1 Problem K33.22

At \( t = 0 \) s, the current in the circuit in the figure is \( I_0 \). At what time is the current \( \frac{1}{2}I_0 \)?

**Solution:** The inductor opposes any change in the current through it. In an LR circuit, the time constant is \( \tau = L/R \). The current drops exponentially with this time constant.

\[
I(t) = I_0 e^{-t/\tau} = I_0 e^{-tR/L} = \frac{1}{2}I_0
\]

\[
e^{-tR/L} = 1/2
\]

\[
-tR/L = \ln(1/2)
\]

\[
tR/L = \ln 2
\]

\[
t = L \ln 2/R = (50 \times 10^{-3} \text{ H}) \ln 2 / (500 \Omega)
\]

\[
= 6.93 \times 10^{-5} \text{ s}
\]

2 Energy within an L-C circuit

Consider an L-C circuit with capacitance \( C \), inductance \( L \), and no voltage source. As a function of time, the charge on the capacitor is \( Q(t) \) and the current through the inductor is \( I(t) \). Assume that the circuit has no resistance and that at one time the capacitor was charged.

**Part A.** As a function of time, what is the energy \( U_L(t) \) stored in the inductor? (In terms of \( L \) and \( I(t) \).)

**Solution:** The energy in an inductor is \( U_L(t) = \frac{1}{2}LI(t)^2 \).

**Part B.** As a function of time, what is the energy \( U_C(t) \) stored in the capacitor? (In terms of \( C \) and \( Q(t) \).)

**Solution:** The energy in a capacitor is \( U_C(t) = \frac{1}{2}Q(t)^2 \).

**Part C.** What is the total energy \( U_{\text{total}} \) stored in the circuit? (In terms of the maximum current \( I_0 \) and inductance \( L \).)

**Solution:** To solve this, we have to know how the current in an L-C circuit behaves. If there is an initial current flowing somehow, it will begin charging the capacitor. The inductor will cause this current to keep flowing until the charge builds up enough to oppose the current, at which point the current will stop and the capacitor will be charged. Then, the capacitor will force the current in the opposite direction while the inductor tries to keep the current constant. In the end, the current will flow back and forth sinusoidally through the circuit, charging the opposite plates of the capacitor. The mathematical derivation is in the book, and the current is

\[
I(t) = I_0 \cos(\omega t)
\]

Without an external voltage source, the frequency will be the resonant frequency of the capacitor and inductor, \( \omega = 1/\sqrt{LC} \). This tells us \( U_L(t) \), but we need the charge on the capacitor. Either integrate the current (because it is the derivative of \( Q(t) \) or use the phasor diagram. The current phasor is to the right. The voltage across the capacitor is \( V_c = X_CI_0 = I_0/\omega C \) and the direction is down. The charge on the capacitor is

\[
Q(t) = CV_C(t) = C \frac{I_0}{\omega C} \cos(\omega t - \pi/2) = \frac{I_0}{\omega} \sin(\omega t)
\]
The total energy is then

\[ U_{\text{total}} = U_L(t) + U_C(t) = \frac{1}{2} LI_0^2 \cos^2 (\omega t) + \frac{1}{2C} I_0^2 \omega^2 \sin^2 (\omega t) \]

\[ = \frac{1}{2} LI_0^2 \cos^2 (\omega t) + \frac{1}{2} LI_0^2 C \sin^2 (\omega t) \]

\[ = \frac{1}{2} LI_0^2 \]

Notice that with no resistance to dissipate the energy, the energy is constant.

### 3 A Radio Tuning Circuit

A radio can be tuned into a particular station frequency by adjusting the capacitance in an L-C circuit. Suppose that the minimum capacitance of a variable capacitor in a radio is 4.11 pF.

**Part A.** What is the inductance \( L \) of a coil connected to this capacitor if the oscillation frequency of the L-C circuit is 1.58 MHz, corresponding to one end of the AM radio broadcast band, when the capacitor is set to its minimum capacitance?

**Solution:** The inductance is found from the resonant frequency formula

\[ \omega^2 = (2\pi f)^2 = \frac{1}{LC} \quad \Rightarrow \quad L = \frac{1}{(2\pi f)^2 C} = \frac{1}{(2\pi 1.58 \times 10^6 \text{ Hz})^2 (4.11 \times 10^{-12} \text{ F})} = 2.47 \times 10^{-3} \text{ H} \]

**Part B.** The frequency at the other end of the broadcast band is 0.533 MHz. What is the maximum capacitance \( C_{\text{max}} \) of the capacitor if the oscillation frequency is adjustable over the range of the broadcast band?

**Solution:** The capacitance is found from the same formula.

\[ C = \frac{1}{(2\pi f)^2 L} = \frac{1}{(2\pi 0.533 \times 10^6 \text{ Hz})^2 (2.47 \times 10^{-3} \text{ H})} = 3.61 \times 10^{-11} \text{ F} \]
4 Problem K35.36

Evaluate $V_R$ in the figure at various EMF frequencies.

**Solution:** First, let’s think about the phasor diagram. In the series circuit, we know the current is constant, so we draw a current phasor in some direction with length $I_0$. Then we draw the $V_R$ phasor in the same direction as the current phasor, with a value of $V_R = I_0 R$. Then, draw the $V_C$ phasor at a right angle to $V_R$ and in the trailing direction (because voltage lags in a capacitor), which is clockwise from $V_R$. The length is $V_C = X_C I_0 = \frac{I_0}{\omega C}$. Since there is no inductance, we can use the pythagorean theorem to find the total voltage, which is the vector sum of the voltage phasors of the series components.

$$V_{total} = \sqrt{V_R^2 + V_C^2} = \sqrt{(I_0 R)^2 + \left( \frac{I_0}{\omega C} \right)^2} = I_0 \sqrt{R^2 + \frac{1}{\omega^2 C^2}} = \mathcal{E}_0$$

Now that we know $I_0$, we can find $V_R$.

$$V_R = I_0 R = \frac{\mathcal{E}_0 R}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}}$$

Make sure you understand how this formula came about. This is the important part. Now for the numbers.

**Part A.** $f = 100$ Hz

**Solution:** $\omega = 2\pi f = 200\pi \text{s}^{-1}$ and $V_R = \frac{(10 \text{ V})(100 \text{ Ω})}{\sqrt{(100 \text{ Ω})^2 + \left( \frac{10 \text{ V}}{200\pi \text{s}^{-1}} \right)^2 \left( \frac{1.59 \times 10^{-6} \text{ F}}{1.59 \times 10^{-6} \text{ F}} \right)^2}} = 0.994 \text{ V}$

**Part B.** $f = 300$ Hz

**Solution:** $\omega = 300 \cdot 2\pi \text{s}^{-1}$ and $V_R = 2.87 \text{ V}$.

**Part C.** $f = 1000$ Hz

**Solution:** $\omega = 1000 \cdot 2\pi \text{s}^{-1}$ and $V_R = 7.07 \text{ V}$.

**Part D.** $f = 3000$ Hz

**Solution:** $\omega = 3000 \cdot 2\pi \text{s}^{-1}$ and $V_R = 9.49 \text{ V}$.

**Part E.** $f = 10,000$ Hz

**Solution:** $\omega = 10000 \cdot 2\pi \text{s}^{-1}$ and $V_R = 9.95 \text{ V}$.

Notice that the lower frequencies are “blocked” by the capacitor while the high frequencies get through to the resistor.