



Bounds for higher topological complexity of real projective space implied by BP

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Abstract

We use Brown–Peterson cohomology to obtain lower bounds for the higher topological complexity, $TC_k(\mathbb{R}P^{2m})$, of real projective spaces, which are often much stronger than those implied by ordinary mod-2 cohomology.

Keywords Brown–Peterson cohomology · Topological complexity · Real projective space

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1 Introduction and main results

In [8], Farber introduced the notion of topological complexity, $TC(X)$, of a topological space X . This can be interpreted as one less than the minimal number of rules, called *motion planning rules*, required to tell how to move between any two points of X .¹ This became central in the field of topological robotics when X is the space of configurations of a robot or system of robots. This was generalized to higher topological complexity, $TC_k(X)$, by Rudyak [10]. This can be thought of as one less than the number of rules required to tell how to move consecutively between any k specified points of X [10, Remark 3.2.7]. In [2], the study of $TC_k(P^n)$ was initiated, and this was continued in [6], where the best lower bounds implied by mod-2 cohomology were obtained. Here, P^n denotes real projective space.

Since $TC_2(P^n)$ is usually equal to the immersion dimension [9] and a sweeping family of strong nonimmersion results was obtained using Brown–Peterson cohomology, $BP^*(-)$, in [3], one is led to apply BP to obtain lower bounds for $TC_k(P^n)$ for $k > 2$. In this paper, we obtain a general result, Theorem 1.1, which implies lower bounds in

¹ Farber’s original definition did not include the “one less than” part, but most recent papers have defined it as we have done here.

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many cases, and then, focus in Theorem 1.2 on a particular family of cases, in which we show our bound is often much stronger than the results implied by mod-2 cohomology.

The general result is obtained from the known information about the BP-cohomology algebra of products of real projective spaces. It gives conditions under which nonzero classes of a certain form can be found. Here and throughout, $\nu(-)$ denotes the exponent of 2 in an integer.

Theorem 1.1 *Let $k \geq 3$ and $r \geq 0$. Suppose that there are positive integers a_1, \dots, a_{k-1} whose sum is $km - (2^k - 1)2^r$, such that*

$$\nu \left(\prod_{i=1}^{k-1} \binom{a_i}{j_i} \right) \geq 2^r \quad (1.1)$$

for all j_1, \dots, j_{k-1} with $j_i \leq m$ and $\sum_{i=1}^{k-1} j_i \geq (k-1)m - (2^k - 1)2^r$. Suppose also that

$$\nu \left(\sum_{\ell} \prod_{i=1}^{k-1} \binom{a_i}{m-\ell_i} \right) = 2^r, \quad (1.2)$$

where $\ell = (\ell_1, \dots, \ell_{k-1})$ ranges over all $(k-1)$ -tuples of the k distinct numbers 2^{r+t} , $0 \leq t \leq k-1$. Then

$$\mathrm{TC}_k(P^{2m}) \geq 2km - (2^k - 1)2^{r+1}.$$

Theorem 1.1 applies in many cases, but we shall focus on one family. Here and throughout, $\alpha(-)$ denotes the number of 1s in the binary expansion of an integer.

Theorem 1.2 *Suppose that $k \geq 3$, $r \geq k-3$, and $m = A \cdot 2^r$ with $A \geq 2^{k-1}$. Then*

$$\mathrm{TC}_k(P^{2m}) \geq 2km - (2^k - 1)2^{r+1}$$

if

(a) $k = 3$ and either

- (i) $A \equiv 5 \pmod{8}$ and $\alpha(A) = 2^r + 2$, or
- (ii) $A \equiv 2 \pmod{4}$ and $\alpha(A) = 2^r + 2$; or

(b) $k \geq 4$ and either

- (i) $A \equiv 6 \pmod{8}$ and $\alpha(A) = 2^r + 2$, or
- (ii) $A \equiv 3 \pmod{8}$ and $\alpha(A) = 2^r + 3$.

We prove Theorems 1.1 and 1.2 in Sect. 2. In Sect. 3, we describe more specifically some families of particular values of (m, k, r) to which this result applies, and the extent to which these results are much stronger than those implied by mod-2 cohomology. In Sect. 4, we prove that the cohomology-implied bounds for $\mathrm{TC}_k(P^n)$ are constant for long intervals of values of n . In these intervals, the BP-implied bounds become much stronger than those implied by cohomology.

2 Proofs of main theorems

In this section, we prove Theorems 1.1 and 1.2. The first step, Theorem 2.1, follows suggestions of Jesús González, and is similar to work in [2]. We are very grateful to González for these suggestions. There are canonical elements X_1, \dots, X_k in $\text{BP}^2((P^n)^k)$, where $(P^n)^k$ is the Cartesian product of k copies of P^n .

Theorem 2.1 *If $(X_1 - X_k)^{a_1} \cdots (X_{k-1} - X_k)^{a_{k-1}} \neq 0 \in \text{BP}^*((P^n)^k)$, then*

$$\text{TC}_k(P^n) \geq 2a_1 + \cdots + 2a_{k-1}.$$

Proof Let $(P^n)^{[0,1]}$ denote the space of paths in P^n , and

$$P_{n,k} = (S^n)^k / ((z_1, \dots, z_k) \sim (-z_1, \dots, -z_k))$$

a projective product space [5]. The quotient map $P_{n,k} \xrightarrow{\pi} (P^n)^k$ is a $(\mathbb{Z}_2)^{k-1}$ -cover, classified by a map $(P^n)^k \xrightarrow{\mu} B((\mathbb{Z}_2)^{k-1}) = (P^\infty)^{k-1}$. The map $(P^n)^{[0,1]} \xrightarrow{p} (P^n)^k$ defined by

$$\sigma \mapsto \left(\sigma(0), \sigma\left(\frac{1}{k-1}\right), \dots, \sigma\left(\frac{k-2}{k-1}\right), \sigma(1) \right)$$

lifts to a map $(P^n)^{[0,1]} \xrightarrow{\tilde{p}} P_{n,k}$ [2, (3.2)]. A definition of $\text{TC}_k(P^n)$ is as the sectional category $\text{secat}(p)$. The lifting \tilde{p} implies that $\text{secat}(p) \geq \text{secat}(\pi)$.

Let $G = (\mathbb{Z}_2)^{k-1}$ and $B_t G = (*^{t+1}G)/G$, where $*^{t+1}G$ denotes the iterated join of $t+1$ copies of G . Note that $B_t G$ is the t th stage in Milnor's construction of BG , with a map $i_t : B_t G \rightarrow BG$. By [11, Thm 9, p. 86], as described in [2, (4.1)], μ lifts to a map $(P^n)^k \xrightarrow{\tilde{\mu}} B_{\text{secat}(\pi)} G$:

$$\begin{array}{ccc} (P^n)^{[0,1]} & \xrightarrow{\tilde{p}} & P_{n,k} & & B_{\text{secat}(\pi)} G \\ & \searrow p & \downarrow \pi & \nearrow \tilde{\mu} & \downarrow i_{\text{secat}(\pi)} \\ & & (P^n)^k & \xrightarrow{\mu} & BG = (P^\infty)^{k-1} \end{array}$$

By [2, Prop 3.1], μ classifies $(p_1^*(\xi) \otimes p_k^*(\xi)) \oplus \cdots \oplus (p_{k-1}^*(\xi) \otimes p_k^*(\xi))$, and so, by [1, Prop 3.6], the induced homomorphism

$$\text{BP}^*((P^\infty)^{k-1}) \xrightarrow{\mu^*} \text{BP}^*((P^n)^k)$$

satisfies $\mu^*(X_i) = u_i(X_i - X_k)$ for $1 \leq i \leq k-1$, with u_i a unit. Since $\mu^* = \tilde{\mu}^* i_{\text{secat}(\pi)}^*$ and $B_t G$ is t -dimensional, $\mu^*(X_1^{a_1} \cdots X_{k-1}^{a_{k-1}}) = 0$ if $2a_1 + \cdots + 2a_{k-1} > \text{secat}(\pi)$. The theorem now follows, since $\prod (X_i - X_k)^{a_i} \neq 0$ implies that $\mu^*(\prod X_i^{a_i}) \neq 0$, which implies

$$\sum 2a_i \leq \text{secat}(\pi) \leq \text{secat}(p) = \text{TC}_k(P^n).$$

□

We use this to prove Theorem 1.1.

Proof of Theorem 1.1 Let I denote the ideal $(v_0, \dots, v_k) \subset \mathbf{BP}^*$. Recall $v_0 = 2$ and $|v_i| = 2(2^i - 1)$. In $\mathbf{BP}^*(X)$, let F_s denote the \mathbf{BP}^* -submodule $I^s \cdot \mathbf{BP}^*(X)$. It follows from [12, 2.2], [4, Cor 2.4], and [7, Thm 1.5] that, in $\mathbf{BP}^*((P^{2m})^k)$, for $r \geq 0$ and integers j_1, \dots, j_k

$$2^{2r} X_1^{j_1} \cdots X_k^{j_k} \equiv v_k^{2r} \sum X_1^{j_1+\ell_1} \cdots X_k^{j_k+\ell_k} \pmod{F_{2r+1}}, \quad (2.1)$$

where the sum is taken over all permutations (ℓ_1, \dots, ℓ_k) of $\{2^r, \dots, 2^{r+k-1}\}$. (An analogous result was derived in BP-homology in [7], following similar, but not quite so complete, results in [4, 12], which also discussed the dualization to obtain BP-cohomology results.)

The result follows from Theorem 2.1 once we show that

$$(X_1 - X_k)^{a_1} \cdots (X_{k-1} - X_k)^{a_{k-1}} \neq 0 \in \mathbf{BP}^{2km - (2^k - 1)2^{r+1}}((P^{2m})^k).$$

This expands as $\sum_{j_1, \dots, j_{k-1}} \pm \binom{a_1}{j_1} \cdots \binom{a_{k-1}}{j_{k-1}} X_1^{j_1} \cdots X_{k-1}^{j_{k-1}} X_k^{km - (2^k - 1)2^r - j_1 - \cdots - j_{k-1}}$, for values of j_1, \dots, j_{k-1} described in Theorem 1.1. By (2.1) and (1.1), this equals, mod F_{2r+1} :

$$v_k^{2r} \sum_{j_1, \dots, j_{k-1}} \sum_{\ell} \pm 2^{-2r} \binom{a_1}{j_1} \cdots \binom{a_{k-1}}{j_{k-1}} X_1^{j_1+\ell_1} \cdots X_{k-1}^{j_{k-1}+\ell_{k-1}} X_k^{km - j_1 - \ell_1 - \cdots - j_{k-1} - \ell_{k-1}} \quad (2.2)$$

with $\ell = (\ell_1, \dots, \ell_{k-1})$ as in (1.2). Note here that $\ell_k = 2^{r+k} - 2^r - \ell_1 - \cdots - \ell_{k-1}$. The terms in (2.2) are 0 unless the exponent of each X_i equals m , since, otherwise, there would be a factor X^p with $p > m$. We are left with

$$\left(\sum_{\ell} \pm 2^{-2r} \binom{a_1}{m-\ell_1} \cdots \binom{a_{k-1}}{m-\ell_{k-1}} \right) v_k^{2r} X_1^m \cdots X_k^m$$

with $(\ell_1, \dots, \ell_{k-1})$ as above, and this is nonzero by the hypothesis (1.2) and the fact, as was noted in [12], that by the (proven) Conner–Floyd conjecture, $v_k^h X_1^m \cdots X_k^m \neq 0$ for any nonnegative integer h . \square

In the following proof of Theorem 1.2, we will often use without comment Lucas’s Theorem regarding binomial coefficients mod 2, and that

$$v \binom{m}{n} = \alpha(n) + \alpha(m-n) - \alpha(m), \quad \text{and} \quad \alpha(x-1) = \alpha(x) - 1 + v(x). \quad (2.3)$$

Proof of Theorem 1.2 We explain the proof when $k \geq 4$ and $A \equiv 6 \pmod{8}$, and then describe the minor changes required when $A \equiv 3 \pmod{8}$ or $k = 3$. We apply Theorem 1.1 with

$$a_i = m - (2^k - 1)2^{r-i}, \quad 1 \leq i \leq k-3, \quad a_{k-2} = m,$$

$$\text{and } a_{k-1} = 2m - (2^k - 1)2^{r-(k-3)}.$$

For (1.1), we show

$$v\binom{a_{k-1}}{j} \geq 2^r \text{ if } (k-1)m - (2^k - 1)2^r - (a_1 + \cdots + a_{k-2}) \leq j \leq m.$$

Thus, we are considering $v\binom{2m-(2^k-1)2^{r-(k-3)}}{j}$ with $m - (2^k - 1)2^{r-(k-3)} \leq j \leq m$. By symmetry and (2.4), we may restrict to $m - (2^{k-1} - 1)2^{r-(k-3)} \leq j \leq m$. Let $m = (8B + 6)2^r$ with $\alpha(B) = 2^r$. We first restrict to j 's divisible by $2^{r-(k-3)}$; let $j = 2^{r-(k-3)}h$. Now, we are considering $v\binom{(8B+6)2^{k-2}-2^k+1}{h}$ with $2^{k-3}(8B + 6) - (2^{k-1} - 1) \leq h \leq 2^{k-3}(8B + 6)$. Lemma 2.2 with $t = k - 2$ shows that $v\binom{(8B+6)2^{k-2}-2^k+1}{h} \geq \alpha(B)$ for the required values of h . The proof for arbitrary j (in the required range) follows from the easily proved fact that

$$\text{for } 0 < \delta < 2^k, \quad v\binom{N \cdot 2^k}{M \cdot 2^k \pm \delta} > v\binom{N \cdot 2^k}{M \cdot 2^k}. \quad (2.4)$$

Now, we prove (1.2). We divide the top and bottom of the binomial coefficients by $2^{r-(k-3)}$; this does not change the exponent. The tops are now

$$2^{k-3}A - (2^k - 1)2^{k-4}, \dots, 2^{k-3}A - (2^k - 1)2^0, 2^{k-3}A, 2^{k-2}A - (2^k - 1),$$

and the bottoms are selected from $2^{k-3}A - 2^{k-3}, \dots, 2^{k-3}A - 2^{2k-4}$. All the bottoms except the last one are greater than the first top one. Thus, to get a nonzero product in (1.2), the last bottom must accompany the first top, and after dividing top and bottom by 2^{k-4} , it becomes $\binom{2A-(2^k-1)}{2A-2^k} \equiv 1 \pmod{2}$. Similar considerations work inductively for all but the final two factors, showing that the i th bottom from the end must appear beneath the i th top and gives an odd factor. What remains is

$$\sum \binom{2^{k-3}A}{j} \binom{2^{k-2}A - 2^k + 1}{j'},$$

where (j, j') are the ordered pairs of distinct elements of

$$\left\{ 2^{k-3}A - 2^{k-3}, 2^{k-3}A - 2^{k-2}, 2^{k-3}A - 2^{k-1} \right\}.$$

The $+1$ on top does not affect the exponent of the binomial coefficients, and so we may remove it and then divide tops and bottoms by 2^{k-3} , obtaining $\sum \binom{A}{j} \binom{2A-8}{j'}$, where (j, j') are ordered pairs of $A - 1, A - 2$, and $A - 4$.

If $A \equiv 6 \pmod{8}$, $v\binom{A}{j} = 0$ if $j = A - 2$ or $A - 4$, and is > 0 if $j = A - 1$. In addition, with $A = 8B + 6$, $v\binom{2A-8}{j'} = \alpha(B)$ if $j' = A - 2$, and is $> \alpha(B)$ if $j' = A - 1$ or $A - 4$. Thus, the sum in (1.2) has $v(-) = 2^r$, coming from the single summand corresponding to $(j, j') = (A - 4, A - 2)$.

When $A \equiv 3 \pmod{8}$, the following minor changes must be made in the above argument. Let $A = 8B + 3$. A minimal value of $v\binom{A}{j}^{(a_{k-1})}$ occurs when $j = 2^{r-(k-3)}h$ with $h = 2^{k-3}(8B + 3) - 2^{k-3}$. We obtain $v\binom{16B-2}{8B+2} = \alpha(B) - 1 = 2^r$, since $\alpha(A) = 2^r + 3$. For (1.2), the minimal value $v\binom{A}{j}\binom{2A-8}{j'} = 2^r$ occurs only for $(j, j') = (A - 2, A - 1)$.

Part (a) of Theorem 1.2 follows similarly. We have $a_1 = m$ and $a_2 = 2m - 7 \cdot 2^r$. Then, by the same methods as used above, we show that with m as in the theorem, and P denoting a positive number and I a number which is irrelevant:

- If $m - 7 \cdot 2^r \leq j \leq m$, then $v\binom{2m-7 \cdot 2^r}{j} \geq 2^r$.
- The values $(v\binom{m}{m-2^r}, v\binom{m}{m-2^{r+1}}, v\binom{m}{m-2^{r+2}})$ are $(0, P, 0)$ (resp. $(P, 0, I)$) in case (i) (resp. (ii)) of the theorem.
- The values $(v\binom{2m-7 \cdot 2^r}{m-2^r} - 2^r, v\binom{2m-7 \cdot 2^r}{m-2^{r+1}} - 2^r, v\binom{2m-7 \cdot 2^r}{m-2^{r+2}} - 2^r)$ are $(P, 0, 0)$ (resp. $(0, 0, P)$) in case (i) [resp. (ii)] of the theorem. \square

The following lemma was used above.

Lemma 2.2 *If $t \geq 2$ and $-2^t + 1 \leq d \leq 2^t$, then $v\binom{(8B+2)2^t+1}{(4B+2)2^t+d} \geq \alpha(B)$.*

Proof Using (2.3), we can show

$$v\binom{(8B+2)2^t+1}{(4B+2)2^t+d} = \begin{cases} \alpha(B) + t + 1 - v(d(d-1)) & -2^t + 1 \leq d < 0 \\ \alpha(B) & d = 0, 1 \\ \alpha(B) + t + v(B) + 2 - v(d(d-1)) & 2 \leq d \leq 2^t, \end{cases}$$

from which the lemma is immediate. \square

3 Numerical results

In this section, we compare the lower bounds for $\text{TC}_k(P^{2m})$ implied by BP with those implied by mod-2 cohomology. In [6], the best lower bounds obtainable using mod-2 cohomology were obtained. They are restated here in (4.1). In Table 1, we compare these with the results implied by our Theorems 1.1 and 1.2 for $\text{TC}_3(P^{2m})$ with $32 \leq m < 64$. Results in the BP column are those implied by 1.1, and those indicated with an asterisk are implied by 1.2. It is quite possible that there are additional results implied by Theorem 2.1, since Theorem 1.1 takes into account only one type of implication about nonzero classes in $\text{BP}^*((P^n)^k)$. Note that the BP-bounds are significantly stronger in the second half of the table.

In Table 2, we present another comparison of the results implied by Theorem 1.2 and those implied by ordinary mod-2 cohomology. We consider lower bounds for $\text{TC}_4(P^{2m})$ for $2^{11} \leq m < 2^{12}$. In Table 2, the first column refers to a range of values of m , the second column refers to the number of distinct new results implied by Theorem 1.2 in that range, and the third column refers to the range of the ratio of bounds implied by Theorem 1.2 to those implied by ordinary cohomology. There are

Table 1 Lower bounds for $TC_3(P^{2m})$ implied by $H^*(-)$ and by BP

| m | $H^*(-)$ | BP |
|-----|----------|------|
| 32 | 192 | 152 |
| 33 | 198 | 152 |
| 34 | 204 | 190 |
| 35 | 206 | 190 |
| 36 | 216 | 190 |
| 37 | 222 | 208* |
| 38 | 222 | 214* |
| 39 | 222 | 214* |
| 40 | 240 | 214* |
| 41 | 246 | 232 |
| 42 | 252 | 238* |
| 43 | 254 | 238* |
| 44 | 254 | 238* |
| 45 | 254 | 238* |
| 46 | 254 | 248 |
| 47 | 254 | 248 |
| 48 | 254 | 248 |
| 49 | 254 | 280 |
| 50 | 254 | 286* |
| 51 | 254 | 286* |
| 52 | 254 | 286* |
| 53 | 254 | 304 |
| 54 | 254 | 310 |
| 55 | 254 | 310 |
| 56 | 254 | 310 |
| 57 | 254 | 310 |
| 58 | 254 | 320* |
| 59 | 254 | 320* |
| 60 | 254 | 332* |
| 61 | 254 | 332* |
| 62 | 254 | 332* |
| 63 | 254 | 332* |

Table 2 Ratio of lower bounds for $TC_4(P^{2m})$ implied by Theorem 1.2 to those implied by $H^*(-)$

| m | # | Ratio |
|--------------|----|------------------|
| [2048, 2815] | 29 | [0.9620, 1.0384] |
| [2816, 3071] | 7 | [0.9877, 1.0673] |
| [3072, 3979] | 26 | [0.9783, 1.2700] |
| [3980, 4095] | 1 | 1.2908 |

many other stronger bounds implied by BP via Theorem 1.1, but our focus here is on the one family which we have analyzed for all k and r .

In the range $2816 \leq m \leq 3071$ here, the bound for $\text{TC}_4(P^{2^m})$ implied by mod-2 cohomology is constant at 22,525, while that implied by Theorem 1.2 increases from 22,248 to 24,040. In the longer range $3072 \leq m \leq 4095$ here, the bound for $\text{TC}_4(P^{2^m})$ implied by mod-2 cohomology is constant at 24,573, while that implied by Theorem 1.2 increases from 24,040 to 31,720. Next, we examine what happens in the generalization of this latter range to $\text{TC}_k(P^{2^m})$ for arbitrary k and arbitrary 2-power near the end of the range. In Theorem 4.1, we will show that the bound for $\text{TC}_k(P^{2^m})$ implied by cohomology has the constant value $(k - 1)(2^e - 1)$ for $\lfloor \frac{k-1}{k} \cdot 2^e \rfloor \leq 2m \leq 2^e - 1$.

In this range, the bound implied by Theorem 1.2 will increase from a value approximately equal to the cohomology-implied bound to a value which, as we shall explain, is asymptotically as much greater than the cohomology-implied bound as it could possibly be. The following result gives a result at the end of each 2-power interval, since each e can be written uniquely as $2^r + r + 3 + d$ for $0 \leq d \leq 2^r$. For example, the case $r = 1, d = 0, k = 3$ in this proposition is the 332* next to $m = 60$ in Table 1, and the case $r = 2, d = 3, k = 4$ gives $m = 3980$, the start of the last row of Table 2.

Proposition 3.1 For $r \geq 1$ and $0 \leq d \leq 2^r$, let

$$m = \begin{cases} 2^{r+1}(2^{2^r+2} - 1) & d = 0, k \geq 3 \\ 2^{r+d+2}(2^{2^r+1} - 1) + 2^{r+1} & d > 0, k = 3 \\ 2^{r+d+2}(2^{2^r+1} - 1) + 3 \cdot 2^r & d > 0, k > 3. \end{cases}$$

Then, $\text{TC}_k(P^{2^m}) \geq 2km - (2^k - 1)2^{r+1}$.

Proof It is straightforward to check that the conditions of Theorem 1.2 are satisfied for these values of m and r . □

For m as in Proposition 3.1, the lower bound for $\text{TC}_k(P^{2^m})$ implied by cohomology is $(k - 1)(2^{2^r+r+4+d} - 1)$. One can check that the ratio of the bound in Proposition 3.1 to the cohomology bound is greater than

$$\frac{k}{k - 1} - \frac{1}{2^{2^r+1}}.$$

Since, as was noted in [2], $(k - 1)n \leq \text{TC}_k(P^n) \leq kn$, the largest the ratio of any two estimates of $\text{TC}_k(P^n)$ could possibly be is $k/(k - 1)$. Thus, the BP-bound improves on the cohomology bound asymptotically by as much as it possibly could, as e (hence r) becomes large.

The authors of [2] have particular interest in estimates for $\text{TC}_k(P^{3 \cdot 2^e})$. We shall prove the interesting fact that our Theorems 1.1 and 1.2 improve significantly on the cohomological lower bound for $\text{TC}_3(P^{3 \cdot 2^e})$, but not for $\text{TC}_k(P^{3 \cdot 2^e})$ when $k > 3$.

The bound implied by cohomology (Theorem 4.1) is

$$\text{TC}_k(P^{3 \cdot 2^e}) \geq (k - 1)(2^{e+2} - 1). \tag{3.1}$$

Table 3 Ratio of lower bounds for $TC_3(P^{3 \cdot 2^e})$ implied by Theorem 3.2 to those implied by $H^*(-)$

| e | r | d | m | BP-bound | H^* -bound | Ratio |
|-----|-----|-----|------|----------|--------------|----------|
| 6 | 1 | 0 | 92 | 524 | 510 | 1.027 |
| 7 | 1 | 1 | 180 | 1052 | 1022 | 1.029 |
| 8 | 1 | 2 | 356 | 2108 | 2046 | 1.030 |
| 9 | 2 | 0 | 760 | 4504 | 4094 | 1.100 |
| 10 | 2 | 1 | 1512 | 9016 | 8190 | 1.101 |
| 11 | 2 | 2 | 3016 | 18,040 | 16382 | 1.101 |
| 22 | 3 | 8 | | | | 1.1235 |
| 23 | 4 | 0 | | | | 1.124994 |

Since $2km - (2^k - 1)2^{r+1} \leq (k - 1)(2^{e+2} - 1)$ if $k \geq 4$ and $m \leq 3 \cdot 2^{e-1}$ (and $r \geq 0$), Theorem 1.1 cannot possibly improve on (3.1) if $k \geq 4$. For BP to possibly improve on (3.1) when $k \geq 4$, a much more delicate analysis of $BP^*((P^n)^k)$ would have to be performed, involving new ways of showing that classes are nonzero, and then using Theorem 2.1.

However, Theorem 1.2 implies a lower bound for $TC_3(P^{3 \cdot 2^e})$ which is asymptotically 9/8 times the bound in (3.1).

Theorem 3.2 *Let $r \geq 1$, $0 \leq d \leq 2^r$, and $e = 2^r + r + d + 3$. Then*

$$TC_3(P^{3 \cdot 2^e}) \geq 9 \cdot 2^e - 3 \cdot 2^{r+3+d} - 2^{r+1}.$$

Proof One easily checks that, with e as in the theorem, $m = 3 \cdot 2^{e-1} - 2^{r+2+d} + 2^{r+1}$ satisfies the hypothesis of Theorem 1.2(a)(ii), and that Theorem 1.2 then implies $TC_3(P^{2m}) \geq 9 \cdot 2^e - 3 \cdot 2^{r+3+d} - 2^{r+1}$, implying this theorem since bounds for $TC_k(P^n)$ implied by Theorem 2.1 apply to P^N for $N \geq n$. \square

In Table 3, we compare the bounds for $TC_3(P^{3 \cdot 2^e})$ implied by Theorem 3.2 and by (3.1) for various values of e . Every e has a unique r and d . The m -column is the value of $m < 3 \cdot 2^{e-1}$ which appears in the proof of Theorem 3.2. The “BP-bound” column is the bound for $TC_3(P^{3 \cdot 2^e})$ given by Theorem 3.2, and the “ H^* -bound” column is given by (3.1). The final column is the ratio of the BP-bound to the H^* -bound, which approaches 1.125 as e gets large.

Using different choices of a_1 and a_2 (found by computer), Theorem 1.1 can do somewhat better for $TC_3(P^{3 \cdot 2^e})$ than Theorem 1.2, but it does not seem worthwhile to try to find the best result implied by Theorem 1.1 for all e , since no pattern is apparent. For e from 7 to 11, the lower bounds for $TC_3(P^{3 \cdot 2^e})$ implied by Theorem 1.1 are, respectively, 1072, 2224, 4516, 9068, and 18,284. For example, when $e = 11$, it is about 1.4% better than that implied by Theorem 3.2 and 11.6% better than that implied by cohomology. For one who wishes to check this result when $e = 11$, use $m = 3066$, $r = 3$, and $a_1 = 3287$ in Theorem 1.1. The values of $v_{\binom{a_1}{m-2^{r+\epsilon}}}$ (resp. $v_{\binom{a_2}{m-2^{r+\epsilon}}}$) for $\epsilon = 0, 1, 2$ are (5, 6, 7) [resp. (6, 6, 3)].

4 $\text{TC}_k(P^n)$ result implied by mod-2 cohomology, in a range

In this section, we prove that the lower bound for $\text{TC}_k(P^n)$ implied by cohomology is constant in the last $\frac{2}{k}$ portion of the interval between successive 2-powers. This generalizes the behavior seen in Table 1 ($k = 3$) or Table 2 ($k = 4$). In the previous section, we showed that the bound implied by BP rises in this range to a value nearly $k/(k-1)$ times that of the cohomology bound, which is as much as it possibly could.

Recall from [2] or [6] that $\text{zcl}_k(P^n)$ is the lower bound for $\text{TC}_k(P^n)$ implied by mod-2 cohomology. It is an analog of Theorem 2.1, except that classes are in grading 1 rather than grading 2. Here, we prove the following new result about $\text{zcl}_k(P^n)$.

Theorem 4.1 *For $k \geq 3$ and $e \geq 2$, $\text{zcl}_k(P^n) = (k-1)(2^e - 1)$ for $\lfloor \frac{k-1}{k} \cdot 2^e \rfloor \leq n \leq 2^e - 1$.*

Note that, since $(k-1)n \leq \text{zcl}_k(P^n) \leq kn$ (by [2] or [6]), this interval of constant $\text{zcl}_k(P^n)$ is as long as it could possibly be.

Proof We rely on [6, Thm 1.2], which can be interpreted to say that, with n_t denoting $n \bmod 2^t$:

$$\text{zcl}_k(P^n) = kn - \max(2^{\nu(n+1)} - 1, kn_t - (k-1)(2^t - 1)), \quad (4.1)$$

with the max taken over all t for which the initial bits of $n \bmod 2^t$ begin a string of at least two consecutive 1s. That $\text{zcl}_k(P^{2^e-1}) = (k-1)(2^e - 1)$ is immediate from (4.1). Since $\text{zcl}_k(P^n)$ is an increasing function of n , it suffices to prove

$$\text{if } n = \lfloor \frac{k-1}{k} \cdot 2^e \rfloor, \quad \text{then } \text{zcl}_k(P^n) = (k-1)(2^e - 1). \quad (4.2)$$

The case $k = 3$ is slightly special, since the binary expansion of $n = \lfloor 2^{e+1}/3 \rfloor$ does not have any consecutive 1s. For this n , (4.1) implies that $\text{zcl}_3(P^n) = 3n + 1 - 2^{\nu(n+1)} = 2^{e+1} - 2$, as desired. From now on, we assume $k > 3$ in this proof.

One part that we must prove is

$$kn - 2^{\nu(n+1)} + 1 \geq (k-1)(2^e - 1) \quad (4.3)$$

if n is as in (4.2). Write $2^e = Ak - \delta$ with $0 \leq \delta \leq k-1$. Then, $n = 2^e - A$, and the desired inequality reduces to $k - \delta \geq 2^{\nu(A-1)}$, since $\nu(A-1) = \nu(2^e - A + 1)$. If $A - 1 = 2^t u$ with u odd, then $k - \delta = 2^e - 2^t u k \geq 2^t$, since $k - \delta > 0$, proving the inequality. \square

The rest of the proof requires the following lemma.

Lemma 4.2 *Let k be odd, and \mathbf{e} the multiplicative order of 2 mod k . Thus, \mathbf{e} is the smallest positive integer, such that k divides $2^{\mathbf{e}} - 1$. Let $m = (k-1)\frac{2^{\mathbf{e}}-1}{k}$, and let B be the binary expansion of m . If $t = \alpha\mathbf{e} + \beta$ with $0 \leq \beta < \mathbf{e}$, then the binary expansion of $\lfloor (k-1)2^t/k \rfloor$ consists of the concatenation of α copies of B , followed by the first β bits of B . In addition, the binary expansion of $\lfloor (2^{\nu k} - 1)2^{v+t}/(2^{\nu k}) \rfloor$ with k odd equals that of $\lfloor (k-1)2^t/k \rfloor$ preceded by ν 1s. If $k \geq 4$, B begins with at least two 1s.*

Table 4 Binary expansions B of numbers appearing in lemma

| k | e | B |
|-----|-----|--------------------|
| 9 | 6 | 111000 |
| 11 | 10 | 1110100010 |
| 13 | 12 | 111011000100 |
| 15 | 4 | 1110 |
| 17 | 8 | 11110000 |
| 19 | 18 | 111100101000011010 |
| 21 | 6 | 111100 |
| 23 | 11 | 11110100110 |

Proof Let $f_t = (k - 1)2^t/k$. Then, letting $\{f\} = f - \lfloor f \rfloor$ denote the fractional part of f :

$$\lfloor f_{t+1} \rfloor = \begin{cases} 2\lfloor f_t \rfloor & \text{if } \{f_t\} < 1/2 \\ 2\lfloor f_t \rfloor + 1 & \text{if } \{f_t\} \geq 1/2. \end{cases}$$

This shows that, as t increases, the binary expansions of the $\lfloor f_t \rfloor$ are just initial sections of subsequent ones. They start with at least two 1s when $k \geq 4$, since $\lfloor 2^2(k - 1)/k \rfloor = 3$.

If e is as in the lemma, then

$$\frac{(k - 1)2^{t+e}}{k} - \frac{(k - 1)2^t}{k} = 2^t \frac{(k - 1)(2^e - 1)}{k},$$

showing that adding this e to the exponent just appends B in front of the binary expansion. Regarding $2^v k$, note that

$$\frac{(2^v k - 1)2^{t+v}}{2^v k} = (2^v - 1)2^t + \frac{(k - 1)2^t}{k},$$

which shows the appending of 1s in front. □

In Table 4, we list some values of B , the binary expansion of m , for the m associated with k as in Lemma 4.2.

The property (4.5) says roughly that the beginning of B has more 1s than anywhere else in B .

For any $k > 3$ and $n = \lfloor \frac{k-1}{k} \cdot 2^e \rfloor$ as in (4.2), Eqs. (4.1) and (4.3) imply that

$$\text{zcl}_k(P^n) \geq kn - (kn - (k - 1)(2^e - 1)) = (k - 1)(2^e - 1),$$

with equality if for all t for which the initial bits of $n \bmod 2^t$ begin a string of at least two consecutive 1s:

$$kn_t - (k - 1)(2^t - 1) \leq kn - (k - 1)(2^e - 1).$$

This is equivalent to

$$1 - \frac{1}{k} \leq \frac{n - n_t}{2^e - 2^t}. \quad (4.4)$$

By the lemma, if k is odd (resp. even), the RHS of (4.4) is the same as (resp. greater than) it would be if (n, e) is replaced by (m, \mathbf{e}) , with notation as in the lemma, provided that $t \leq \mathbf{e}$. Note that equality holds in (4.4) if (n, e, t) is replaced by $(m, \mathbf{e}, 0)$. Hence, again using the lemma for cases in which $t > \mathbf{e}$, (4.4) will follow from its validity if (n, e) is replaced by (m, \mathbf{e}) , and, since $1 - \frac{1}{k} = \frac{m}{2^e - 1}$, this reduces to showing

$$\frac{m_t}{2^t - 1} \leq \frac{m}{2^e - 1}. \quad (4.5)$$

Let $q = \frac{2^e - 1}{k} = 2^e - 1 - m$ and $q_t = 2^t - 1 - m_t$ its reduction mod 2^t . Now, the desired inequality reduces to $\frac{q_t}{2^t - 1} \geq \frac{q}{2^e - 1} = \frac{1}{k}$, i.e., $kq_t \geq 2^t - 1$. We can prove the validity of this last inequality as follows. Write $q = q_t + 2^t\alpha$, for an integer α . Then

$$2^e - 1 = kq = kq_t + 2^t\alpha k.$$

Reducing mod 2^t gives the desired result. \square

Remark 4.3 It appears that the stronger inequality $kq_t \geq 3 \cdot 2^t - 1$ holds when $q = \frac{2^e - 1}{k}$, but we do not need it, and it seems much harder to prove.

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