Math 242 fall 2008 notes on problem session for week of 11-10-08
This is a short overview of problems that we covered.

1. Find a QR factorization of $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Apply Gram-Schmidt process to the columns. $\boldsymbol{w}_1 = (1,0,1), \boldsymbol{w}_2 = (1,0,0), \boldsymbol{w}_3 = (2,1,0)$.
   For notation space here write vectors as row vectors instead of column vectors.

   - $\boldsymbol{u}_1' = \boldsymbol{w}_1 = (1,0,1)$ and $r_{11} = \|\boldsymbol{u}_1'\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$: $\boldsymbol{u}_1 = \frac{\boldsymbol{u}_1'}{\|\boldsymbol{u}_1'\|} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$
   - $r_{12} = \langle \boldsymbol{w}_2, \boldsymbol{u}_1 \rangle = (1,0,0) \cdot (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}$: $\boldsymbol{u}_2' = \boldsymbol{w}_2 - \langle \boldsymbol{w}_2, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 = (1,0,0) - \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}) = (\frac{1}{2}, 0, \frac{-1}{2})$: $r_{22} = \|\boldsymbol{u}_2'\| = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \sqrt{\frac{1}{2}}$: $\boldsymbol{u}_2 = \frac{\boldsymbol{u}_2'}{\|\boldsymbol{u}_2'\|} = (\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$
   - $r_{13} = \langle \boldsymbol{w}_3, \boldsymbol{u}_1 \rangle = (2,1,0) \cdot (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}) = \sqrt{2}$; $r_{23} = \langle \boldsymbol{w}_3, \boldsymbol{u}_2 \rangle = (2,1,0) \cdot (\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}) = \sqrt{2}$; $\boldsymbol{u}_3' = \boldsymbol{w}_3 - \langle \boldsymbol{w}_3, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 - \langle \boldsymbol{w}_3, \boldsymbol{u}_2 \rangle \boldsymbol{u}_2 = (2,1,0) - \sqrt{2}(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}) - \sqrt{2}(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}) = (0,1,0)$: $r_{33} = \|\boldsymbol{u}_3'\| = \sqrt{0^2+1^1+0^2} = 1$: $\boldsymbol{u}_3 = \frac{\boldsymbol{u}_1'}{\|\boldsymbol{u}_3'\|} = (0,1,0)$.

   Now use the r_{ij} from above (with $r_{ij} = 0$ for $i > j$) and put the $\boldsymbol{u}_i$ as columns of Q

   $$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & 0 & 1 \end{pmatrix} = QR.$$

2. A square matrix A is skew symmetric if $A^T = -A$. For such a matrix show that it Cayley transform $Q = (I-A)^{-1}(I+A)$ is orthogonal (it can be shown that $(I-A)^{-1}$ will exist).

   Observe first that if A and B commute, $AB = BA$ then
   $(A+B)(A-B) = A^2 - AB + BA - B^2 = A^2 - B^2 = A^2 - BA + AB - B^2 = (A+B)(A-B)$.

   In particular, we have $(I-A)(I+A) = (I+A)(I-A)$. We will write M^{-T} for the transpose of the inverse which is equal to the inverse of the transpose. We also use $I^T = I$ and $A^T = -A$.

   Now $QQ^T = [(I-A)^{-1}(I+A)][(I-A)^{-1}(I+A)]^T = [(I-A)^{-1}(I+A)][(I+A)^T(I-A)^{-T}] = [(I-A)^{-1}(I+A)][(I+A^T)(I-A^T)^{-1}] = [(I-A)^{-1}(I+A)][(I-A)(I+A)^{-1}] = (I-A)^{-1}[(I+A)(I-A)](I+A)^{-1} = (I-A)^{-1}[(I-A)(I+A)](I+A)^{-1} = II = I$.

3. Show that the inverse of an orthogonal matrix is orthogonal. Given $Q^T = Q^{-1}$ we have $(Q^{-1})^T = (Q^T)^T = Q = (Q^{-1})^{-1}$ showing that Q^{-1} is orthogonal.

4. If Q is orthogonal show that $\|Q\boldsymbol{x}\| = \|\boldsymbol{x}\|$ for all $\boldsymbol{x}$.
$\|Q\boldsymbol{x}\| = (Q\boldsymbol{x})^T(Q\boldsymbol{x}) = \boldsymbol{x}^T Q^T Q\boldsymbol{x} = \boldsymbol{x}^T I\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{x} = \|\boldsymbol{x}\|$.

   Show the converse: If $\|Q\boldsymbol{x}\| = \|\boldsymbol{x}\|$ for all $\boldsymbol{x}$ for all $\boldsymbol{x}$ then Q is orthogonal. Use the following fact: If K, L are symmetric matrices and $\boldsymbol{x}^T K\boldsymbol{x} = \boldsymbol{x}^T L\boldsymbol{x}$ for all $\boldsymbol{x}$ then $K = L$.

   We have for all $\boldsymbol{x}$ that $\boldsymbol{x}^T Q^T Q\boldsymbol{x} = (Q\boldsymbol{x})^T(Q\boldsymbol{x}) = \|Q\boldsymbol{x}\| = \|\boldsymbol{x}\| = \boldsymbol{x}^T I\boldsymbol{x}$

   Since $Q^T Q$ and I are symmetric the fact above shows that $Q^T Q = I$ and hence Q is orthogonal.

5. Prove the fact stated above: If K, L are symmetric matrices and $\boldsymbol{x}^T K\boldsymbol{x} = \boldsymbol{x}^T L\boldsymbol{x}$ for all $\boldsymbol{x}$ then $K = L$. Write $\boldsymbol{e}_j$ for the j^{th} column of the identity matrix (of appropriate size). Note that $\boldsymbol{e}_i^T M\boldsymbol{e}_j = m_{ij}$. This follows since $\boldsymbol{e}_i^T(M\boldsymbol{e}_j)$ is the i^{th} entry of $M\boldsymbol{e}_j$ and $M\boldsymbol{e}_j$ is the j^{th} column of M. Thus $k_{ii} = \boldsymbol{e}_i^T K\boldsymbol{e}_i = \boldsymbol{e}_i^T K\boldsymbol{e}_i = l_{ii}$. So the diagonal entries of K and L are equal. Then for $i \neq j$, using $k_{ij} = k_{ji}$ since K is symmetric, we have $(\boldsymbol{e}_i + \boldsymbol{e}_j)^T K(\boldsymbol{e}_i + \boldsymbol{e}_j) = \boldsymbol{e}_i^T K\boldsymbol{e}_i + \boldsymbol{e}_i^T K\boldsymbol{e}_j + \boldsymbol{e}_j^T K\boldsymbol{e}_i + \boldsymbol{e}_j^T K\boldsymbol{e}_j = k_{ii} + k_{ij} + k_{ji} + k_{jj} = k_{ii} + 2k_{ij} + k_{jj}$. Similarly $(\boldsymbol{e}_i + \boldsymbol{e}_j)^T L(\boldsymbol{e}_i + \boldsymbol{e}_j) = l_{ii} + 2l_{ij} + l_{jj}$. Since these are equal we have $k_{ii} + 2k_{ij} + k_{jj} = l_{ii} + 2l_{ij} + l_{jj}$ and since $k_{ii} = l_{ii}$ and $k_{jj} = l_{jj}$ we get $k_{ij} = l_{ij}$ showing that $K = L$.