28. Assume that the equilateral triangles have side length $s$. The heuristic starts with a minimum spanning tree. Since the distance between any two points is at least $s$, a minimum spanning tree has weight at least $s(2m - 2)$. Thus the spanning tree consisting of the path $1, 2m - 1, 2, 2m - 2, 3, 2m - 3, \ldots, m + 2, m - 1, m + 1, m$ is minimum. With this, the only two odd degree vertices are 1 and $m$ and the heuristic adds the edge between them with weight $s(m - 1)$. The result is a Hamiltonian path, so there is no shortcutting and the heuristic produces a Hamiltonian path of length $s(2m - 2 + m - 1) = s(3m - 3)$. A minimum weight Hamiltonian tour has weight at least $s(2m - 1)$ as each edge has weight at least $s$ and there are $2m - 1$ edges. Given $\epsilon$, take $m > \frac{3}{4\epsilon} + \frac{1}{2}$, then it can be checked (with some straightforward algebra) that $-3/2 \geq -\epsilon(2m - 1)$. Thus, for such $m$ we have $3m - 3 = (2m - 1) + .5(2m - 1) - 3/2 \geq (2m - 1) + (.5 - \epsilon)(2m - 1)$ as needed.

29. (6.2.4) We give three different proofs:

Proof 1: Let $M^*$ be a maximum matching and $M$ maximal. $M^*$ has $2|M^*|$ vertices as ends and $M$ has $2|M|$. So at least $2|M^*| - 2|M|$ vertices of $M^*$ are not ends of edges of $M$. If $|M| < |M^*|/2$ then $2|M^*| - 2|M| > |M^*|$ and since more than $|M^*|$ vertices are not covered by $M$, both ends of some edge of $M^*$ are not covered by $M$. This edge could then be added to $M$ contradicting maximality.

Proof 2: By maximality, the vertices that are ends of edges in $M$ cover all edges (an edge not covered could be added to $M$). Thus there is a vertex cover of size at most $2|M|$. So $2|M| \geq \beta(G)$. Since also we have $\beta(G) \geq \alpha'(G)$ (weak duality) we get $2|M| \geq \alpha'(G)$.

Proof 3: Let $M^*$ be a maximum matching and $M$ maximal. Consider the symmetric difference $M \triangle M^*$. There are at least $|M^*| - |M|$ augmenting paths in this symmetric difference. Since $M$ is maximal none of these paths consists of a single edge from $M^*$. Thus each augmenting path contains at least one edge from $M$ and we get $|M| \geq |M^*| - |M| \Rightarrow |M| \geq |M^*|/2$.

30. (6.2.24) Show that Tutte’s condition holds. This implies the existence of a 1-factor. Note that $G - S$ is connected for $|S| < r$ and in particular that $G$ is connected. If $1 \leq |S| < r$ then odd$(G - S) \leq 1$ and we have odd$(G - S) \leq |S|$. Since $G$ is connected and has even order odd$(G_S) = 0$ when $S = \emptyset$ and we have odd$(G_S) \leq |S|$ for $S = \emptyset$. For $|S| \geq r$ construct a bipartite graph $H$ with parts $S = \{v_1, v_2, \ldots, v_s\}$ and the components $C_1, C_2, \ldots, C_t$ with $t = odd(G - S)$. Put an edge between $v_i$ and $C_j$ if there is at least one edge between $v_i$ and a vertex of $C_j$. Since there is no $K_{1,r+1}$ the degree of each $v_i$ in $H$ is at most $r$. Since deleting fewer
than $r$ vertices does not disconnect the graph, the degree of each $C_j$ in $H$ is at least $r$. If $H$ has $e$ edges, counting the edges in two ways we get $sr \geq e \geq tr$. So $s \geq t$, which is $|S| \geq odd(G_S)$. So Tutte’s condition holds in all cases.

31. (6.3.6) (a) Consider the matrix

$$
\begin{array}{cccc}
4 & 4 & 4 & 3 \\
1 & 1 & 4 & 3 \\
1 & 4 & 5 & 3 \\
5 & 6 & 4 & 7 \\
5 & 3 & 6 & 8 \\
\end{array}
$$

We can find an initial cover by taking $u_i = \max_j row_i$ for all $i$, yielding

$$
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
6 & 2 & 2 & 2 & 3 & 0 \\
4 & 3 & 3 & 0 & 1 & 0 \\
5 & 4 & 1 & 0 & 2 & 0 \\
9 & 4 & 3 & 5 & 2 & 0 \\
8 & 3 & 5 & 2 & 0 & 5 \\
\end{array}
$$

where the underlined zeros correspond to a matching. If we let $R = \{\emptyset\}$ and $T = \{3, 4, 5\}$ the minimum $\epsilon$ of the uncovered elements is equal to 1. Thus, we decrease $u_i$ by 1, for all $i$ and increase $v_3, v_4, v_5$ by 1, yielding

$$
\begin{array}{c|cccc}
0 & 0 & 1 & 1 & 1 \\
5 & 1 & 1 & 2 & 3 & 0 \\
3 & 2 & 2 & 0 & 1 & 0 \\
4 & 3 & 0 & 0 & 2 & 0 \\
8 & 3 & 2 & 5 & 2 & 0 \\
7 & 2 & 4 & 2 & 0 & 5 \\
\end{array}
$$

Here, we let $R = \{\emptyset\}$ and $T = \{2, 3, 4, 5\}$. This gives $\epsilon = 1$, so we decrease $u_i$ by 1 for all $i$ and increase $v_2, v_3, v_4, v_5$ by 1, yielding the final solution

$$
\begin{array}{c|cccc}
0 & 1 & 2 & 2 & 2 \\
4 & 0 & 1 & 2 & 3 & 0 \\
2 & 1 & 2 & 0 & 1 & 0 \\
3 & 2 & 0 & 0 & 2 & 0 \\
7 & 2 & 2 & 5 & 2 & 0 \\
6 & 1 & 4 & 2 & 0 & 5 \\
\end{array}
$$

where $c(u, v) = 29 = w(M)$. 
(b) Consider the matrix

\[
\begin{array}{cccc}
7 & 8 & 9 & 8 \\
8 & 7 & 6 & 7 \\
9 & 6 & 5 & 4 \\
8 & 5 & 7 & 6 \\
7 & 6 & 5 & 5
\end{array}
\]

We can find an initial cover by taking \( u_i = \max_j \text{row}_i \) for all \( i \), yielding

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 \\
9 & 2 & 1 & 0 & 1 & 2 \\
8 & 0 & 1 & 2 & 1 & 2 \\
9 & 0 & 3 & 4 & 5 & 3 \\
8 & 0 & 3 & 1 & 2 & 4 \\
7 & 0 & 1 & 2 & 2 & 2
\end{array}
\]

If we let \( R = \{\emptyset\} \) and \( T = \{1, 3\} \), we have \( \epsilon = 1 \), so we decrease \( u_i \) by 1 for all \( i \) and increase \( v_1, v_3 \) by 1, yielding

\[
\begin{array}{cccc}
1 & 0 & 1 & 0 \\
8 & 2 & 0 & 0 \\
7 & 0 & 2 & 0 \\
8 & 0 & 2 & 4 \\
7 & 0 & 2 & 1 \\
6 & 0 & 0 & 2
\end{array}
\]

Then, if we let \( R = \{\emptyset\} \) and \( T = \{1, 2, 3, 4\} \), we have \( \epsilon = 1 \), so we decrease \( u_i \) by 1 for all \( i \) and increase \( v_1, v_2, v_3, v_4 \) by 1, yielding

\[
\begin{array}{cccc}
2 & 1 & 2 & 1 \\
7 & 2 & 0 & 0 \\
6 & 0 & 0 & 2 \\
7 & 0 & 2 & 4 \\
6 & 0 & 2 & 1 \\
5 & 0 & 0 & 2
\end{array}
\]

Finally, we let \( R = \{1, 2, 5\} \) and \( T = \{1\} \), giving \( \epsilon = 1 \), and yielding the final solution

\[
\begin{array}{cccc}
3 & 1 & 2 & 1 \\
7 & 2 & 0 & 0 \\
6 & 0 & 0 & 2 \\
6 & 0 & 1 & 3 \\
5 & 0 & 1 & 0 \\
5 & 0 & 0 & 2
\end{array}
\]
where \( c(u, v) = w(M) = 36. \)

(b) Consider the matrix

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
6 & 7 & 8 & 7 & 2 \\
1 & 3 & 4 & 4 & 5 \\
3 & 6 & 2 & 8 & 7 \\
4 & 1 & 3 & 5 & 4 \\
\end{bmatrix}
\]

We can find an initial cover by taking \( u_i = \max_j \text{row}_j \) for all \( i \), yielding

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 5 \\
4 & 3 & 2 & 1 & 0 & \\
8 & 2 & 1 & 0 & 1 & 6 \\
5 & 4 & 2 & 1 & 1 & 0 \\
8 & 5 & 2 & 6 & 0 & 1 \\
5 & 1 & 4 & 2 & 0 & 1 \\
\end{array}
\]

If we let \( R = \{\emptyset\} \) and \( T = \{3, 4, 5\} \), we have \( \epsilon = 1 \), so we adjust the cover and the excess graph, accordingly, yielding

\[
\begin{array}{cccccc}
0 & 0 & 1 & 1 & 1 & 4 \\
4 & 3 & 2 & 2 & 1 & 0 \\
7 & 1 & 0 & 0 & 1 & 6 \\
4 & 3 & 1 & 1 & 1 & 0 \\
7 & 4 & 1 & 6 & 0 & 1 \\
4 & 0 & 3 & 2 & 0 & 1 \\
\end{array}
\]

Then, if we let \( R = \{2, 4\} \) and \( T = \{4, 5\} \), we have \( \epsilon = 1 \), leading to the final solution

\[
\begin{array}{cccccc}
0 & 0 & 1 & 2 & 2 & 3 \\
2 & 1 & 1 & 1 & 0 & 7 \\
1 & 0 & 0 & 1 & 6 & 3 \\
2 & 0 & 0 & 1 & 0 & 6 \\
3 & 0 & 5 & 0 & 1 & 4 \\
0 & 3 & 2 & 0 & 1 & 4 \\
\end{array}
\]

where \( c(u, v) = w(M) = 28 \)

32. (6.3.8) Create a bipartite graph with parts \( \{u_1, u_2, \ldots, u_m\} \) and \( \{v_1, v_2, \ldots, v_n\} \) and the weight on edge \( u_iv_j \) equal to \( \max\{0, x_i + y_j - t\} \). The weights are the overtime, if any, for the corresponding pairing of routes. Thus we solve the problem by finding a minimum weight perfect matching.
Label so that \( x_1 \leq x_2 \leq \cdots \leq x_n \) and \( y_1 \geq y_2 \geq \cdots \geq y_n \). To show that a best solution is to pair the \( i^{th} \) shortest with the \( i^{th} \) longest we need to show that matching using edges \( u_iv_i \) for \( i = 1, 2, \ldots, n \) is minimum. We will show this using induction.

If \( n = 1 \) the result is trivial. For \( n > 1 \) if \( u_1v_1 \) is in the matching delete this edge and use induction on the remaining edges. We will show that there exists a minimum matching pairing \( u_1v_1 \) and the apply induction as in the previous sentence to establish the result. Assume that \( u_iv_j \) and \( v_1u_i \) are in the minimum matching with weight \( c^* \).

Switching these edges to match \( u_1v_1 \) and \( u_iv_j \) yields a matching with weight \( c \) such that

\[
c^* - c = \max\{0, x_1 + y_j - t\} + \max\{0, x_i + y_1 - t\} - \max\{0, x_1 + y_1 - t\} - \max\{0, x_i + y_j - t\}
\]

(as all other weights remain unchanged). With \( x_1 \leq x_i \) and \( y_1 \geq y_j \) we get

Case 1: If \( x_i + y_1 \leq t \) then each of \( x_1 + y_j, x_i + y_j, x_1 + y_j \) is at most \( t \). In this case the weights on all four edges are 0 and \( c^* - c = 0 \).

Case 2: \( x_i + y_j \leq t \). Then \( x_1 + y_j \leq t \) and the weights on edges \( u_1y_j \) and \( u_1y_i \) are 0. As also \( x_1 + y_i \leq x_i + y_i \) the weight on edge \( u_1v_1 \) is at most that on edge \( u_1y_i \) and \( c^* - c \geq 0 \).

Case 3: \( x_1 + y_1 \leq t \). Then \( x_1 + y_j \leq t \) and the weights on edges \( u_1v_1 \) and \( u_1v_1 \) are 0. As \( x_i + y_j \leq x_i + y_1 \) the weight on edge \( u_1v_1 \) is at most that on edge \( u_1y_i \) and \( c^* - c \geq 0 \).

Case 4: None of the above. Then The weights on edges \( u_1v_i, u_1v_j, u_1v_i \) are respectively \( x_1 + y_1 - t, x_i + y_j - t, x_i + y_1 - t \) and \( c^* - c = (\max\{0, x_1 + y_j - t\} + (x_i + y_1 - t) - (x_1 + y_1 - t) - (x_i + y_j - t) = (\max\{0, x_1 + y_j - t\}) - (x_i + y_j - t) \geq 0 \).

Thus in each case switching does not increase the weight and we get a minimum matching using the edge \( u_1v_1 \) and as noted above by induction the result follows.

33. (6.3.11) Form a bipartite graph with bipartition \( U = \{u_1, u_2, \ldots, u_n\} \) and \( V = \{v_{rs} | r = 1, 2, \ldots, k \) and \( s = 1, 2, \ldots, k_r \}. \) Put the weight on edge \( u_iv_{rs} \) to be \( t \) if seminar \( r \) is the \( i^{th} \) highest seminar on the list of student \( i \). A minimum weight perfect matching is stable under this definition of stable. (Here we put student \( i \) in seminar \( r \) if \( u_i \) is matched to \( v_{rs} \) for some \( s \).) If student \( i \) in seminar \( r \) and student \( i' \) in seminar \( r' \) want to switch then \( i \) prefers \( r' \) and \( i' \) prefers \( r \) so for any \( s, s', s'', s''' \) we have \( weight(u_iv_{rs}) > weight(u_iv_{rs'}) \) and also \( weight(u_{i'}v_{r's''}) > weight(u_{i'}v_{r's'''}) \). Thus the matching after the switch has lower weight, a contradiction.

Note - one can also check that a ‘greedy’ approach, going through the list of students and assigning the highest ranked seminar with an available slot will produce a stable matching.