we require $x^2 - 4x + 4 + y^2 \leq 1$, hence $(x-2)^2 + y^2 \leq 1$, which is a disk of radius 1 with all $x > 0$. For $x \leq 0$, we require $x^2 + 4x + 4 + y^2 \leq 1$, hence $(x+2)^2 + y^2 \leq 1$, which is a disk of radius 1 with all $x < 0$. Thus R consists of two disks of radius 1, and has area 2π.

27. $(1, 7)$ and $(5, 3)$. [22, 38] We have $350 + x + 407 + 10y = 36j$ for some j. Since $757 = 36\cdot 21 + 1$, we have $10y + x = 36k - 1$ for some k. Since $0 \leq 10y + x \leq 99$, we have $10y + x = 35$ or 71.

28. $\cos(18^{o})$. [14, 2] Let the radius of the circle equal 1. The 10-gon consists of 20 right triangles with hypotenuse 1 and an angle of 18 degrees. Thus the area of the 10-gon is $20\cdot\frac{1}{2}\sin(18)\cos(18) = 10\sin(18)\cos(18)$. Similarly the area of the 20-gon is

$20\sin(9)\cos(9)$. By the double angle formula, this latter equals $10\sin(18)$. Thus the ratio is $\cos(18)$.

29. $(\sqrt{5}-1)/2$. [7, 1] A lies on the intersection of the curve with the line $y = x$ and so its x coordinate satisfies $x = k\sqrt{x}$, hence $x = k^2$. The side length s of the second square satisfies $s = k\sqrt{k^2 - s}$. Thus $s^2 + k^2 s - k^4 = 0$ and hence $s = \frac{1}{2}(-k^2 + \sqrt{k^4 + 4k^4})$. The desired ratio s/k^2 equals $\frac{1}{2}(-1 + \sqrt{1+4})$.

30. 111888. [15, 21] The multiples of 999 have the form $1000d - d$. If $d \le 100$, this has a 9 in the 100's position. If $101 \le d \le 110$, it has a 9 in the 10's position. If $d = 111$, it has a 9 in the 1's position. If $d = 112$, it equals 111888.

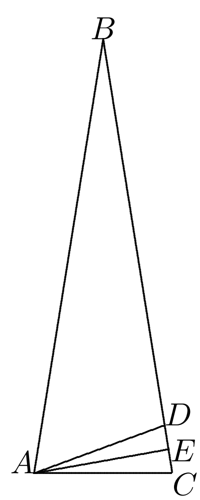
31. 9. [16, 29] Clearly no 1-digit number works. If $n = 10a + b$ is a 2-digit number satisfying the required property, then $10a + b = ab + a + b$ and so $b = 9$, yielding the nine numbers 19, 29,. . . ,99. If $n = 100a + 10b + c$ is a 3-digit number, we would need $100a + 10b + c = abc + a + b + c$ and so $99a + 9b = abc$. Since $bc \le 81$, this has no solutions. A similar argument will show that there will be no solutions if n has more than three digits.

32. 21. [9, 3] For the point (x, y), this sum is $(x+3)^2+(y+1)^2+(x+1)^2+y^2+(x-1)^2+(y-2)^2+(x-1)^2+(y-3)^2 = 4x^2+4x+12+4y^2-8y+14 = (2x+1)^2+11+4(y-1)^2+10$. The minimum value of this occurs when $x = -1/2$ and $y = 1$, and is 21.

33. $3x + 1$. [4, 5] The remainder must be a first-degree polynomial $ax + b$. We have $x^3+x^6+x^9+x^{27} = q(x)(x^2-1)+ax+b$. Letting $x = 1$ yields $4 = a+b$, while letting $x = -1$ yields $-2 = -a + b$. We solve these equations, obtaining $a = 3$ and $b = 1$.

34. -6. [13, 6] Let the roots be $a-d$, a, and $a+d$. The sum of the roots is the coefficient of $-x^2$; hence $a = 1$. The negative of the product of the roots equals the constant term. Thus $d^2 - 1 = 8$. Hence the roots are -2, 1, and 4. Then k is the sum of products of pairs of roots, so equals $-2 - 8 + 4$.

35. 9. [15, 24] Let AE be the altitude from A, and let h denote its length. Note that triangle ACD is isosceles, and so $ED = EC = 1/2$. We have $h^2 = 3^2 - (\frac{1}{2})^2$, and $AB^2 = h^2 + (\frac{17}{2})^2$. Thus $AB^2 = 9 + \frac{16 \cdot 18}{4} = 81$.

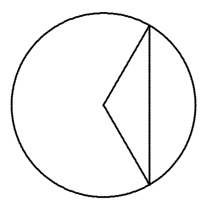


36. 6. [8, 4] We expand

$$(100+1)^{20} = 1 + 20\cdot 100 + \binom{20}{2}100^2 + \binom{20}{3}100^3 + \binom{20}{4}100^4 + \cdots .$$

Neither the first three terms nor the omitted terms at the end will affect the ninth digit from the right. The fourth and fifth terms are 1140000000 and 484500000000. The desired digit is $1+5$.

37. [2, 1] $\frac{4\pi}{3} + \frac{\sqrt{3}}{2}$. Their overlap is twice the area A to the right of the vertical line in the diagram below. This vertical line is at radius 1/2. Thus the indicated angle is 120°, and so the area of the wedge subtended by the angle is $\pi/3$. Then A equals $\pi/3$ minus the area of the indicated triangle, which is $\frac{1}{2}\frac{\sqrt{3}}{2}$. Thus the area of overlap is $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$. Finally the area of the union is 2π minus the overlap.



38. 49 and 288. [9, 1] We need that $n(n+1)$ is 2 times a perfect square. Since n and $n+1$ contain no common divisors, one must be a square and the other twice a square. The square must be odd. We examine the first few odd squares m^2 to see whether m^2-1 or m^2+1 is 2 times a square. For m^2-1, we obtain 8, 24, 48, 80, 120, 168, 224, and 288. For m^2+1, we obtain 10, 26, 50, 82, 122, 170, 226, and 290. Of these 8, 50, and 288 are 2 times a perfect square. The pairs $(n, n+1)$ are $(8,9)$, $(49,50)$, and $(288,289)$.

39. 75, 96. [11, 1] Let R (resp. r) denote the radius of the large (resp. small) circle. Then $R = r + 21$. Let O denote the center of the small circle. Referring to the diagram on your exam sheet, the right triangle OFB implies $r^2 = (R-24)^2 + 21^2$, and hence $r^2 = (r-3)^2 + 441$, which implies $6r = 450$.

40. $2\sqrt{25+12\sqrt{3}}$. [3, 0] Let s be the desired length, and place the triangle with $A = (s\frac{\sqrt{3}}{2}, \frac{s}{2})$, $B = (s\frac{\sqrt{3}}{2}, -\frac{s}{2})$, and $C = (0,0)$. If $P = (x,y)$, we have $x^2+y^2 = 100$, $(x - \frac{\sqrt{3}}{2}s)^2 + (y + \frac{s}{2})^2 = 64$, and $(x - \frac{\sqrt{3}}{2}s)^2 + (y - \frac{s}{2})^2 = 36$. The second and third imply $sy = 14$. Substitute this and the first into the second and get $sx = (s^2+50)/\sqrt{3}$. Multiply the first equation by s^2 and substitute our expressions for sx and sy, obtaining an equation which reduces to $s^4 - 200s^2 + 3088 = 0$. Get $s^2 = 100 + \sqrt{6912}$.