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The KO^*-rings of BT^m, the Davis–Januszkiewicz spaces and certain toric manifolds

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The other eight authors dedicate this work to the memory of their friend, teacher, and coauthor, Mark Mahowald

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ABSTRACT

This paper contains an explicit computation of the KO^*-ring structure of an m-fold product of $\mathbb{C}P^\infty$, the Davis–Januszkiewicz spaces and of toric manifolds which have trivial Sq^2-homology. A key ingredient is the stable splitting of the Davis–Januszkiewicz spaces given by Bahri et al. (2009, 2010) [6,7].

1. Introduction

The term "toric manifolds" in this paper refers to the topological spaces whose detailed information may be found in [15,11] and a brief description is given in Section 6. These spaces are also called "quasitoric manifolds" and include the class of all non-singular projective toric varieties.

An n-torus T^n acts on a toric manifold M^{2n} with quotient space a simple polytope P^n having m codimension-one faces (facets). Associated to P^n is a simplicial complex K_P on vertices $\{v_1, v_2, \ldots, v_m\}$ with each v_i corresponding to a single facet F_i of P^n. The set $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ is a simplex in K_P if and only if $F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k} \neq \varnothing$.

The classifying space of the real n-torus T^n is denoted by BT^n. Associated to the torus action is a Borel-space fibration

$$M^{2n} \longrightarrow ET^n \times_{T^n} M^{2n} \xrightarrow{p} BT^n. \tag{1.1}$$

Of course here, $BT^n = \mathbb{C}P^\infty \times \mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty$, ($n$ factors).

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The homotopy type of the Borel space $ET^n \times_{T^n} M^{2n}$ depends on K_P only. It is referred to as the Davis–Januszkiewicz space of K_P and is denoted by the symbol $\mathcal{DJ}(K_P)$. More generally, a Davis–Januszkiewicz space exists for *any* simplicial complex K; Section 5 contains more details about this generalization. The convention following is adopted throughout.

Convention: All generalized homology and cohomology theories are considered *reduced*.

It is known for any complex-oriented cohomology theory E^* that,

$$E^*(\mathcal{DJ}(K_P)) = E^*(BT^m)/I_{SR}^E \tag{1.2}$$

where I_{SR}^E is an ideal in $E^*(BT^m)$ described next. In this context, the two-dimensional generators of the graded ring $E^*(BT^m)$ are denoted by $\{v_1, v_2, \ldots, v_m\}$. The ideal I_{SR}^E is generated by all square-free monomials $v_{i_1}v_{i_2}\cdots v_{i_s}$ corresponding to $\{v_{i_1}, v_{i_2}, \ldots, v_{i_s}\} \notin K_P$. The ring (1.2) is called the E^*-Stanley–Reisner ring.

For a toric manifold M^{2n} an argument, based on the collapse of the Atiyah–Hirzebruch–Serre spectral sequence for (1.1), yields an isomorphism of E^*-algebras

$$E^*(M^{2n}) \cong E^*(\mathcal{DJ}(K_P))\big/J^E \tag{1.3}$$

where the ideal J^E is generated by $p^*(E^2(BT^n))$ and therefore by the E-theory Chern classes of certain associated line bundles, ([12], page 18 and [13], page 6).

For the case of non-singular compact projective toric varieties and E equal to ordinary singular cohomology with integral coefficients ($E = H\mathbb{Z}$), this is the celebrated result of Danilov and Jurkiewicz [14]. The $E = H\mathbb{Z}$ version for the topological generalization of compact smooth toric varieties, is due to Davis and Januszkiewicz [15]. For certain singular toric varieties, the results of [14,23] cover also the case $E = H\mathbb{Q}$.

The question of an analogue of (1.3) for a non-complex-oriented theory arises naturally. The obvious first candidate is KO-theory. The ring structure of

$$KO^*(BT^m) = KO^*\left(\prod_{i=1}^{m} \mathbb{C}P^\infty\right) \tag{1.4}$$

does not seem to appear in the literature for $m > 2$. The thesis of Dobson [17] investigates the ring structure for the case $m = 2$. The KO^*-algebra structure of $KO^*(\mathbb{C}P^n)$ is deduced in [13] from the seminal work of Fujii [18]. There, the calculation is extended to $KO^*(\mathbb{C}P^\infty)$ in the context of the theorem of Wood, which is applied also in Section 2.

The fact that $KO^*(BT^m)$ is torsion free and concentrated in even degree, was known to D.W. Anderson, whose thesis [2] appeared in 1964.

Two different presentations for the ring $KO^*(BT^m)$ are given in Sections 3 and 4. Following Atiyah and Segal [4], these provide a description of the completion of the representation ring $RO(T^m)$ at the augmentation ideal. The calculation herein may be interpreted in that context, along the lines of Anderson [3]. In particular, the fact that the complexification map and the realification map

$$c\colon KO^*(BT^m) \longrightarrow KU^*(BT^m) \tag{1.5}$$

$$r\colon KU^*(BT^m) \longrightarrow KO^*(BT^m) \tag{1.6}$$

are injective and surjective respectively, is used throughout. This follows from the Bott exact sequence

$$\cdots \to KO^{*+1}(X) \xrightarrow{\cdot e} KO^*(X) \xrightarrow{\chi} KU^{*+2}(X) \xrightarrow{r} KO^{*+2}(X) \to \cdots \tag{1.7}$$

where χ is complexification (1.5) followed by multiplication by the Bott element v^{-1}. Since $KO^*(BT^m)$ is concentrated in even degree, the Bott sequence implies that the realification map r is surjective and complexification c is injective. They are related by

$$(r \circ c)(x) = 2x \quad \text{and} \quad (c \circ r)(x) = x + \bar{x}. \tag{1.8}$$

The complexity of the calculation is a result of the fact that the realification map r is *not* a ring homomorphism. The first presentation generalizes the methods of [17]. A companion result is given for $KO^*(\bigwedge_{i=1}^m \mathbb{C}P^\infty)$. The second approach produces generators better suited to the task of giving a description of $KO^*(\mathcal{DJ}(K_P))$ in terms of $KO^*(BT^m)$. The results of [5] are used then to give a description of $KO^*(M^{2n})$ analogous to (1.3) for any toric manifold which has no Sq^2-homology.

The group structure of $KO^*(\bigwedge_{i=1}^m \mathbb{C}P^\infty)$ is much more accessible than the ring structure. The Adams spectral sequence yields a concise description in terms of the groups $KU^*(\bigwedge_{i=1}^k \mathbb{C}P^\infty)$ with fewer smash product factors. This is discussed in the next section.

2. The group structure of $KO^*(\bigwedge_{i=1}^m \mathbb{C}P^\infty)$

2.1. The ko-homology

The calculation begins with the determination of $ko_*(BT^m)$, the connective ko-homology corresponding to the spectrum bo. The main tool is the Adams spectral sequence. It is used in conjunction with the following equivalence, which is well known among homotopy theorists and extends a result of Wood. Let bu denote the spectrum corresponding to connective complex k-theory.

Theorem 2.1. *There is an equivalence of spectra*

$$\bigvee_{k=0}^{\infty} \Sigma^{4k+2} bu \longrightarrow bo \wedge \mathbb{C}P^\infty.$$

Proof. Background information about the Adams spectral sequence in connection with ko-homology calculations may be found in [5] or [9]. A change of rings theorem implies that the E_2-term of the Adams spectral sequence for $ko_*(\mathbb{C}P^\infty) = \pi_*(bo \wedge \mathbb{C}P^\infty)$ depends on the $\mathcal{A}_1$-module structure of $H^*(\mathbb{C}P^\infty; \mathbb{Z}_2)$ where $\mathcal{A}_1$ denotes the sub-algebra of the Steenrod algebra $\mathcal{A}$ generated by Sq^1 and Sq^2.

As an $\mathcal{A}_1$-module, $H^*(\mathbb{C}P^\infty; \mathbb{Z}_2)$ is a sum of shifted copies of $H^*(\mathbb{C}P^2; \mathbb{Z}_2)$. Consequently, the E_2-term of the spectral sequence is a sum of shifted copies of $\mathrm{Ext}^{s,t}_{\mathcal{A}_1}(H^*(\mathbb{C}P^2; \mathbb{Z}_2), \mathbb{Z}_2)$ and so has classes in even degree only. Hence, the spectral sequence collapses. A non-trivial class in each $\pi_{4k+2}(bo \wedge \mathbb{C}P^\infty)$ is represented in the E_2-term by a generator of dimension $4k+2$ in filtration zero. The η-extension on this class is trivial as the E_2-term is zero in odd degree. So the map

$$S^{4k+2} \longrightarrow bo \wedge \mathbb{C}P^\infty \tag{2.1}$$

extends over a $(4k+4)$-cell e^{4k+4} attached by the Hopf map η. This gives a map

$$\Sigma^{4k}\mathbb{C}P^2 = S^{4k+2} \cup_\eta e^{4k+4} \longrightarrow bo \wedge \mathbb{C}P^\infty \tag{2.2}$$

which extends to

$$s\colon \bigvee_{k=0}^{\infty} \Sigma^{4k}\mathbb{C}P^2 \longrightarrow bo \wedge \mathbb{C}P^\infty. \tag{2.3}$$

Smashing with bo and composing with the product map $bo \wedge bo \xrightarrow{\mu} bo$ gives

$$bo \wedge \left(\bigvee_{k=0}^{\infty} \Sigma^{4k}\mathbb{C}P^2\right) \xrightarrow{1\wedge s} bo \wedge bo \wedge \mathbb{C}P^\infty \xrightarrow{\mu\wedge 1} bo \wedge \mathbb{C}P^\infty. \tag{2.4}$$

The equivalence of spectra $\Sigma^2 bu \to bo \wedge \mathbb{C}P^2$, due to Wood and cited in [1] (page 206), is used next to give a map

$$g\colon \bigvee_{k=0}^{\infty} \Sigma^{4k+2} bu \longrightarrow bo \wedge \mathbb{C}P^\infty. \tag{2.5}$$

This map induces an isomorphism of stable homotopy groups and hence gives the required equivalence of spectra. □

Remark. An equivalence of the form (2.4) follows also from the methods of [19] and the fact that twice the Hopf bundle over $\mathbb{C}P^\infty$ is a Spin bundle and therefore ko-orientable.

The next corollary follows immediately.

Corollary 2.2. *There is an isomorphism of graded abelian groups*

$$\bigoplus_{k=0}^{\infty} \Sigma^{4k+2}\left(KU_*\left(\bigwedge_{i=2}^{m} \mathbb{C}P^\infty\right)\right) \longrightarrow KO_*\left(\bigwedge_{i=1}^{m} \mathbb{C}P^\infty\right).$$

Notice here that the summands on the left hand side are the underlying groups of a tensor product of divided power algebras each of which is dual to a polynomial algebra.

Recall next that there are classes $e \in ko_1$, $\alpha \in ko_4$, $\beta \in ko_8$ so that

$$ko_* = \mathbb{Z}[e, \alpha, \beta]/\langle 2e, e^3, e\alpha, \alpha^2 - 4\beta\rangle \tag{2.6}$$

and a class $v \in ku_2$ so that

$$ku_* = \mathbb{Z}[v]. \tag{2.7}$$

Remark 2.3. An examination of the E_2-term of the Adams spectral sequence for $ko_*(\mathbb{C}P^2)$ reveals that the action of ko_* on $ko_*(\mathbb{C}P^2) \cong ku_*$ is given by

$$e \cdot 1 = e \cdot v = 0, \qquad \alpha \cdot 1 = 2v^2, \qquad \beta \cdot 1 = v^4. \tag{2.8}$$

This coincides with the module action of ko_* on ku_* given by the "complexification" map, ([17], page 16).

2.2. From ko-homology to KO-cohomology

One consequence of the calculation above is that the Bott element β acts as a monomorphism on $ko_*(\bigwedge_{i=1}^{s} \mathbb{C}P^\infty)$ and so can be inverted to get the periodic KO-homology of $\bigwedge_{i=1}^{s} \mathbb{C}P^\infty$.

Proposition 2.4. *There is an isomorphism of abelian groups*

$$\bigoplus_{k=0}^{\infty} KU_{*+4k+2}\left(\bigwedge_{i=2}^{m} \mathbb{C}P^\infty\right) \longrightarrow KO_*\left(\bigwedge_{i=1}^{m} \mathbb{C}P^\infty\right).$$

Proof. The result follows from Corollary 2.2 and Remark 2.3. □

The proof of Theorem 2.1 works equally well in the dual situation. Let $D(\mathbb{C}P^{2n})$ denote the S-dual of $\mathbb{C}P^{2n}$. Aside from a dimensional shift, $H^*(D(\mathbb{C}P^{2n}); \mathbb{Z}_2)$, as an $\mathcal{A}_1$-module, is isomorphic to a sum of suspended copies of $H^*(\mathbb{C}P^2; \mathbb{Z}_2)$. So, the Adams spectral sequence for $\pi_*(bo \wedge D(\mathbb{C}P^{2n}))$ collapses for dimensional reasons. The argument of Theorem 2.1 goes through essentially unchanged to give an equivalence of spectra

$$g : \bigvee_{k=1}^{n} \Sigma^{-4k} bu \longrightarrow bo \wedge D(\mathbb{C}P^{2n}). \tag{2.9}$$

The next lemma, which follows directly from the discussion in Section 2.1, records the fact that (2.9) is natural with respect to the inclusion

$$\mathbb{C}P^{2n} \xrightarrow{\subset} \mathbb{C}P^{2(n+1)}.$$

Lemma 2.5. *The diagram following commutes*

$$\begin{array}{ccc} \bigvee_{k=1}^{n} \Sigma^{-4k} bu & \xrightarrow{g} & bo \wedge D(\mathbb{C}P^{2n}) \\ \uparrow{\scriptstyle \phi} & & {\scriptstyle \psi}\uparrow \\ \bigvee_{k=1}^{n+1} \Sigma^{-4k} bu & \xrightarrow{g} & bo \wedge D(\mathbb{C}P^{2(n+1)}) \end{array} \tag{2.10}$$

where the map ϕ collapses $\Sigma^{-4(n+1)} bu$ to a point and ψ is induced by the inclusion

$$\mathbb{C}P^{2n} \xrightarrow{\subset} \mathbb{C}P^{2(n+1)}. \quad \square$$

The duality result from [1, Proposition 5.6], implies

$$D\left(\bigwedge_{i=1}^{m} \mathbb{C}P^{2n}\right) \simeq \bigwedge_{i=1}^{m} D(\mathbb{C}P^{2n}). \tag{2.11}$$

From this follows an isomorphism of abelian groups, analogous to Proposition 2.4 for finite projective spaces,

$$\bigoplus_{k=1}^{n} KU_{*-4k}\left(D\left(\bigwedge_{i=2}^{m} \mathbb{C}P^{2n}\right)\right) \longrightarrow KO_*\left(D\left(\bigwedge_{i=1}^{m} \mathbb{C}P^{2n}\right)\right) \tag{2.12}$$

and so an isomorphism

$$\bigoplus_{k=1}^{n} KU^{*+4k}\left(\bigwedge_{i=2}^{m} \mathbb{C}P^{2n}\right) \longrightarrow KO^*\left(\bigwedge_{i=1}^{m} \mathbb{C}P^{2n}\right). \tag{2.13}$$

The next result extends (2.13) to $\bigwedge_{i=1}^{m} \mathbb{C}P^\infty$.

Proposition 2.6. *There are isomorphisms*

$$KO^*\left(\bigwedge_{i=1}^{m} \mathbb{C}P^\infty\right) \cong \lim_{\overleftarrow{n}} KO^*\left(\bigwedge_{i=1}^{m} \mathbb{C}P^{2n}\right)$$

and

$$\bigoplus_{k=1}^{\infty} KU^{*+4k}\left(\bigwedge_{i=2}^{m} \mathbb{C}P^\infty\right) \cong \lim_{\overleftarrow{n}} \bigoplus_{k=1}^{n} KU^{*+4k}\left(\bigwedge_{i=2}^{m} \mathbb{C}P^{2n}\right).$$

Proof. It follows from the calculations above that the maps in the inverse limit arising from

$$KO^*\left(\bigwedge_{i=1}^{m} \mathbb{C}P^{2(n+1)}\right) \longrightarrow KO^*\left(\bigwedge_{i=1}^{m} \mathbb{C}P^{2n}\right)$$

and

$$\bigoplus_{k=1}^{n+1} KU^{*+4k}\left(\bigwedge_{i=2}^{m} \mathbb{C}P^{2(n+1)}\right) \longrightarrow \bigoplus_{k=1}^{n} KU^{*+4k}\left(\bigwedge_{i=2}^{m} \mathbb{C}P^{2n}\right)$$

(induced from the maps ψ and ϕ of diagram (2.10)) are all surjective. Thus the Mittag-Leffler condition is satisfied and the $\lim_{\overleftarrow{n}}{}^1$ terms are zero. □

Finally, Lemma 2.5 implies that the isomorphisms (2.13) are compatible with the maps in the inverse limits and so yield the main result of this section.

Theorem 2.7. *There is an isomorphism of graded abelian groups*

$$\bigoplus_{k=1}^{\infty} KU^{*+4k}\left(\bigwedge_{i=2}^{m} \mathbb{C}P^\infty\right) \longrightarrow KO^*\left(\bigwedge_{i=1}^{m} \mathbb{C}P^\infty\right).$$

3. The algebra $KO^*(BT^m)$

This section contains the first of two descriptions of the algebra $KO^*(BT^m)$. It extends the calculation done in [17] for the case $m = 2$. Section 3.4 is an addendum to this section which incorporates pertinent observations made by the referee about the calculation.

3.1. Notation and statement of results

Here, as in Section 1,

$$BT^m \cong \prod_{i=1}^{m} \mathbb{C}P^\infty.$$

The two sets of generators presented for $KO^*(BT^m)$ have contrasting advantages. The first description yields generators which are slightly complicated but the relations among them are fairly straightforward. This situation is reversed in the second description.
The complexification and realification maps, (1.5) and (1.6), are denoted again by c and r respectively.

Let $\alpha \in KO^{-4}$ and $\beta \in KO^{-8}$ be the elements arising from (2.6), for which $\alpha^2 = 4\beta$. Let $v \in KU^{-2}$ be the Bott element, which satisfies $r(v^2) = \alpha$ and $c(\alpha) = 2v^2$. Let BT_+^m denote the disjoint union of BT^m with a point. The generators of $KU^0(BT_+^m)$ are denoted by x_i for $i = 1, \ldots, m$ so that $KU^0(BT_+^m) \cong \mathbb{Z}[[x_1, \ldots, x_m]]$.
More notation is established next.

Definition 3.1. Consider the set $N = \{1, \ldots, m\}$ and let $S \subseteq N$. Then

(1) set $\min(S) = \min\{i : i \in S\}$,
(2) let $|S|$ denote the cardinality of S,
(3) for $s \in \{0, 1, 2\}$, let $X_S^{(s)} = r(v^s \prod_{i \in S} x_i) \in KO^{-2s}(BT^m)$,
(4) let $X_S = X_S^{(0)}$ and $X_i = X_{\{i\}} = r(x_i)$,
(5) for $s \in \{0, 1\}$, let $X_\varnothing^{(s)} = 1 + (-1)^s$,
(6) for $s \in \{0, 1\}$, let $M_S^{(s)} = X_S^{(s)} \cdot \prod_{i \in N \setminus S} X_i$ and $M_S = M_S^{(0)}$.
(7) $\widehat{BT}^m = \bigwedge_{i=1}^{m} \mathbb{C}P^\infty$ and
(8) $\overline{\mathbb{Z}}[[-]]$ denotes the augmentation ideal of a power series ring.

Theorem 3.2. *There is an isomorphism of graded rings*

$$KO^*(BT^m) \cong \mathbb{Z}[\gamma^{\pm 1}] \otimes \overline{\mathbb{Z}}[[X_S, X_S^{(1)} : \varnothing \neq S \subseteq N]]/\sim$$

where γ is an element with $|\gamma| = -4$ satisfying $2\gamma = \alpha$ and $\gamma^2 = \beta$. Here $\sim$ refers to the two families of relations (I) *and* (II).

(I) *If A, B and C are disjoint subsets of N and $0 \leq s, t \leq 1$, then*

$$X^{(s)}_{A\cup B}X^{(t)}_{A\cup C} = \prod_{i\in A} X_i \cdot \left[\sum_{T\subseteq A} X^{(s+t)}_{T\cup B\cup C} + (-1)^{s+|A\cup B|}\sum_{S\subseteq B}(-1)^{|S|}\left(\prod_{i\in S}X_i\right)X^{(s+t)}_{C\cup B\setminus S}\right].$$

Here B, C, S, T may be empty, $X_S^{(2)} = \gamma X_S$ and products over empty sets are considered equal to 1.

(II) *For $i < \min(S)$, $|S| > 1$ and $s \in \{0, 1\}$,*

$$X_iX_S^{(s)} = (-1)^s \sum_{T\subseteq S}\left[(-1)^{|T|}\left(\prod_{j\in S\setminus T}X_j\right)\cdot X^{(s)}_{\{i\}\cup T}\right] + X^{(s)}_{\{i\}\cup S}\,.$$

Again, T may be empty.

In particular, $KO^*(BT^m)$ is a finite direct sum of free KO^*-modules over power series rings.

Remark 3.3. The element γ is introduced here for notational convenience. It arises naturally in the Adams spectral sequence and has the property that $\gamma r(x) = r(v^2x)$ for all $x \in KU^0(BT^m)$. The use of γ may be removed in Theorem 3.2 and in Corollary 3.4, by allowing the choice of the exponent s in Definition 3.1 to be unrestricted.

Relations (I) allow the elimination of all products $X_U^{(u)}X_V^{(v)}$ with $|U|$ and $|V|$ both greater than 1, reducing everything to products of X_i's times at most one $X_W^{(w)}$ with $|W| > 1$. Relations (II) allow the elimination of $X_iX_W^{(w)}$ with $|W| > 1$ and $i < \min(W)$. Notice that for a product $X_{i_1}X_{i_2}\cdots X_{i_k}X_W^{(w)}$, (II) need be performed once only for just one X_{i_j} with minimal i_j.

The next corollary is now immediate.

Corollary 3.4. *Every element of $KO^*(BT^m)$ can be expressed as a formal sum of terms from*

$$\mathcal{G}_1 = \left\{\gamma^j\left(\prod_{i\,=\,\min(S)}^{m} X_i^{e_i}\right)X_S^{(s)} : S\subseteq N,\ S\neq\varnothing,\ e_i\geq 0,\ j\in\mathbb{Z} \text{ and } s\in\{0,1\}\right\}.$$

The example below follows easily from Theorem 3.2.

Example 3.5. For $s \in \{0, 1\}$, $KO^{-(4j+2s)}(BT^2)$ has a basis

$$\mathcal{G}_1 = \left\{\gamma^jX_2^{e_2}X_2^{(s)},\ \gamma^jX_1^{e_1}X_2^{e_2}X_1^{(s)},\ \gamma^jX_1^{e_1}X_2^{e_2}X_{\{1,2\}}^{(s)} : e_1, e_2 \geq 0\right\}.$$

The following relations determine all products among these basis elements. Here $s \in \{0, 1\}$ and $i \in \{1, 2\}$. Recall $X_S^{(2)} = \gamma X_S$.

$$\begin{aligned}
&X_{\{1,2\}}X_{\{1,2\}} = X_1X_2(X_{\{1,2\}} + X_1 + X_2 + 4)\\
&X^{(s)}_{\{1,2\}}X^{(1)}_{\{1,2\}} = X_1X_2(X^{(s+1)}_{\{1,2\}} + X_1^{(s+1)} + X_2^{(s+1)})\\
&X_i^{(1)}X_i^{(1)} = \gamma(X_i^2 + 4X_i)\\
&X_1^{(s)}X_2^{(1)} = 2X^{(s+1)}_{\{1,2\}} - X_2X_1^{(s+1)}\\
&X_1^{(1)}X^{(1)}_{\{1,2\}} = \gamma X_1(2X_2 + X_{\{1,2\}}).
\end{aligned}$$

The first two relations are of type (I); the last three are of type (II).

Remark. The case $m = 2$ is done in [17]. The example above agrees with Proposition 8.2.20 in [17] after certain typographical errors are corrected. (These include replacing all the equal signs with minus signs and correcting the formula for "$w_{2i}w_{2j}$" so that it is consistent with Lemma 8.2.8 in the same document.)

A closely related result gives $KO^*(\widehat{BT}^m)$.

Theorem 3.6. *$KO^{-(4j+2s)}(\widehat{BT}^m)$ is a free module over $\mathbb{Z}[[X_1, \ldots, X_m]]$ on*

$$\left\{\gamma^jM_S^{(s)} : 1 \in S \subseteq N\right\}.$$

The product $M_{S_1}^{(s)}M_{S_2}^{(t)}$ can be computed in terms of this basis from the relations (I) *and* (II) *of Theorem 3.2.*

Relations (I) and (II) of Theorem 3.2 are proved next. This is followed by an identification of the terms given by Theorem 3.6 with those appearing in Theorem 2.7. Finally, Theorem 3.2 is derived from Theorem 3.6.

3.2. The proof of relations (I) and (II)

The complexification map c is injective and so it suffices to prove relations (I) and (II) after c is applied. For convenience, the relations will be verified in the ring $KU^*(BT^m)$ with the classes $\{z_i = \sqrt{1+x_i} : i = 1, \dots, m\}$ adjoined.

$$\begin{aligned}
c(X_S^{(s)}) &= v^s \prod_{i\in S} x_i + \overline{v}^s \prod_{i\in S} \overline{x}_i \\
&= v^s \left(\prod_{i\in S} x_i\right)\left(1 + (-1)^{s+|S|} \prod_{i\in S} \frac{1}{1+x_i}\right) \\
&= v^s \prod_{i\in S}\left(\frac{x_i}{\sqrt{1+x_i}}\right)\cdot\left(\prod_{i\in S}\sqrt{1+x_i} + (-1)^{s+|S|}\prod_{i\in S}\frac{1}{\sqrt{1+x_i}}\right) \\
&= v^s \prod_{i\in S}\left(\frac{z_i^2-1}{z_i}\right)\cdot\left(\prod_{i\in S} z_i + (-1)^{s+|S|}\prod_{i\in S}\frac{1}{z_i}\right) \\
&= v^s \prod_{i\in S}\left(z_i - \frac{1}{z_i}\right)\cdot\left(\prod_{i\in S} z_i + (-1)^{s+|S|}\prod_{i\in S}\frac{1}{z_i}\right).
\end{aligned}$$

More notation is introduced next.

Definition 3.7. Let A, S, and T be disjoint subsets of N. Then

1. Let
$$w_{A,S,T}^{(s)} := \frac{\left(\prod\limits_{j\in A} z_j^2\right)\left(\prod\limits_{j\in S} z_j\right)}{\prod\limits_{j\in T} z_j} + (-1)^{s+|S\cup T|}\frac{\prod\limits_{j\in T} z_j}{\left(\prod\limits_{j\in A} {z_j}^2\right)\left(\prod\limits_{j\in S} z_j\right)},$$
2. for $A = \varnothing$, set $w_{S,T}^{(s)} = w_{\varnothing,S,T}^{(s)}$ and for $T = \varnothing$, $w_S^{(s)} = w_{S,\varnothing}^{(s)}$,
3. set $w_i = w_{\{i\}}^{(0)} = z_i - \dfrac{1}{z_i}$.

Notice that in this new notation, the calculation above is

$$c(X_S^{(s)}) = v^s\left(\prod_{i\in S} w_i\right) w_S^{(s)}.$$

Recall that if A, B and C are disjoint subsets of N and $0 \le s, t \le 1$, relation (I) is

$$X_{A\cup B}^{(s)} X_{A\cup C}^{(t)} = \prod_{i\in A} X_i \cdot \left[\sum_{T\subseteq A} X_{T\cup B\cup C}^{(s+t)} + (-1)^{s+|A\cup B|}\sum_{S\subseteq B}(-1)^{|S|}\left(\prod_{i\in S} X_i\right) X_{C\cup B\setminus S}^{(s+t)}\right]. \tag{3.1}$$

Applying c and dividing both sides by $v^{s+t}\big(\prod_{i\in A} w_i^2\big)\big(\prod_{i\in B\cup C} w_i\big)$ makes (3.1) equivalent to

$$w_{A\cup B}^{(s)} w_{A\cup C}^{(t)} = \sum_{T\subseteq A}\left[\left(\prod_{i\in T} w_i\right) w_{T\cup B\cup C}^{(s+t)}\right] + (-1)^{s+|A\cup B|}\sum_{S\subseteq B}(-1)^{|S|}\left(\prod_{i\in S} w_i\right) w_{C\cup B\setminus S}^{(s+t)}. \tag{3.2}$$

The left hand side of (3.2) is checked easily to satisfy

$$w_{A\cup B}^{(s)} w_{A\cup C}^{(t)} = w_{A,B\cup C,\varnothing}^{(s+t)} + (-1)^{s+|A\cup B|} w_{C,B}^{(s+t)}. \tag{3.3}$$

The first term on the right hand side of (3.2) satisfies

$$\sum_{T\subseteq A}\left[\left(\prod_{i\in T} w_i\right) w_{T\cup B\cup C}^{(s+t)}\right] = \sum_{T\subseteq A}\left(\sum_{R\subseteq T}(-1)^{|T\setminus R|} w_{R,B\cup C,\varnothing}^{(s+t)}\right), \tag{3.4}$$

where $R \subseteq T$ is defined by the fact that $T \setminus R$ is the set of i's in T for which the second term of $w_i = z_i - \frac{1}{z_i}$ is chosen in the product $\prod_{i\in T} w_i$. With T satisfying $R \subseteq T \subseteq A$, the order of summation on the right hand side of (3.4) is rearranged to give

$$\sum_{T\subseteq A}\left(\sum_{R\subseteq T}(-1)^{|T\setminus R|}w^{(s+t)}_{R,B\cup C,\varnothing}\right)=\sum_{R\subseteq A}w^{(s+t)}_{R,B\cup C,\varnothing}\left(\sum_{T\subseteq A}(-1)^{|T\setminus R|}\right). \tag{3.5}$$

Now

$$\sum_{T\subseteq A}(-1)^{|T\setminus R|}=\sum_{j=0}^{|A\setminus R|}(-1)^j\binom{|A\setminus R|}{j}. \tag{3.6}$$

Here, the binomial coefficient on the right hand side counts the number of sets T, $R\subseteq T\subseteq A$ satisfying $|T\setminus R|=j$. Notice that the right hand side is zero unless $A=R$ in which case it equals 1. So now (3.4) implies that the first term on the right of (3.2) is $w^{(s+t)}_{A,B\cup C,\varnothing}$ which is the first term on the right hand side of (3.3).

The second term on the right hand side of (3.2) is analyzed similarly. With S satisfying $U\subseteq S\subseteq B$,

$$(-1)^{s+|A\cup B|}\sum_{S\subseteq B}(-1)^{|S|}\left(\prod_{i\in S}w_i\right)w^{(s+t)}_{C\cup B\setminus S}=(-1)^{s+|A\cup B|}\sum_{S\subseteq B}\left((-1)^{|S|}\sum_{U\subseteq S}(-1)^{|U|}w^{(s+t)}_{C\cup B\setminus U,U}\right) \tag{3.7}$$

where here $S\setminus U$ is the set of i's in S for which the second term of $w_i=z_i-\frac{1}{z_i}$ is chosen in the product $\prod_{i\in S}w_i$. Continuing as above,

$$\begin{aligned}(-1)^{s+|A\cup B|}\sum_{S\subseteq B}\left((-1)^{|S|}\sum_{U\subseteq S}(-1)^{|U|}w^{(s+t)}_{C\cup B\setminus U,U}\right)&=(-1)^{s+|A\cup B|}\sum_{U\subseteq B}\left[w^{(s+t)}_{C\cup B\setminus U,U}\left(\sum_{S}(-1)^{|S\setminus U|}\right)\right]\\&=(-1)^{s+|A\cup B|}\sum_{U\subseteq B}\left[w^{(s+t)}_{C\cup B\setminus U,U}\sum_{j=0}^{|B\setminus U|}(-1)^j\binom{|B\setminus U|}{j}\right]\\&=(-1)^{s+|A\cup B|}w^{(s+t)}_{C,B}\end{aligned} \tag{3.8}$$

which is the second term on the right hand side of (3.3). This completes the proof of the relations (I).

The verification of relations (II) is next. For $i<\min(S)$, $|S|>1$ and $s\in\{0,1\}$, the second set of relations is

$$X_iX^{(s)}_S=(-1)^s\sum_{T\subseteq S}\left[(-1)^{|T|}\left(\prod_{j\in S\setminus T}X_j\right)\cdot X^{(s)}_{\{i\}\cup T}\right]+X^{(s)}_{\{i\}\cup S}\ .$$

Applying c to both sides and dividing by $\prod_{j\in\{i\}\cup S}w_j$ makes relations (II) equivalent to

$$w_iw^{(s)}_S=(-1)^s\sum_{T\subseteq S}\left[(-1)^{|T|}w^{(s)}_{\{i\}\cup T}\prod_{j\in S\setminus T}w_j\right]+w^{(s)}_{\{i\}\cup S}. \tag{3.9}$$

Definition 3.7 implies immediately that

$$w_iw^{(s)}_S=-\,w^{(s)}_{S,\{i\}}+\,w^{(s)}_{\{i\}\cup S} \tag{3.10}$$

and so it remains to show that

$$(-1)^s\sum_{T\subseteq S}\left[(-1)^{|T|}w^{(s)}_{\{i\}\cup T}\prod_{j\in S\setminus T}w_j\right]=-\,w^{(s)}_{S,\{i\}}. \tag{3.11}$$

To this end and using the fact that $i<\min(S)$ implies $i\notin S$,

$$\begin{aligned}(-1)^s\sum_{T\subseteq S}\left[(-1)^{|T|}w^{(s)}_{\{i\}\cup T}\prod_{j\in S\setminus T}w_j\right]&=(-1)^s\sum_{T\subseteq S}\left[(-1)^{|T|}\sum_{B\subseteq S\setminus T}(-1)^{|B|}w^{(s)}_{\{i\}\cup S\setminus B,B}\right]\\&=(-1)^s\sum_{B\subseteq S}\left[(-1)^{|B|}w^{(s)}_{\{i\}\cup S\setminus B,B}\sum_{T\subseteq S\setminus B}(-1)^{|T|}\right]\\&=(-1)^s\sum_{B\subseteq S}\left[(-1)^{|B|}w^{(s)}_{\{i\}\cup S\setminus B,B}\sum_{j=0}^{S\setminus B}(-1)^j\binom{|S\setminus B|}{j}\right]\\&=(-1)^{s+|S|}w^{(s)}_{\{i\},S}=-w^{(s)}_{S,\{i\}}\end{aligned}$$

where, as in (3.6), $\sum_{j=0}^{S\setminus B}(-1)^j\binom{|S\setminus B|}{j}=0$ unless $S=B$ in which case it equals 1.

3.3. The proof of Theorems 3.2 and 3.6

First, the additive generators appearing in Theorem 3.6 are identified with the KU generators given by Theorem 2.7. Choose generators $y_i \in KU^0(\prod_{i=2}^m \mathbb{C}P^\infty)$ so that

$$KU^*\left(\bigwedge_{i=2}^m \mathbb{C}P^\infty\right) \cong \mathbb{Z}[v^{\pm 1}][[y_2,\ldots,y_m]]\cdot(y_2\cdots y_m). \tag{3.12}$$

Theorem 2.7 can be written as

$$KO^*\left(\bigwedge_{i=1}^m \mathbb{C}P^\infty\right) \cong \bigoplus_{k=1}^{\infty} KU^{*+4k}\left(\bigwedge_{i=2}^m \mathbb{C}P^\infty\right) \cong \mathbb{Z}[v^{\pm 1}][[z,y_2,\ldots,y_m]]\cdot(zy_2\cdots y_m) \tag{3.13}$$

where z is given grading equal to 4. The fact that the realification map r increases filtration in the Adams spectral sequence by 1, v has filtration 1 and γ filtration 2, allows the determination of the Adams filtration of the generators described in Theorem 3.6. This then makes possible a comparison of generators via the Adams spectral sequence, modulo terms of higher filtration.

Lemma 3.8. *Let S be a set satisfying $1 \in S \subseteq N$. In the description (3.13), the element*

$$v^{2j+s}z^{e_1}y_2^{2e_2}\cdots y_m^{2e_m}\left(\prod_{i\in N\setminus S} y_i\right)zy_2\cdots y_m \in KU^{-(4j+2s)+4(e_1+1)}\left(\bigwedge_{i=2}^m \mathbb{C}P^\infty\right)$$

corresponds to the element

$$X_1^{e_1}\cdots X_m^{e_m}\gamma^j M_S^{(s)} \in KO^{-(4j+2s)}(\widehat{BT}^m) = KO^{-(4j+2s)}\left(\bigwedge_{i=1}^m \mathbb{C}P^\infty\right),$$

modulo terms of higher filtration in the Adams spectral sequence. $\square$

Lemma 3.8 allows now the use of Theorem 2.7 to conclude that the generators given by Theorem 3.6 must span $KO^{-(4j+2s)}(\widehat{BT}^m)$ and be linearly independent.

The proof of Theorem 3.2 from Theorem 3.6 uses the homotopy equivalence

$$\Sigma(Y_1\times Y_2\times\ldots\times Y_m) \longrightarrow \Sigma\left(\bigvee_{S\subseteq N}\left(\bigwedge_{i\in S} Y_i\right)\right), \tag{3.14}$$

where $Y_i = \mathbb{C}P^\infty$ for all $i \in N$. Theorem 3.6 is applied to each wedge summand $\bigwedge_{i\in S} Y_i$ to conclude that for $S = \{i_1, i_2, \ldots, i_{|S|}\}$, $KO^{-(4j+2s)}(\bigwedge_{i\in S} Y_i)$ is a free module over $\mathbb{Z}[[X_{i_1},\ldots,X_{i_{|S|}}]]$ on $\left\{\gamma^j M_T^{(s)} : 1 \in T \subseteq S\right\}$. Assembling these generators over all $S \subseteq N$, $S \neq \varnothing$ produces the generators in Theorem 3.2. The multiplicative relations (I) and (II) have been checked.

3.4. Addendum: an interpretation of the calculation

The authors are grateful to an anonymous referee for suggesting the formulation of a topological result implicit in the calculations of this section.

Proposition 3.9. *The image of the complexification map*

$$c: KO^*(BT^m) \longrightarrow KU^*(BT^m)$$

is the ring of conjugate invariants in $KU^*(BT^m)$.

Proof. Let ξ denote the conjugation operator on $KU^*(BT^m)$. It follows from the Bott sequence (1.7) that the image of c is the kernel of the composite rv^{-1}. This is same as the kernel of the map crv^{-1} because c is a monomorphism. Then

$$crv^{-1} = (1+\xi)v^{-1} = v^{-1} + \xi v^{-1} = v^{-1} - v^{-1}\xi = v^{-1}(1-\xi).$$

The result follows now because multiplication by v^{-1} is an isomorphism. $\square$

4. A second set of generators for the algebra $KO^*(BT^m)$

4.1. Notation and statement of results

As usual,

$$KU^* \cong \mathbb{Z}[v, v^{-1}] \quad \text{with } v \in KU^{-2}$$
$$KO^* \cong \mathbb{Z}[e, \alpha, \beta, \beta^{-1}]/(2e, e^3, e\alpha, \alpha^2 - 4\beta)$$

where $e \in KO^{-1}$, $\alpha \in KO^{-4}$ and $\beta \in KO^{-8}$. As in Section 3, denote by x_i, $i = 1, \ldots, n$, the generators of $KU^0(BT^n)$. It is convenient to write

$$KU^0(BT^m) \cong \mathbb{Z}[[x_1, \ldots, x_m, \bar{x}_1, \ldots, \bar{x}_m]]/(x_i\bar{x}_i + x_i + \bar{x}_i). \tag{4.1}$$

Let $I = (i_1, i_2, \ldots, i_m)$ and $J = (j_1, j_2, \ldots, j_m)$ with $i_k \geq 0, j_k \geq 0$ for $k = 1, \ldots, m$. For $s \in \mathbb{Z}$ and r the realification map (1.6), set

$$[I, J]^{(s)} := r\left(v^s x_1^{i_1} x_2^{i_2} \ldots x_m^{i_m} (\bar{x}_1)^{j_1} (\bar{x}_2)^{j_2} \ldots (\bar{x}_m)^{j_m}\right)$$

in $KO^{-2s}(BT^m)$. If $s = 0$, the notation $[I, J]$ is used instead of $[I, J]^{(0)}$.

Theorem 4.1. *The classes $[I, J]^{(s)}$ satisfy the relations:*

(A) $[I, J]^{(s)} = (-1)^s[J, I]^{(s)}$

(B) $[I, J]^{(s)} = -[I', J]^{(s)} - [I, J']^{(s)}$

where, for $I = (i_1, \ldots, i_k, \ldots, i_m)$, $J = (j_1, \ldots, j_k, \ldots, j_m)$ with $i_k \cdot j_k \neq 0$,

$$I' = (i_1, \ldots, i_k - 1, \ldots, i_m) \quad \text{and} \quad J' = (j_1, \ldots, j_k - 1, \ldots, j_m).$$

(C) $[I, J]^{(s)} \cdot [H, K]^{(t)} = [I + H, J + K]^{(s+t)} + (-1)^s[J + H, I + K]^{(s+t)}$

where the product here is in $KO^*(BT^m)$.

Remark. Formula (C) is symmetric because relation (A) implies

$$(-1)^s[J + H, I + K]^{(s+t)} = (-1)^t[I + K, J + H]^{(s+t)}.$$

Proof of Theorem 4.1. Relations (A) follow immediately from (1.8) by applying complexification followed by realification. Relations (B) follow by recalling that $x = -\bar{x}(1 + x)$ and decomposing $x^i\bar{x}^{j-1}$ as

$$\begin{aligned} x^i\bar{x}^{j-1} &= x^{i-1}x\bar{x}^{j-1} \\ &= x^{i-1}\bar{x}^{j-1}(-\bar{x}(1 + x)) \\ &= -x^{i-1}\bar{x}^j - x^i\bar{x}^j \end{aligned}$$

which gives $x^i\bar{x}^j = -x^{i-1}\bar{x}^j - x^i\bar{x}^{j-1}$. To see relations (C), the complexification monomorphism c is applied to both sides:

$$\begin{aligned} c\left([I, J]^{(s)} \cdot [H, K]^{(t)}\right) &= c\left([I, J]^{(s)}\right) \cdot c\left([H, K]^{(t)}\right) \\ &= [v^s x_1^{i_1} x_2^{i_2} \ldots x_m^{i_m} (\bar{x}_1)^{j_1} (\bar{x}_2)^{j_2} \ldots (\bar{x}_m)^{j_m} + (-1)^s v^s (\bar{x}_1)^{i_1} (\bar{x}_2)^{i_2} \ldots (\bar{x}_m)^{i_m} x_1^{j_1} x_2^{j_2} \ldots x_m^{j_m}] \\ &\quad \cdot [v^t x_1^{h_1} x_2^{h_2} \ldots x_m^{h_m} (\bar{x}_1)^{k_1} (\bar{x}_2)^{k_2} \ldots (\bar{x}_m)^{k_m} + (-1)^t v^t (\bar{x}_1)^{h_1} (\bar{x}_2)^{h_2} \ldots (\bar{x}_m)^{h_m} x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}] \\ &= v^{s+t} x_1^{i_1+h_1} x_2^{i_2+h_2} \ldots x_m^{i_m+h_m} (\bar{x}_1)^{j_1+k_1} (\bar{x}_2)^{j_2+k_2} \ldots (\bar{x}_m)^{j_m+k_m} \\ &\quad + (-1)^{s+t} v^{s+t} x_1^{j_1+k_1} x_2^{j_2+k_2} \ldots x_m^{j_m+k_m} (\bar{x}_1)^{i_1+h_1} (\bar{x}_2)^{i_2+h_2} \ldots (\bar{x}_m)^{i_m+h_m} \\ &\quad + (-1)^s v^{s+t} x_1^{j_1+h_1} x_2^{j_2+h_2} \ldots x_m^{j_m+h_m} (\bar{x}_1)^{i_1+k_1} (\bar{x}_2)^{i_2+k_2} \ldots (\bar{x}_m)^{i_m+k_m} \\ &\quad + (-1)^t v^{s+t} x_1^{i_1+k_1} x_2^{i_2+k_2} \ldots x_m^{i_m+k_m} (\bar{x}_1)^{j_1+h_1} (\bar{x}_2)^{j_2+h_2} \ldots (\bar{x}_m)^{j_m+h_m} \\ &= c\left([I + H, J + K]^{(s+t)}\right) + c\left((-1)^s[J + H, I + K]^{(s+t)}\right) \\ &= c\left([I + H, J + K]^{(s+t)}\right) + c\left((-1)^t[I + K, J + H]^{(s+t)}\right). \quad \square \end{aligned}$$

Remark 4.2. The elements $X_S^{(s)}$ of Definition 3.1 are related to the classes $[I, J]^{(s)}$ by

$$X_S^{(s)} = [(\epsilon(1), \epsilon(2), \ldots, \epsilon(m)), (0, 0, \ldots, 0)]^{(s)}$$

where ϵ is the characteristic function of S.

Next, a distinguished class of elements $[I, J]^{(s)} \in KO^{-2s}(BT^m)$ is selected.

For $I = (i_1, i_2, \ldots, i_m)$ and $J = (j_1, j_2, \ldots, j_m)$, with all $i_k \geq 0, j_k \geq 0$, set

$$\mathcal{G}_2 := \left\{[I, J]^{(s)} : I \cdot J = 0 \text{ and } i_l \geq j_l \text{ if } i_k + j_k = 0 \text{ for } k < l\right\} \tag{4.2}$$

where here, $I \cdot J$ denotes the dot product of vectors and $I \cdot J = 0$ is interpreted to mean that for all k, $i_k = 0$ or $j_k = 0$. The KO^*-module structure is described easily. Recall that

$$KO^* \cong \mathbb{Z}[e, \alpha, \beta, \beta^{-1}]/(2e, e^3, e\alpha, \alpha^2 - 4\beta).$$

Lemma 4.3. *The KO^*-module action on $KO^*(BT^m)$ is given by*

$$e \cdot ([I], [J])^{(s)} = 0$$
$$\alpha \cdot ([I], [J])^{(s)} = 2([I], [J])^{(s+2)}$$
$$\beta \cdot ([I], [J])^{(s)} = ([I], [J])^{(s+4)}.$$

Proof. The complexification monomorphism c is applied to both sides of these relations. The result follows then from the identities $c(e) = 0,\ c(\beta) = v^4$ and $c(\alpha) = 2v^2$ from [17], Lemma 2.0.3. □

Theorem 4.4. *Every element of $KO^*(BT^m)$ can be expressed as a formal sum of terms from $\mathcal{G}_2$.*

Proof. The classes $v^s x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m} (\bar{x}_1)^{j_1} (\bar{x}_2)^{j_2} \cdots (\bar{x}_m)^{j_m}$ generate $KU^*(BT^m)$ as a power series ring. The realification map r is onto by (1.7). So, the classes $[I, J]^{(s)} \in KO^{-2s}(BT^m)$ generate $KO^*(BT^m)$ as a KO^*-module. Relations (A) and (B) in Theorem 4.1 imply that every element $[I, J]^{(s)} \in KO^{-2s}(BT^m)$ can be written as a linear combination of elements in $\mathcal{G}_2$. A product of two elements in $\mathcal{G}_2$ is not given explicitly in terms of elements of $\mathcal{G}_2$ by relation (C) but repeated applications of relation (A) and (B) reduce the result of (C) to a linear combination of elements of $\mathcal{G}_2$. □

Remark. Lemma 4.3 and the proof of Theorem 4.4 describe the KO^*-algebra structure of $KO^*(BT^m)$. In particular, as noted in Section 1, this result and Theorem 3.2 both describe the completion of the representation ring $RO(T^m)$ at the augmentation ideal.

5. The Davis–Januszkiewicz spaces

5.1. The Davis–Januszkiewicz space associated to a simplicial complex

In Section 1, the Davis–Januszkiewicz space $\mathcal{DJ}(K_P)$, associated to simple polytope P, was defined in terms of a toric manifold M^{2n}. More generally, a Davis–Januszkiewicz space $\mathcal{DJ}(K)$ can be constructed for any simplicial complex K by means of the *generalized moment-angle complex/polyhedral product* construction $Z(K; (X, A))$ of [15,11,16,6,7]. A description of the space $\mathcal{DJ}(K)$ follows.

Definition 5.1. Let K be a simplicial complex with m vertices. Identify simplices $\sigma \in K$ as increasing subsequences of $[m] = (1, 2, 3, \ldots, m)$. The Davis–Januszkiewicz space $\mathcal{DJ}(K)$ is defined by

$$\mathcal{DJ}(K) = Z(K; (\mathbb{C}P^\infty, *)) \ \subseteq\ BT^m = \prod_{i=1}^{m} \mathbb{C}P^\infty$$

where $*$ represents the basepoint and

$$Z(K; (\mathbb{C}P^\infty, *)) = \bigcup_{\sigma \in K} D(\sigma)$$

with

$$D(\sigma) = \prod_{i=1}^{m} W_i\ , \quad \text{where } W_i = \begin{cases} \mathbb{C}P^\infty & \text{if} \quad i \in \sigma \\ * & \text{if} \quad i \in [m] - \sigma. \end{cases} \tag{5.1}$$

A toric manifold M^{2n} is specified by a simple n-dimensional polytope and a *characteristic function* on its facets as described in [15]. Equivalently, M^{2n} can be realized as a quotient. The characteristic function corresponds to a specific choice of subtorus $T^{m-n} \subseteq T^m$ which acts freely on the moment-angle complex $Z(K_P; (D^2, S^1))$ to give

$$M^{2n} \cong Z(K_P; (D^2, S^1))/T^{m-n}.$$

This description of M^{2n} yields an equivalence of Borel constructions

$$ET^m \times_{T^m} Z(K_P; (D^2, S^1)) \simeq ET^n \times_{T^n} \left(Z(K_P; (D^2, S^1))/T^{m-n}\right) \cong ET^n \times_{T^n} M^{2n} = \mathcal{D}\mathcal{J}(K_P). \tag{5.2}$$

Moreover, for any simplicial complex K, there is an equivalence [15,11,16],

$$ET^m \times_{T^m} Z(K; (D^2, S^1)) \cong Z(K; (\mathbb{C}P^\infty, *)). \tag{5.3}$$

It follows that for $K = K_P$, the three descriptions of $\mathcal{D}\mathcal{J}(K_P)$ given by (5.2) and (5.3) agree up to homotopy equivalence.

5.2. The KO^*-rings of the Davis–Januszkiewicz spaces

It is well known that (1.2) extends to $\mathcal{D}\mathcal{J}(K)$ and so, for any complex-oriented cohomology theory E^*

$$E^*(\mathcal{D}\mathcal{J}(K)) \cong E^*(BT^m)/I^E_{SR} \tag{5.4}$$

where I^E_{SR} is the Stanley–Reisner ideal described in Section 1. A related but more general result can be found in [7], Theorem 2.35. Also from [7], the geometric results following will prove useful for the computation of $KO^*(\mathcal{D}\mathcal{J}(K))$ in this section. Below, increasing subsequences of $[m] = (1, 2, 3, \ldots, m)$ are denoted by $\sigma = (i_1, i_2, \ldots, i_k)$, $\tau = (i_1, i_2, \ldots, i_t)$ and $\omega = (i_1, i_2, \ldots, i_s)$ and $X_{i_j} = \mathbb{C}P^\infty$ for all i_j.

Theorem 5.2. *The Davis–Januszkiewicz space $\mathcal{D}\mathcal{J}(K)$ splits stably as follows.*

$$\Sigma\big(\mathcal{D}\mathcal{J}(K)\big) \xrightarrow{\simeq} \Sigma\Bigg(\bigvee_{\sigma\in K} X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_k}\Bigg).$$

Moreover, there is a cofibration sequence

$$\Sigma\big(\mathcal{D}\mathcal{J}(K)\big) \xrightarrow{i} \Sigma\Bigg(\bigvee_{\tau\in[m]} X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_t}\Bigg) \xrightarrow{q} \Sigma\Bigg(\bigvee_{\omega\notin K} X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_s}\Bigg),$$

where the map i is split.

A particular case of (5.4) is given by $E^* = KU^*$, so

$$KU^*(\mathcal{D}\mathcal{J}(K)) \cong KU^*(BT^m)/I^{KU}_{SR}. \tag{5.5}$$

Remark 5.3. Notice that in the representation of $KU^0(BT^m)$ given in (4.1), the monomials generating the ideal I^{KU}_{SR} could equally well contain a generator x_i or its conjugate $\bar{x}_i$.

Theorem 5.2 and the results of Section 2 imply that $KO^*(\mathcal{D}\mathcal{J}(K))$ is concentrated in even degrees. The Bott sequence (1.7) implies then that the realification map

$$r\colon KU^*(\mathcal{D}\mathcal{J}(K)) \longrightarrow KO^*(\mathcal{D}\mathcal{J}(K))$$

is onto and that the complexification map

$$c\colon KO^*(\mathcal{D}\mathcal{J}(K)) \longrightarrow KU^*(\mathcal{D}\mathcal{J}(K))$$

is a monomorphism. The goal of the remainder of this section is to use the generators $\mathcal{G}_2$ of Section 4 to describe the ring $KO^*(\mathcal{D}\mathcal{J}(K))$.

Let K be a simplicial complex on m vertices. For $I = (i_1, i_2, \ldots, i_m)$ as in Section 4, set

$$\epsilon(I) = \{k : i_k \neq 0\} \subseteq [m].$$

Let SR_{KO} denote the ideal in $KO^*(BT^m)$ generated by the set

$$\{[I, J]^{(s)} \in \mathcal{G}_2 : \epsilon(I) \cup \epsilon(J) \notin K\}, \tag{5.6}$$

where again, simplices of K are identified as increasing subsequences of $[m] = (1, 2, 3, \ldots, m)$. The notation SR_{KO} for the KO Stanley–Reisner ideal is more appropriate than I^{KO}_{SR} as it is structurally different from that for a complex-oriented theory. Next, the ideal SR_{KO} is related to $r(I^{KU}_{SR})$. The non-multiplicativity of the map r makes necessary a preliminary lemma.

Lemma 5.4. *The abelian group $r(I^{KU}_{SR})$ is the kernel of the map*

$$i^*\colon KO^*(BT^m) \longrightarrow KO^*(\mathcal{D}\mathcal{J}(K)). \tag{5.7}$$

Proof. With reference to the notation of Theorem 5.2, set

$$\widehat{X}^{\tau} = X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_t} \quad \text{and} \quad \widehat{X}^{\omega} = X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_s}.$$

Recall here that each $X_{i_j} = \mathbb{C}P^{\infty}$. The split cofibration of Theorem 5.2 gives rise to the diagram following

$$\begin{array}{ccccc} KU^{-2s}\big(\mathcal{D}\mathcal{J}(K)\big) & \xleftarrow{i^*} & KU^{-2s}\big(\bigvee_{\tau\in[m]} \widehat{X}^{\tau}\big) & \xleftarrow{q^*} & KU^{-2s}\big(\bigvee_{\omega\notin K} \widehat{X}^{\omega}\big) \\ \downarrow r & & \downarrow r & & \downarrow r \\ KO^{-2s}\big(\mathcal{D}\mathcal{J}(K)\big) & \xleftarrow{i^*} & KO^{-2s}\big(\bigvee_{\tau\in[m]} \widehat{X}^{\tau}\big) & \xleftarrow{q^*} & KO^{-2s}\big(\bigvee_{\omega\notin K} \widehat{X}^{\omega}\big). \end{array} \tag{5.8}$$

The maps i^* are onto and so $KO^*\big(\mathcal{D}\mathcal{J}(K)\big)$ is a quotient of $KO^*(BT^m)$. A diagram chase is needed next. Let $x \in KO^{-2s}\big(\bigvee_{\tau\in[m]} \widehat{X}^{\tau}\big)$ be such that $i^*(x) = 0$. Then, $y \in KO^{-2s}\big(\bigvee_{\omega\notin K} \widehat{X}^{\omega}\big)$ exists satisfying $q^*(y) = x$. Since r is onto, $z \in KU^{-2s}\big(\bigvee_{\omega\notin K} \widehat{X}^{\omega}\big)$ exists with $r(z) = y$. Then

$$r(q^*(z)) = q^*(r(z)) = x.$$

Now $q^*(z) \in I_{SR}^{KU}$ which implies that $x \in r(I_{SR}^{KU})$. Conversely, the commutativity of the left hand half of (5.8) implies that if $x \in r(I_{SR}^{KU})$ then $i^*(x) = 0$, completing the proof. □

Corollary 5.5. *The abelian group $r(I_{SR}^{KU})$ is an ideal in $KO^*(BT^m)$.*

The next proposition allows a characterization of this important ideal in terms of the set of generators (5.6).

Proposition 5.6. *As ideals in $KO^*(BT^m)$*

$$r(I_{SR}^{KU}) = SR_{KO}.$$

Proof. Let $[I, J]^{(s)} \in SR_{KO}$. Since $\epsilon(I) \cup \epsilon(J) \notin K$,

$$[I, J]^{(s)} = r\big(v^s y_{\alpha_1} y_{\alpha_2} \ldots y_{\alpha_k} \mathfrak{m}\big) \tag{5.9}$$

where, in the light of Remark 5.3, each $y_{\alpha_j} = x_{\alpha_j}$ or $\overline{x}_{\alpha_j}$, $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \notin K$ and $\mathfrak{m}$ is a monomial in the classes $x_1, \ldots, x_m, \overline{x}_1, \ldots, \overline{x}_m$. (Notice here that the choice of $\alpha_1, \alpha_2, \ldots, \alpha_k$ in (5.9) may not be unique.) Now $v^s y_{\alpha_1} y_{\alpha_2} \ldots y_{\alpha_k} \mathfrak{m} \in I_{SR}^{KU}$ and so $[I, J]^{(s)} \in r(I_{SR}^{KU})$. Conversely, an element in $r(I_{SR}^{KU})$ is a KO^*-linear combination of elements each of the form $r\big(v^s y_{\alpha_1} y_{\alpha_2} \ldots y_{\alpha_k} \mathfrak{n}\big)$ again with each $y_{\alpha_j} = x_{\alpha_j}$ or $\overline{x}_{\alpha_j}$, $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \notin K$ and $\mathfrak{n}$ is a monomial in the classes $x_1, \ldots, x_m, \overline{x}_1, \ldots, \overline{x}_m$. Now $r\big(v^s y_{\alpha_1} y_{\alpha_2} \ldots y_{\alpha_k} \mathfrak{n}\big) = [I', J']^{(s)}$ for some I' and J' and moreover, $\epsilon(I') \cup \epsilon(J') \notin K$ because $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \notin K$. It follows that $r(I_{SR}^{KU}) \subset SR_{KO}$, proving the converse. □

The main theorem of this section follows.

Theorem 5.7. *There is an isomorphism of graded rings*

$$KO^*(\mathcal{D}\mathcal{J}(K)) \cong KO^*(BT^m)/SR_{KO}.$$

Proof. Proposition 5.6 identifies SR_{KO} as $r(I_{SR}^{KU})$, which is the kernel of the map i^* of (5.7). □

Remark 5.8. In the notation established by Definition 3.1, let $S \subseteq N$, $S \neq \varnothing$ and set

$$\overline{S} = \{\min(S), \min(S) + 1, \ldots, m\}.$$

Then, in terms of the basis $\mathcal{G}_1$, (Corollary 3.4), SR_{KO} is the ideal in $KO^*(BT^m)$ generated (redundantly) by the sets

$$\left\{\gamma^j\Big(\prod_{i\in\overline{S}} X_i\Big) X_S^{(s)} : S \subseteq N,\ S \neq \varnothing,\ j \in \mathbb{Z},\ s \in \{0, 1\} \text{ and } \overline{S} \cup S \notin K\right\},$$

and

$$\big\{\gamma^j X_S^{(s)} : S \subseteq N,\ S \neq \varnothing,\ j \in \mathbb{Z},\ s \in \{0, 1\} \text{ and } S \notin K\big\}.$$

The examples following illustrate calculations in $KO^0(\mathcal{DJ}(K))$ based on Theorem 5.7. The relations of Theorem 4.1 are used with $s = t = 0$ and the elements $[I, J]$ are to be interpreted modulo the ideal of relations SR_{KO}.

Examples 5.9. (1) Let $K = \{\{v_1\}, \{v_2\}\}$ be the simplicial complex consisting of two distinct vertices. Classes of the form $[(i, 0), (0, 0)]$ and $[(0, h), (0, 0)]$ represent $\mathcal{G}_2$ generators of $KO^0(\mathbb{C}P^\infty \times *)$ and $KO^0(* \times \mathbb{C}P^\infty)$ respectively in $KO^0(BT^2)$ as described by (4.2). Now, for i and h not both zero,

$$\begin{aligned}[(i, 0), (0, 0)] \cdot [(0, h), (0, 0)] &= [(i, h), (0, 0)] + [(0, h), (i, 0)] \\ &= 0 \text{ by } (5.6)\end{aligned}$$

which is consistent with the fact that $\mathcal{DJ}(K) = \mathbb{C}P^\infty \vee \mathbb{C}P^\infty$ in this case.
(2) Let $L = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_2, v_4\}, \{v_2, v_3, v_4\}\}$ be the simplicial complex consisting of a 1-simplex wedged to a 2-simplex at the vertex v_2. Here, classes of the form $[(i_1, i_2, 0, 0), (0, 0, 0, 0)]$ and $[(0, h_2, h_3, 0), (0, 0, 0, 0)]$ represent $\mathcal{G}_2$ generators of $KO^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty \times * \times *)$ and $KO^0(* \times \mathbb{C}P^\infty \times \mathbb{C}P^\infty \times *)$ respectively in $KO^0(BT^4)$. Now

$$\begin{aligned}&[(i_1, i_2, 0, 0), (0, 0, 0, 0)] \cdot [(0, h_2, h_{,3}, 0), (0, 0, 0, 0)] \\ &\quad = [(i_1, i_2 + h_2, h_3, 0), (0, 0, 0, 0)] + [(0, h_2, h_3, 0), (i_1, i_2, 0, 0)] = 0 \text{ by } (5.6)\end{aligned}$$

reflecting the fact that $\{v_1, v_2, v_3\} \notin L$. Moreover

$$\begin{aligned}&[(i_1, i_2, 0, 0), (0, 0, 0, 0)] \cdot [(l_1, l_{,2}, 0, 0), (0, 0, 0, 0)] \\ &\quad = [(i_1 + l_1, i_2 + l_2, 0, 0), (0, 0, 0, 0)] + [(l_1, l_2, 0, 0), (i_1, i_2, 0, 0)].\end{aligned} \tag{5.10}$$

Repeated application of relations (A) and (B) in Theorem 4.1 reduce the right hand side of (5.10) to a sum of terms of the form $[(*, *, 0, 0), (*, *, 0, 0)]$ each of which satisfies the $\mathcal{G}_2$ condition for $KO^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. This is consistent with the fact that $KO^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ is a KO^*-subalgebra of $KO^*(\mathcal{DJ}(K))$ corresponding to the simplex $\{v_1, v_2\} \in L$.

5.3. The $\mathrm{CAT}(K)$ approach

Definition 5.1 expresses $\mathcal{DJ}(K)$ as the colimit of an exponential diagram BT^K ([22], where $D(\sigma)$ is written BT^σ), over the category $\mathrm{CAT}(K)$ associated to the posets of faces of K. Since BT^K is a cofibrant diagram, its homotopy colimit is homotopy equivalent to $\mathcal{DJ}(K)$ also. The KO^* version of the Bousfield–Kan spectral sequence [10], studied in [21, Section 3], applies in this case and gives an alternative calculation of $KO^*(\mathcal{DJ}(K))$ in terms of the $\mathrm{CAT}^{op}(K)$-diagram $KO^*(BT^K)$ whose value on each face $\sigma \in K$ is $KO^*(D(\sigma))$. The arguments of [21] apply unchanged and are similar to those of Section 5.2. They imply that the spectral sequence collapses at the E_2-term and is concentrated entirely along the vertical axis. So the edge homomorphism gives an isomorphism

$$KO^*(\mathcal{DJ}(K)) \xrightarrow{\cong} \lim KO^*(BT^K) \tag{5.11}$$

of KO^*-algebras, by analogy with [21, Corollary 3.12].

Informally, the elements of $\lim KO^*(BT^K)$ are considered as finite sequences (u_σ) whose terms $u_\sigma \in KO^*(BT^\sigma)$ are compatible under the inclusions $i\colon BT^\sigma \longrightarrow BT^\tau$ for every $\tau \supset \sigma$. More precisely, the isomorphism (5.11) leads to the conclusion following.

Theorem 5.10. *As KO^*-algebras, $KO^*(\mathcal{DJ}(K))$ is isomorphic to*

$$\left\{(u_\sigma) \in \prod_{\sigma \in K} KO^*(BT^\sigma)\colon i^*(u_\tau) = u_\sigma \text{ for every } \tau \supset \sigma\right\}$$

where the multiplication and KO^*-module structure are defined termwise. □

Theorem 5.10 extends to $E^*(\mathcal{DJ}(K))$ for any arbitrary cohomology theory. The corollary following is complementary to Theorem 5.7.

Corollary 5.11. *The natural homomorphism*

$$\ell\colon KO^*(BT^m) \longrightarrow \lim KO^*(BT^K)$$

is onto with kernel equal to the ideal SR_{KO} of Theorem 5.7.

Proof. The homomorphism ℓ is induced by the projections $KO^*(BT^m) \to KO^*(BT^\sigma)$ as σ ranges over the faces of K, hence it is onto.

Theorem 4.4 describes each summand $KO^*(BT^\sigma)$ of $KO^*(BT^m)$ as generated over KO^* by those elements $[I, J]^{(s)}$ of $\mathcal{G}_2$ for which $\epsilon(I) \cup \epsilon(J) \subseteq \sigma$. Moreover, Theorem 5.10 implies that $\ell([I, J]^{(s)}) = 0$ if and only if $[I, J]^{(s)}$ satisfies $\epsilon(I) \cup \epsilon(J) \notin K$ as in (5.6). So, ℓ maps non-trivial formal sums of elements in $\mathcal{G}_2$ to zero if and only if they lie in SR_{KO}. □

Corollary 5.11 generalizes to an arbitrary cohomology theory and establishes an isomorphism

$$E^*(BT^m)/\ker \ell \longrightarrow E^*(\mathcal{D}\mathcal{J}(K))$$

of E^*-algebras.

It is instructive to revisit Examples 5.9 from this complementary viewpoint.

Examples 5.12. (1) If $K = \{\{v_1\}, \{v_2\}\}$, then $\text{CAT}(K)$ contains the (-1)-simplex $\varnothing$ and two 0-simplices. Theorem 5.10 gives $KO^*(\mathcal{D}\mathcal{J}(K))$ as the KO^*-algebra $KO^*(BT^{\{v_1\}}) \oplus KO^*(BT^{\{v_2\}})$. The homomorphism ℓ of Corollary 5.11 maps the elements $[(i, 0), (0, 0)]$ and $[(0, h), (0, 0)]$ of $KO^0(BT^2)$ to the elements $\big([(i), (0)], 0\big)$ and $\big(0, [(h), (0)]\big)$; in particular, their product is zero.

(2) If $L = \big\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_2, v_4\}, \{v_2, v_3, v_4\}\big\}$, then $\text{CAT}(L)$ contains the (-1)-simplex $\varnothing$, four 0-simplices, four 1-simplices and one 2-simplex. Theorem 5.10 expresses $KO^*(\mathcal{D}\mathcal{J}(K))$ as a certain KO^*-subalgebra of

$$KO^*(BT^{\{v_2\}}) \times KO^*(BT^{\{v_1,v_2\}}) \times KO^*(BT^{\{v_2,v_3,v_4\}}),$$

which may be identified as the pullback

$$KO^*(BT^{\{v_1,v_2\}}) \oplus_{KO^*(BT^{\{v_2\}})} KO^*(BT^{\{v_2,v_3,v_4\}}). \tag{5.12}$$

The elements of (5.12) consist of ordered pairs (u, w), for which $u \in KO^*(BT^{\{v_1,v_2\}})$ and $w \in KO^*(BT^{\{v_2,v_3,v_4\}})$ share a common restriction to $KO^*(BT^{\{v_2\}})$. Pairs are multiplied coordinate-wise; products of the form $(u, 0) \cdot (0, w)$ give $(0, 0) = 0$. For i_1 and h_3 nonzero, the homomorphism ℓ of Corollary 5.11 maps the elements

$$[(i_1, i_2, 0, 0), (0, 0, 0, 0)] \quad \text{and} \quad [(0, h_2, h_3, 0), (0, 0, 0, 0)]$$

of $KO^0(BT^4)$ to the pairs

$$\big([(i_1, i_2), (0, 0)], 0\big) \quad \text{and} \quad \big(0, [(h_2, h_3, 0), (0, 0, 0)]\big)$$

respectively. Their product is zero as required. Similarly, ℓ maps $[(i_1, i_2, 0, 0), (0, 0, 0, 0)]$ and $[(l_1, l_{,2}, 0, 0), (0, 0, 0, 0)]$ to the pairs

$$\big([(i_1, i_2), (0, 0)], 0\big) \quad \text{and} \quad \big([(l_1, l_2), (0, 0)], 0\big)$$

respectively and, their product is $\big([(i_1 + l_1, i_2 + l_2), (0, 0)] + [(l_1, l_2), (i_1, i_2)], 0\big)$.

6. Toric manifolds

6.1. Background

Briefly, a toric manifold M^{2n} is a manifold covered by local charts $\mathbb{C}^n$, each with the standard T^n action, compatible in such a way that the quotient M^{2n}/T^n has the structure of a simple polytope P^n. A *simple* polytope P^n has the property that at each vertex, exactly n facets intersect. Under the T^n action, each copy of $\mathbb{C}^n$ must project to an $\mathbb{R}^n_+$ neighborhood of a vertex of P^n. The construction of Davis and Januszkiewicz ([15], Section 1.5) realizes all such manifolds as follows. Let

$$\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$$

denote the set of facets of P^n. The fact that P^n is simple implies that every codimension-l face F can be written uniquely as

$$F = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_l}$$

where the F_{i_j} are the facets containing F. Let

$$\lambda : \mathcal{F} \longrightarrow \mathbb{Z}^n$$

be a function into an n-dimensional integer lattice satisfying the condition that whenever $F = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_l}$ is a codimension-l face F of P^n, then $\{\lambda(F_{i_1}), \lambda(F_{i_2}), \ldots, \lambda(F_{i_l})\}$ span an l-dimensional submodule of $\mathbb{Z}^n$ which is a direct summand. Next, regarding $\mathbb{R}^n$ as the Lie algebra of T^n, λ associates to each codimension-l face F of P^na rank-l subgroup $G_F \subset T^n$. Finally, let $p \in P^n$ and $F(p)$ be the unique face with p in its relative interior. Define an equivalence relation $\sim$ on $T^n \times P^n$ by $(g, p) \sim (h, q)$ if and only if $p = q$ and $g^{-1}h \in G_{F(p)} \cong T^l$. Then

$$M^{2n} \cong M^{2n}(\lambda) = T^n \times P^n / \sim$$

and, M^{2n} is a smooth, closed, connected, $2n$-dimensional manifold with T^n action induced by left translation ([15], page 423). The projection $\pi : M^{2n} \to P^n$ is induced from the projection $T^n \times P^n \to P^n$. It is noted in [15] that every smooth projective toric variety has this description.

The goal of this section is an analogue of (1.3) for the KO^*-rings of certain toric manifolds M^{2n}. For toric manifolds determined by a simple polytope and a characteristic map on its facets, a description of the KO^*-module structure of $KO^*(M^{2n})$ was given in [5] in terms of $H^*(M^{2n}; \mathbb{Z}_2)$ as a module over $\mathcal{A}_1$, the subalgebra of the Steenrod algebra generated by Sq^1 and Sq^2.

A more refined computation of the KO^*-module structure, for certain families of manifolds M^{2n}, is presented in [20]. The KO^*-ring structure for families of toric manifolds known as Bott towers may be found in [13], without reference to $KO^*(\mathcal{D}\mathcal{J}(K))$.

6.2. *The Steenrod algebra structure of toric manifolds*

Denote by S^0 the $\mathcal{A}_1$-module consisting of a single class in dimension 0 and the trivial action of Sq^1 and Sq^2. Denote by $\mathcal{M}$ the $\mathcal{A}_1$-module with a class x in dimension 0, a class y in dimension 2 and the action given by $\mathrm{Sq}^2(x) = y$.

According to (1.3), $H^*(M^{2n}; \mathbb{Z}_2)$ is concentrated in even degree and so, as an $\mathcal{A}_1$-module, must be isomorphic to a direct sum of suspended copies of the modules S^0 and $\mathcal{M}$. That is, there is a decomposition

$$H^*(M^{2n}; \mathbb{Z}_2) \cong \bigoplus_{i=0}^{n} s_i \Sigma^{2i} S^0 \ \oplus \ \bigoplus_{j=0}^{n-1} m_j \Sigma^{2j} \mathcal{M}, \qquad s_i, m_j \in \mathbb{Z}. \tag{6.1}$$

The numbers s_i and m_j were labeled "BB-numbers" in [13, Section 5]. The Sq^2-homology of M^{2n}, $H^*(M^{2n}; \mathrm{Sq}^2)$, is zero precisely when $s_j = 0$ for all j.

Examples 6.1. The toric manifolds $\mathbb{C}P^{2k}$ are Sq^2-acyclic for any positive integer k.

Examples 6.2. The toric manifolds $\mathbb{C}P^{2k+1}$ have $s_i = 0$ for $i \le k$ and $s_{k+1} = 1$, for any positive integer k. The *terminally odd* Bott towers of [13, Section 5] have $s_1 = 1$ and $s_i = 0$ for $i \ge 2$; the *totally even* towers have $m_j = 0$ for every j.

Examples 6.3. The non-singular toric varieties $X^n(r; a_r, \ldots, a_n)$ constructed in [20] and satisfying $2 \le r \le n$, $a_j \in \mathbb{Z}$ and $n - r$ odd are all Sq^2-acyclic These varieties correspond to n-dimensional fans having $n + 2$ rays.

Remark. The preprint [8] contains a construction of families of toric manifolds derived from a given one. Work is in progress to confirm that this construction can be done in such a way that the family of derived toric manifolds will each be Sq^2-acyclic, though this property might not be satisfied by the initial one.

The next proposition is an immediate consequence of the calculation in [5].

Proposition 6.4. *If M^{2n} is Sq^2-acyclic, then the graded ring $KO^*(M^{2n})$ is concentrated in even degree and has no additive torsion.* $\square$

6.3. *The KO-rings of Sq^2-acyclic toric manifolds*

Recall from (1.1) the Borel fibration for toric manifolds,

$$M^{2n} \xrightarrow{\ i\ } ET^n \times_{T^n} M^{2n} \xrightarrow{\ p\ } BT^n \tag{6.2}$$

with total space $\mathcal{D}\mathcal{J}(K)$.

Theorem 6.5. *For any Sq^2-acyclic toric manifold M^{2n}, there is an isomorphism*

$$KO^*(M^{2n}) \cong KO^*\big(\mathcal{D}\mathcal{J}(K)\big)/r(J^{KU})$$

of KO^*-algebras, where r is the realification map and J^{KU} is the ideal defined in (1.3).

Remark 6.6. Notice that $r(J^{KU})$, which is the realification of the ideal generated by the image of $KU^*(BT^n) \xrightarrow{p^*} KU^*\big(\mathcal{D}\mathcal{J}(K)\big)$, is not the same as J^{KO} which is the ideal generated by $p^*\big(KO^*(BT^n)\big)$; this represents a significant departure from the situation for complex-oriented $E^*(M^{2n})$. As in Lemma 5.4, the non-multiplicativity of the map r implies that $r(J^{KU})$ is not in general an ideal but Theorem 6.5 confirms that $KO^*\big(\mathcal{D}\mathcal{J}(K)\big)/r(J^{KU})$ is multiplicatively closed.

Proof of Theorem 6.5. The Bott sequences (1.7) for M^{2n}, $\mathcal{D}\mathcal{J}(K)$ and BT^n link together to give the commutative diagram following.

$$\begin{tikzcd}
KO^{*-2}(M^{2n}) \arrow[d, "\chi"] & KO^{*-2}\big(\mathcal{D}\mathcal{J}(K)\big) \arrow[l, "i^*"'] \arrow[d, "\chi"'] & KO^{*-2}(BT^n) \arrow[l, "p^*"'] \arrow[d, "\chi"'] \\
KU^*(M^{2n}) \arrow[d, "r \text{ onto}"] & KU^*\big(\mathcal{D}\mathcal{J}(K)\big) \arrow[l, "i^*_{KU} \text{ onto}"'] \arrow[d, "r \text{ onto}"] & KU^*(BT^n) \arrow[l, "p^*"'] \arrow[d, "r \text{ onto}"] \\
KO^*(M^{2n}) & KO^*\big(\mathcal{D}\mathcal{J}(K)\big) \arrow[l, "i^*"'] & KO^*(BT^n) \arrow[l, "p^*"']
\end{tikzcd} \tag{6.3}$$

Recall now that Proposition 6.4 implies that $KO^*(M^{2n})$ is concentrated in even degrees and so all the Bott sequences are short exact. The lower left commutative square in (6.3) implies that the maps i^* are onto. A diagram chase is needed next to identify the kernel of i^*.

Let $z \in KO^*\big(\mathcal{D}\mathcal{J}(K)\big)$ and suppose $i^*(z) = 0$. Since r is onto, $y \in KU^*\big(\mathcal{D}\mathcal{J}(K)\big)$ exists with $r(y) = z$. Then $r(i^*_{KU}(y)) = i^*(z) = 0$. The exactness of the leftmost Bott sequence implies now that $x \in KO^{*-2}(M^{2n})$ exists with $\chi(x) = i^*_{KU}(y)$. The map i^* is onto so $w \in KO^{*-2}\big(\mathcal{D}\mathcal{J}(K)\big)$ exists satisfying $i^*(w) = x$. Then

$$i^*_{KU}(y - \chi(w)) = i^*_{KU}(y) - i^*(\chi(w)) = i^*_{KU}(y) - \chi(i^*(w)) = i^*_{KU}(y) - \chi(x) = 0.$$

So $y - \chi(w) \in \big\langle p^*\big(KU^*(BT^n)\big)\big\rangle$ by (1.3) for $E = KU$. Finally, $r(y - \chi(w)) = r(y) = z$ and so $z \in r\big(\big\langle p^*(KU^*(BT^n))\big\rangle\big)$ as required. □

6.4. *Further examples*

A few simple examples illustrate the fact that the situation is considerably more difficult when M^{2n} is not Sq^2-acyclic. In all that follows, the number s_i and m_j are those defined by (6.1).

Manifolds M^{2n} for which all $m_j = 0$, as is the case for the totally even Bott towers of Examples 6.2, have $KO^*(M^{2n})$ a free KO^*-module. Particularly revealing is the most basic case $M^{2n} = \prod_{k=1}^{n} \mathbb{C}P^1$ with $n = 1$. Recall from Section 4 that

$$KO^* \cong \mathbb{Z}[e, \alpha, \beta, \beta^{-1}]/(2e, e^3, e\alpha, \alpha^2 - 4\beta)$$

with $e \in KO^{-1}$, $\alpha \in KO^{-4}$ and $\beta \in KO^{-8}$.

Example 6.7. The classes $X_1^{(s)} \in KO^{-2s}(\mathbb{C}P^\infty)$ and $X_i^{(0)} = X_i$, specified in Definition 3.1, restrict to classes in $KO^{-2s}(\mathbb{C}P^1)$ which also will be denoted by $X_1^{(s)}$ and X_1. The KO^*-algebra $KO^*(\mathbb{C}P^1)$ is isomorphic to $KO^*[g]/(g^2)$ where $g \in KO^2(\mathbb{C}P^1)$ is the generator arising from the unit of the spectrum KO. In particular

$$e^2 g = X_1 \in KO^{(0)}(\mathbb{C}P^1) \quad \text{and} \quad 2\beta g = X_1^{(3)} \in KO^{-6}(\mathbb{C}P^1).$$

Now $\mathbb{C}P^1$ is the smooth toric variety associated to the simplicial complex $K = \big\{\{v_1\}, \{v_2\}\big\}$ in the manner described in Section 1. Let η denote the universal vector bundle over BT^1. The fibration (6.2) specializes to

$$\mathbb{C}P^1 \xrightarrow{i} S(\eta \oplus \mathbb{R}) \xrightarrow{p} BT^1,$$

the total space of which is the sphere bundle of $\eta \oplus \mathbb{R}$. So $\mathcal{D}\mathcal{J}(K)$ is homotopy equivalent to $\mathbb{C}P^\infty \vee \mathbb{C}P^\infty$. (Of course, this agrees with the description given by (5.3).) The map i includes $\mathbb{C}P^1$ into each wedge summand by pinching the equator.

It follows from [13, Section 4] that

(1) i^* is an epimorphism onto $KO^d(\mathbb{C}P^1)$ for all $d \not\equiv 1, 2 \bmod 8$.
(2) If $d = 1 - 8t$ then $e\beta^t g$ has order 2 but $KO^d(\mathbb{C}P^\infty \vee \mathbb{C}P^\infty) = 0$.
(3) If $d = 2 - 8t$ then $2\beta^t g \in \mathrm{Im}(i^*)$ but $\beta^t g \notin \mathrm{Im}(i^*)$.

These details combined with diagram (6.3) confirm that

$$\mathrm{Im}(i^*) \cong KO^*\big(\mathcal{D}\mathcal{J}(K)\big)/r(J^{KU})$$

in dimensions $\equiv 1, 2 \bmod 8$.

Example 6.7 extends to an analysis of various toric manifolds with a single non-zero s_i but unrestricted m_j.

Example 6.8. The projective space $\mathbb{C}P^{4k+1}$ has $s_{2k+1} = 1$ and all other $s_i = 0$. It is the smooth toric variety associated to the simplicial complex K which is the boundary of the simplex Δ^{4k+1}.

The KO^*-algebra $KO^*(\mathbb{C}P^{4k+1})$ admits $KO^*(S^{8k+2})$ as an additive summand, generated by $h \in KO^{8k+2}(\mathbb{C}P^{4k+1})$ such that $h^2 = 0$. In particular,

$$e^2\beta^k h = X_1^{2k+1} \in KO^0(\mathbb{C}P^{4k+1}) \quad \text{and} \quad 2\beta^{k+1}h = X_1^{2k}X_1^{(3)} \in KO^{-6}(\mathbb{C}P^{4k+1}).$$

It follows from Example 6.7 that

$$i^* : KO^d\big(\mathcal{D}\mathcal{J}(K)\big) \longrightarrow KO^d(\mathbb{C}P^{4k+1})$$

is an epimorphism for $d \not\equiv 1, 2 \bmod 8$. So the cokernel of i^* is isomorphic to the $\mathbb{Z}/2$ vector space generated by the elements $e\beta^t h$ and $\beta^t h$, whereas

$$\mathrm{Im}(i^*) \cong KO^*\big(\mathcal{D}\mathcal{J}(K)\big)/r(J^{KU})$$

in dimensions $\not\equiv 1, 2 \bmod 8$.

Example 6.9. A terminally odd Bott tower M^{2n} has $s_1 = 1$ and all other $s_i = 0$. In this case the simplicial complex K is the boundary of an n-dimensional cross-polytope. As in Example 6.7, it follows that the cokernel of i^* is isomorphic to the $\mathbb{Z}_2$-vector space generated by the elements $e\beta^t g$ and $\beta^t g$ whereas

$$\mathrm{Im}(i^*) \cong KO^*\big(\mathcal{D}\mathcal{J}(K)\big)/r(J^{KU})$$

in dimensions $\not\equiv 1, 2 \bmod 8$.

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