Score Sequences for Tournaments

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Score Sequences of Round Robin Tournaments

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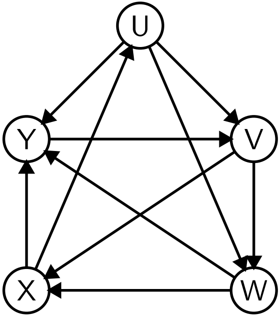
U wins 3 games, V wins 2 games, W wins 2 games, X wins 2 games, Y wins 1 games

Score sequence is (3,2,2,2,1)



Is the following sequence of 25 numbers a score sequence?

22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3

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Try testing *ALL* possible tournaments?

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All of the atoms in the known universe checking a billion tournaments per second

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Try testing *ALL* possible tournaments?

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Use mathematical tools to make the check faster

For 7 players there are $\frac{7(7-1)}{2} = 21$ games in a round robin tournament

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Which of the following are score sequences for a tournament with 7 players?

$(7, 5, 4\frac{1}{3}, 4, 2\frac{3}{7}, 0, -2)$

$(5, 4, 3, 3, 3, 1, 0)$

$(3, 3, 3, 3, 3, 3, 3)$

$(6, 6, 4, 2, 1, 1, 1)$

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$(7, 5, 4\frac{1}{3}, 4, 2\frac{3}{7}, 0, -2)$ NO - Scores must be non-negative integers

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$(3, 3, 3, 3, 3, 3, 3)$

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$(3, 3, 3, 3, 3, 3, 3)$

$(6, 6, 4, 2, 1, 1, 1)$ NO - Last 5 teams must have at least $10 = \frac{5\cdot 4}{2}$ wins

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$(3, 3, 3, 3, 3, 3, 3)$ YES

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Landau (1951) considered tournaments in the context of pecking order in poultry populations

A necessary condition for a sequence $(s_1, s_2, \ldots, s_n)$ of non-negative integers to be the score sequence of a round-robin tournament:

$$\sum_{i \in I} s_i \geq \frac{|I|(|I|-1)}{2} \text{ for any } I \subseteq \{1, 2, \ldots, n\}$$

$$\text{with equality when } I = \{1, 2, \ldots, n\}$$

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The number of wins for any set of teams must be as large as the number of games played between those teams

and

the total number of wins must equal the total number of games played

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Landau's Theorem: these necessary conditions are also sufficient

If the conditions hold there is a tournament with the given score sequence

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If the conditions hold there is a tournament with the given score sequence

If not a score sequence then there is a set of teams violating these obvious conditions

The sequence

$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$

can be checked by hand in a few minutes. It is not a score sequence

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$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$

Not a score sequence

Last 10 teams have 44 wins in $45 = \frac{10 \cdot 9}{2}$ games

Landau's Theorem:
A sequence $(s_1, s_2, \ldots, s_n)$ of non-negative integers is a score sequence of a round-robin tournament if and only if

$$\sum_{i \in I} s_i \geq \binom{|I|}{2} \text{ for any } I \subseteq \{1, 2, \ldots, n\}$$

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What if we allow ties?

What if the score is 3 points for a win, 1 for a tie and 0 for a loss (world cup soccer scoring)?

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What is $\binom{|I|}{2}$?

$$\binom{13}{2} = \frac{13 \cdot 12}{2} = 8 \text{ choose } 2$$

$$=$$

Number of 2 element subsets of $\{1, 2, \ldots, 13\}$

Binomial coefficients $\binom{n}{k} = n$ choose k

$$=$$

number of k elements subsets of $\{1, 2, \ldots, n\}$

$$\binom{13}{3} = \frac{13 \cdot 12 \cdot 11}{3 \cdot 2}$$

$$\binom{13}{5} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2}$$

Binomial coefficients - ‘Pascal’s Triangle’

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

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1							
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1	6	15	20	15	6	1	
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Hayluda Bhattotpala (India around 1000)
Al-Karaji and Kayyam (Persia around 1050)
Yang Hui (China around 1350)
Tartaglia (Italy around 1550)
Pascal (France around 1650)

Binomial identity: $\binom{7}{3} = \binom{6}{2} + \binom{6}{3}$

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

"Proof":

The $\binom{7}{3}$=35 size 3 subsets of $\{A, B, C, D, E, F, G\}$

$=$

The $\binom{6}{2} = 15$ subsets including A $+$ The $\binom{6}{3} = 20$ subsets avoiding A

$$\begin{array}{rlllllllll} & 1 \\ & 1 & 1 \\ & 1 & 2 & 1 \\ 8 = & 1 & 3 & 3 & 1 \\ & 1 & 4 & 6 & 4 & 1 \\ & 1 & 5 & 10 & 10 & 5 & 1 \\ & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \end{array}$$

	1							
	1	1						
	1	2	1					
$8 =$	1	3	3	1				
$16 =$	1	4	6	4	1			
	1	5	10	10	5	1		
	1	6	15	20	15	6	1	
	1	7	21	35	35	21	7	1

$$
\begin{array}{rcllllllll}
 & & 1 \\
 & & 1 & 1 \\
 & & 1 & 2 & 1 \\
8 & = & 1 & 3 & 3 & 1 \\
16 & = & 1 & 4 & 6 & 4 & 1 \\
32 & = & 1 & 5 & 10 & 10 & 5 & 1 \\
 & & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\
 & & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1
\end{array}
$$

$$
\begin{array}{rcl}
1 &=& 1 \\
2 &=& 1\ 1 \\
4 &=& 1\ 2\ 1 \\
8 &=& 1\ 3\ 3\ 1 \\
16 &=& 1\ 4\ 6\ 4\ 1 \\
32 &=& 1\ 5\ 10\ 10\ 5\ 1 \\
64 &=& 1\ 6\ 15\ 20\ 15\ 6\ 1 \\
128 &=& 1\ 7\ 21\ 35\ 35\ 21\ 7\ 1
\end{array}
$$

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\begin{aligned}
1 &= 1\\
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\end{aligned}
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Row sums are powers of 2

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128 &=& 1\ 7\ 21\ 35\ 35\ 21\ 7\ 1
\end{array}
$$

Row sums are powers of 2

"Proof": $128 = 2^7$, number of subsets of $\{1, 2, \ldots, 7\}$
row sums over choices of subset size

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
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1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

1							
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1	5	10	10	5	1		
1	6	15	20	15	6	1	
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Diagonal sums are binomial coefficients:

$1 + 3 + 6 + 10 + 15 = 35$

Diagonal sums are binomial coefficients:

"Proof":

1							
1	1						
1	2	1					
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1							
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"Proof":

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

=

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

=

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

	1							
	1	1						
	1	2	1					
$3 =$	1	3	3	1				
	1	4	6	4	1			
	1	5	10	10	5	1		
	1	6	15	20	15	6	1	
	1	7	21	35	35	21	7	1

	1							
	1	1						
	1	2	1					
3 =	1	3	3	1				
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\end{array}
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Anti-diagonal sums are Fibonacci numbers

1	=	1									
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Anti-diagonal sums are Fibonacci numbers

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

1	=	1								
1	=	1	1							
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$F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = 0, F_1 = 1.$

1	=	1								
1	=	1	1							
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$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

$F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = 0, F_1 = 1.$

"Proof": Use binomial identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Landau's Theorem via systems of linear inequalities

Landau's Theorem via systems of linear inequalities

- Possible score sequence $(s_1, s_2, \ldots, s_n)$
- For each integral pair $1 \leq i < j \leq n$ define a variable $x_{i,j}$ with $x_{i,j} = 1$ if i beats j and $x_{i,j} = 0$ if i losses to j
- There is a tournament with the given score sequence if and only if the following has a solution:

$$\sum_{i<j}(1 - x_{i,j}) + \sum_{j<k} x_{j,k} = s_j \text{ for } j = 1, 2, \ldots, n$$
$$x_{i,j} \in \{0, 1\} \text{ for all } i < j$$

Relax to $0 \leq x_{i,j} \leq 1$

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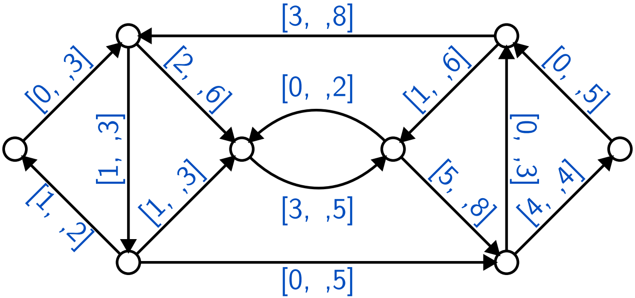
$$x_{i,j} \in \{0, 1\} \text{ for all } i < j$$

Relax to $0 \leq x_{i,j} \leq 1$

For scores $(6, 6, 4, 2, 1, 1, 1)$ equation for $s_3 = 4$ is
$(1 - x_{1,3}) + (1 - x_{2,3}) + x_{3,4} + x_{3,5} + x_{3,6} + x_{3,7} = 4$

Circulation in a network: flow between lower and upper bounds satisfying flow conservation at each node



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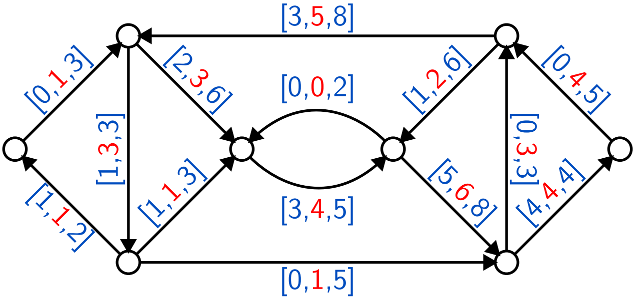

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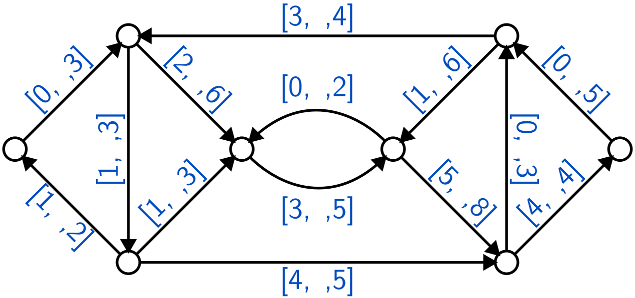
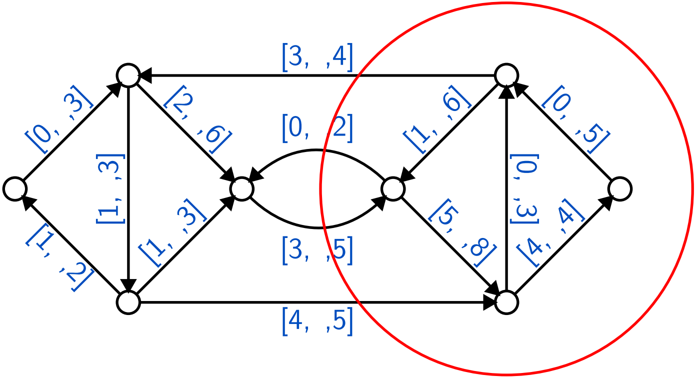
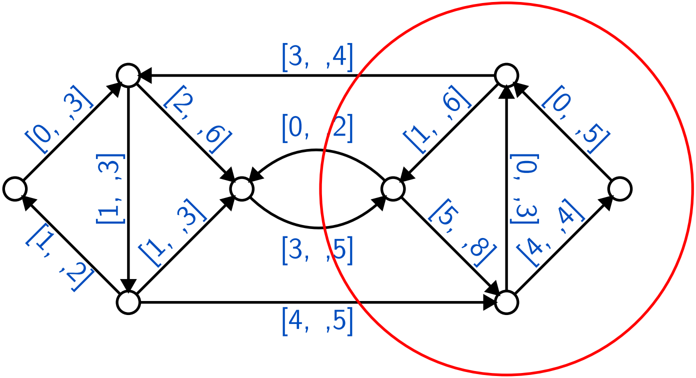




Requirements in $= 3 + 4 = 7 > 6 = 4 + 2 =$ capacity out



Requirements in $= 3 + 4 = 7 > 6 = 4 + 2 =$ capacity out

No circulation

Hoffman (1956)

A necessary condition for a circulation:

for for every set of nodes:

capacities out $\geq$ the requirements in

(sum of upper bounds) $\geq$ (sum of lower bounds in)

Hoffman (1956)

A necessary condition for a circulation:
for for every set of nodes:
capacities out $\geq$ the requirements in
(sum of upper bounds) $\geq$ (sum of lower bounds in)

Hoffman's Circulation Theorem (1956): These necessary conditions are also sufficient

If the conditions hold there is a circulation

Hoffman (1956)

A necessary condition for a circulation:
for for every set of nodes:
capacities out $\geq$ the requirements in
(sum of upper bounds) $\geq$ (sum of lower bounds in)

Hoffman's Circulation Theorem (1956): These necessary conditions are also sufficient

If the conditions hold there is a circulation

If there is no circulation the is some set with capacities out $<$ requirements in

Hoffman's Circulation Theorem via systems of linear inequalities

- Network with upper bounds $u_{i,j}$ and lower bounds $l_{i,j}$ for arcs i, j
- For each arc i, j define a variable $x_{i,j}$ which will correspond to the amount of flow.
- There is a circulation if and only if the following has a solution:

$$\sum_{i,j \in A} x_{i,j} = \sum_{j,k \in A} x_{j,k} \text{ for each node } j$$
$$l_{i,j} \leq x_{i,j} \leq u_{i,j} \text{ for each arc } i, j$$

Equations force flow conservation
inequalities enforce lower and upper bounds

Hoffman's Circulation inequalities

$$\sum_{i,j \in A} -x_{i,j} + \sum_{j,k \in A} x_{j,k} = 0 \text{ for each node } j$$

$$l_{i,j} \leq x_{i,j} \leq u_{i,j} \text{ for each arc } i, j$$

Landau's score sequence inequalities

$$-(s_j + j - 1) + \sum_{i<j} -x_{i,j} + \sum_{j<k} x_{j,k} = 0 \text{ for } j = 1, 2, \ldots, n$$

$$0 \leq x_{i,j} \leq 1 \text{ for all } i < j$$

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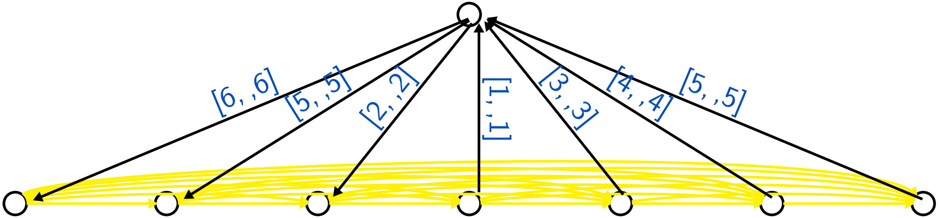
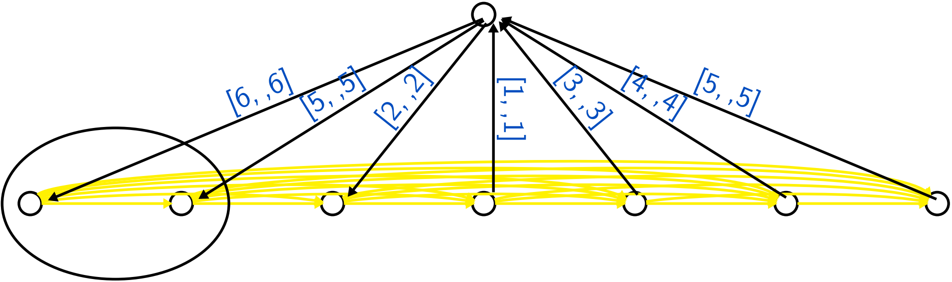
Almost the same: Create new vertex with flows to j forced to be $s_j - j + 1$

$\Rightarrow$

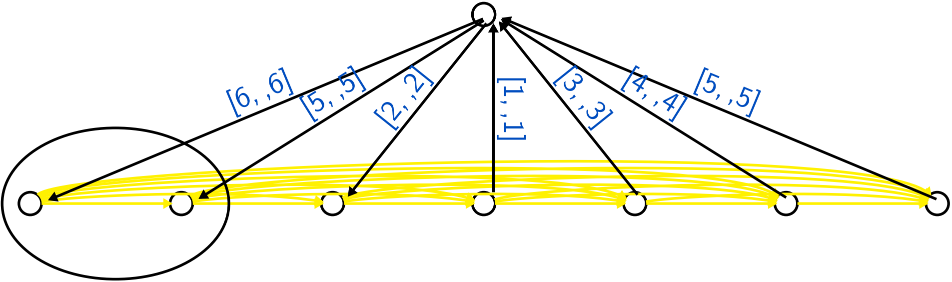
Landau's Theorem as a special case of Hoffman's Circulation Theorem



Yellow arcs left to right, lower bound 0, upper bound 1

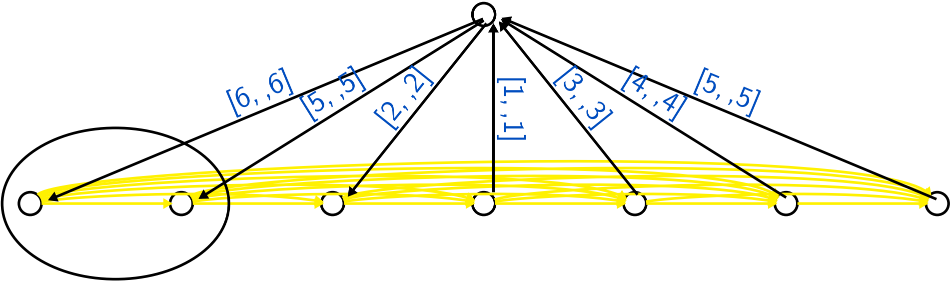


Yellow arcs left to right, lower bound 0, upper bound 1
Requirements in $= 11 > 10 =$ capacities out



Yellow arcs left to right, lower bound 0, upper bound 1
Requirements in $= 11 > 10 =$ capacities out

No circulation



Yellow arcs left to right, lower bound 0, upper bound 1
Requirements in $= 11 > 10 =$ capacities out

No circulation

Corresponds to $(6, 6, 4, 2, 1, 1, 1)$
Not a score sequence

Do these have nonnegative solutions?

$$\begin{aligned} x+2y &= 3 \\ 4x+5y &= 6 \end{aligned} \qquad \begin{aligned} x+2y &= 3 \\ 4x+8y &= 12 \end{aligned} \qquad \begin{aligned} x+2y &= 3 \\ 4x+8y &= 6 \end{aligned}$$

Do these have nonnegative solutions?

$$\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned} \qquad \begin{aligned} x + 2y &= 3 \\ 4x + 8y &= 12 \end{aligned} \qquad \begin{aligned} x + 2y &= 3 \\ 4x + 8y &= 6 \end{aligned}$$

$x = -1,\ y = 2$ — no

line $x + 2y = 3$ — yes

Has no solution. Why not?

Do these have nonnegative solutions?

$\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned}$	$\begin{aligned} x + 2y &= 3 \\ 4x + 8y &= 12 \end{aligned}$	$\begin{aligned} x + 2y &= 3 \\ 4x + 8y &= 6 \end{aligned}$
$x = -1,\ y = 2$ no	line $x + 2y = 3$ yes	Has no solution Why not?

Intersection of two lines

May be a point, a line or parallel lines

Do these have nonnegative solutions?

$$\begin{aligned} x + y + 2z &= 3 \\ 5x + 8y + 13z &= 21 \\ x - y + z &= 0 \end{aligned} \qquad \begin{aligned} x + y + 2z &= 13 \\ 5x + 8y + 13z &= 21 \\ x - 3y - 3z &= 1 \end{aligned}$$

Do these have nonnegative solutions?

$$\begin{aligned} x + y + 2z &= 3 \\ 5x + 8y + 13z &= 21 \\ x - y + z &= 0 \end{aligned}$$

$$\begin{aligned} x + y + 2z &= 13 \\ 5x + 8y + 13z &= 21 \\ x - 3y - 3z &= 1 \end{aligned}$$

$x = 0,\ y = z = 1$
yes

no
Why not?

This system has no nonnegative solution

$$\begin{aligned} x + y + 2z &= 13 \\ 5x + 8y + 13z &= 21 \\ x - 3y - 3z &= 1 \end{aligned}$$

This system has no nonnegative solution

$$\begin{array}{rrcrcrcl} \color{red}{-2} & x & + & y & + & 2z & = & 13 \\ \color{red}{1} & 5x & + & 8y & + & 13z & = & 21 \\ \color{red}{2} & x & - & 3y & - & 3z & = & 1 \end{array}$$

Multiply equations by (-2), 1, 2 respectively

This system has no nonnegative solution

$$\begin{array}{rrrrrrrrl} -2 & x & + & y & + & 2z & = & 13 \\ 1 & 5x & + & 8y & + & 13z & = & 21 \\ 2 & x & - & 3y & - & 3z & = & 1 \end{array}$$

Multiply equations by (-2), 1, 2 respectively
Add resulting equations

$$\begin{array}{rrrrrrr} -2x & - & 2y & - & 4z & = & -26 \\ 5x & + & 8y & + & 13z & = & 21 \\ 2x & - & 6y & - & 6z & = & 2 \end{array}$$

This system has no nonnegative solution

$$\begin{array}{rrcrcrcl} -2 & x & + & y & + & 2z & = & 13 \\ 1 & 5x & + & 8y & + & 13z & = & 21 \\ 2 & x & - & 3y & - & 3z & = & 1 \end{array}$$

Multiply equations by (-2), 1, 2 respectively
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Result is

$$5x + 0y + 3z = -3$$

Every solution has at least one of x, y, z negative

This system has no nonnegative solution

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Result is

$$5x + 0y + 3z = -3$$

Every solution has at least one of x, y, z negative
Farkas' Lemma: Either a linear system has a nonnegative solution *or* there are multipliers showing an inconsistency

Farkas' Lemma

Either a linear system has a nonnegative solution

OR

There are multipliers showing inconsistency

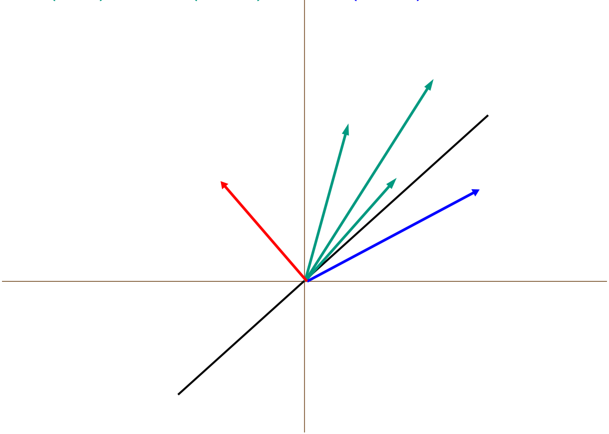
-2	$x + y + 2z = 13$	$\Rightarrow 3x + 6y + 9z \leq -5$
1	$5x + 8y + 13z = 21$	

Rewrite

$$\begin{array}{rrcrcrcl} -2 & x & + & y & + & 2z & = & 13 \\ 1 & 5x & + & 8y & + & 13z & = & 21 \end{array} \quad \Rightarrow 3x + 6y + 9z \leq -5$$

as

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix} x + \begin{pmatrix} 1 \\ 8 \end{pmatrix} y + \begin{pmatrix} 2 \\ 13 \end{pmatrix} z = \begin{pmatrix} 13 \\ 21 \end{pmatrix}$$



Farkas' Lemma

Either $\begin{pmatrix} 13 \\ 21 \end{pmatrix}$ is in the cone generated by

$$\left\{ \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 8 \end{pmatrix}, \begin{pmatrix} 2 \\ 13 \end{pmatrix} \right\}$$

OR

There is a separating hyperplane

Multipliers showing inconsistency provide the normal to the hyperplane forming an angle less than 90 degree with the 'columns' and greater than 90 degrees with the right hand side

$$\begin{array}{rrrrrrrr} -2 & x & + & y & + & 2z & = & 13 \\ 1 & 5x & + & 8y & + & 13z & = & 21 \end{array} \quad \Rightarrow 3x + 6y + 9z \leq -5$$

Set up systems for circulations and score sequences. if no solution, the 'multipliers' are $0, 1$ and produce violations of necessary conditions.

Set up systems for circulations and score sequences. if no solution, the 'multipliers' are $0, 1$ and produce violations of necessary conditions.

Farkas' Lemma for nonnegative solutions to linear systems of equations

$\Downarrow$

Hoffman's Circulation Theorem

$\Downarrow$

Landau's Theorem for Score Sequences