Solutions to Homework 910
Combinatorics (Math 446) Fall 2004 Lehigh University

2.1.9 (a) For induction the basis is \( \hat{F}_0 = 1 = \binom{0}{0} \) and \( \hat{F}_1 = 1 + 0 = \binom{1}{0} + \binom{0}{1} \). Then using induction, Pascal’s identity and the recurrence for \( \hat{F} \) we get for \( n \geq 2 \)

\[
\hat{F}_n = \hat{F}_{n-1} + \hat{F}_{n-2} \\
= \sum_{j=0}^{n-1} \binom{n-1-j}{j} + \sum_{k=0}^{n-2} \binom{n-2-k}{k} \\
= \sum_{j=0}^{n-1} \binom{n-1-j}{j} + \sum_{k=1}^{n-1} \binom{n-1-(k+1)}{(k+1)-1} \\
= 1 + \sum_{i=1}^{n-1} \left( \binom{n-1-i}{i} + \binom{n-1-i}{i-1} \right) \\
= \sum_{i=0}^{n} \binom{n-i}{i}
\]

Combinatorially, \( \hat{F}_n \) counts the number of 1,2 sequences summing to \( n \). Partition the sequences by the number of 2’s in the sequence. If there are \( i \) 2’s then there are \( n-i \) terms in the sequence, of which \( i \) are 2’s. So we pick the locations for the 2’s in \( \binom{n-i}{i} \) ways. Summing over \( i \) gives the identity.

(b) For induction the basis is \( \hat{F}_{n+2} = 2 + 1 = 1 + \hat{F}_0 \). Then, using induction and the recurrence for \( \hat{F}_n \) we get for \( n \geq 1 \)

\[
\hat{F}_{n+2} = \hat{F}_{n+1} + \hat{F}_n = (1 + \sum_{i=0}^{n-1} \hat{F}_i) + \hat{F}_n = 1 + \sum_{i=0}^{n} \hat{F}_i
\]

Combinatorially, \( \hat{F}_{n+2} \) counts the number of 1,2 sequences summing to \( n + 2 \). There is one sequence of all 1’s. Partition the remaining sequences by the first occurrence of a 2. If the first 2 appears after \( i \) 1’s then the sequence begins with \( i \) 1’s and a 2 with the rest of the sequence an arbitrary 1,2 sequence summing to \( (n+2) - i - 2 = n - i \); there are \( \hat{F}_{n-i} \) of these. Summing over \( i \) we get \( \hat{F}_{n+2} = 1 + \sum_{i=0}^{n} \hat{F}_{n-i} = 1 + \sum_{i=0}^{n} \hat{F}_i \).

2.2.4 The homogeneous part \( a_n = 3a_{n-1} - 2a_{n-2} \) has characteristic polynomial \( x^2 - 3x + 2 = (x-1)(x-2) \) with roots 1, 2. From proposition 2.2.10 we see that there is a particular solution of the form \( P(n)r^n2^n \) where \( r = 1 \) and \( P(n) \) is a polynomial with degree 1, i.e., a constant, call it \( P \). Thus we get that our solution is of the form \( a_n = c_1(1^n) + c_2(2^n) + Pn2^n = c_1 + c_22^n + Pn2^n \). From \( a_0 = a_1 = 1 \) we get \( a_2 = 3a_1 - 2a_0 + 2^2 = 3(1) - 2(1) + 4 = 5. \) Then using the general form we get \( 1 = a_0 = c_1 + c_2; 1 = a_1 = c_1 + 2c_2 + 2P \) and \( 5 = a_2 = c_1 + 4c_2 + 8P \). This system of 3 equations in 3 unknowns has solution \( c_1 = 5, c_2 = -4, P = 2 \) and we get \( a_n = 5 - 4 \cdot 2^n + 2n2^n = 5 - 4 \cdot 2^n + n2^{n+1} \).
Using generating functions we get

\[
A(x) = \sum_{n=0}^{\infty} a_n x^n \\
= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \\
= 1 + x + \sum_{n=2}^{\infty} (3a_{n-1} - 2a_{n-2} + 2^n) x^n \\
= 1 + x + 3x \left( \sum_{n=1}^{\infty} a_{n-1} x^{n-1} \right) - 2x^2 \left( \sum_{n=0}^{\infty} a_{n-2} x^{n-2} \right) + \sum_{n=2}^{\infty} 2^n x^n \\
= 1 + 3x(A(x) - 1) - 2x^2 A(x) + (2x)^2 \sum_{m=0}^{\infty} (2x)^m \\
= 1 - 2x + (3x - 2x^2) A(x) + \frac{4x^2}{1 - 2x}
\]

Then solving for \( A(x) \) we get

\[
A(x) = \frac{1}{1 - 3x + 2x^2} \left( 1 - 2x + \frac{4x^2}{1 - 2x} \right) = \frac{1}{(1 - 2x)(1 - x)} \left( 1 - 2x + \frac{4x^2}{1 - 2x} \right)
\]

Then using partial fractions we get

\[
A(x) = \frac{1}{1 - x} + \frac{4x^2}{(1 - 2x)^2(1 - x)} = \frac{5}{1 - x} + \frac{-6}{1 - 2x} + \frac{2}{(1 - 2x)^2}.
\]

The coefficient of \( x^n \) in \( 1/(1 - x) \) is 1, the coefficient of \( x^n \) in \( 1/(1 - 2x) \) is \( 2^n \) and the coefficient of \( x^n \) in \( 1/(1 - 2x)^2 \) is \( (n + 1)2^n \) (the last using \( 1/(1 - z)^t = \sum_{z \geq 0} \binom{t + z - 1}{t - 1} z^s \) with \( z = 2x \) and \( t = 2 \) and \( s = n \)). Hence the coefficient of \( x^n \) is \( a_n = (5)(1) + (-6)(2^n) + (2)(n + 1)2^n = 5 - 4 \cdot 2^n + n2^{n+1} \).

3.1.6 Define \( a \prec b \) if and only if \( b - a \geq 2 \). Each \( k \) subset of \([n]\) with no consecutive integers corresponds to a sequence \( 1 \leq a_1 \prec a_2 \prec \cdots \prec a_k \leq n \). Let \( g_1 = a_1, g_{k+1} = n - a_k \) and for \( i = 2, 3, \ldots, k \) let \( g_i = a_i - a_{i-1} \). The properties of the \( a_j \) imply that the \( g_i \) are non-negative integers and in addition \( g_1 \geq 1 \) and \( g_2, g_3, \ldots, g_{k-1} \) are at least 2. Note also that \( \sum_{i=1}^{k+1} g_i = a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_k - a_{k-1}) + (n - a_k) = n \). So by looking at these gaps and recalling what we did for multisets we could directly determine \( b_{n,k} \) (which we won’t do here) and we can also use generating functions, getting the function as we did for multisets, with terms of the form \( x^2 + x^3 + \cdots \) corresponding to \( g_i \geq 2 \) and then use the expansion for multisets to evaluate as follows:

\[
\sum_{n \geq 0} b_{n,k} x^n = (x + x^2 + \cdots)(x^2 + x^3 + \cdots)^{k-1}(1 + x + x^2 + \cdots) = \frac{x^{2k-1}}{(1 - x)^{k+1}}
\]

and

\[
\frac{x^{2k-1}}{(1 - x)^{k+1}} = x^{2k-1} \sum_{m=0}^{\infty} \binom{m + (k + 1) - 1}{(k + 1) - 1} x^m = x^{2k-1} \sum_{m=0}^{\infty} \binom{m + k}{k} x^m.
\]
Then $b_{n,k}$ is the coefficient of $x^n$ which is the coefficient of $x^{n-2k+1}$ in the last sum which is $\binom{n-2k+1+k}{k} = \binom{n+k+1}{k}$. 

3.3.9 See also example 3.3.8 in the text for a different presentation of the solutions to the questions involving $a_{n,k}$.

(a) In the solution to 1.1.33 on the previous homework we showed $a_{n,k} = k! \binom{n+k-1}{k}$. In a similar manner to the version 1 proof we see that placing the unlabelled flags corresponds to positive integral solutions to $\sum_{i=1}^n x_i = k$ with the $x_i$ indicating how many flags are on pole $i$ (and hence positive in this case). This is $\binom{k-1}{n-1}$ and for each such solution we can label the flags in $k!$ ways so we get $b_{n,k} = \binom{k-1}{n-1} k!$.

(b) The exponential generating function for number of ways to place the flags on a single flagpole is $(0! + 1! \frac{x}{1!} + 2! \frac{x^2}{2!} + 3! \frac{x^3}{3!} + \cdots) = 1 + x + x^2 + x^3 + \cdots = 1/(1-x)$ and if at least one flag must be on the pole the initial $0! = 1$ term is omitted and we get $(x + x^2 + x^3 + \cdots) = x/(1-x)$. Thus for $n$ poles we get

$$A(x) = \sum_{k \geq 0} \frac{a_{n,k}}{k!} x^k = \left( \frac{1}{1-x} \right)^n$$

and

$$B(x) = \sum_{k \geq 0} \frac{b_{n,k}}{k!} x^k = \left( \frac{x}{1-x} \right)^n.$$

(c) From (b) we see that $a_{n,k}/k!$ is the coefficient of $x^k$ in $(1/(1-x))^n$ which is $\binom{n+k-1}{k}$ and hence $a_{n,k} = k! \binom{n+k-1}{k}$. Also from (b) we see that $b_{n,k}/k!$ is the coefficient of $x^k$ in $(x/(1-x))^n$ which is the coefficient of $x^{k-n}$ in $(1/(1-x))^n$ which is $\binom{n+k-n-1}{k-n-1} = \binom{k-1}{n-1}$ and hence $b_{n,k} = k! \binom{k-1}{n-1}$.

3.4.9 (a) The Ferrers diagram for a partition of $2r+k$ into $r+k$ parts has $r+k$ rows so $r+k$ of the dots are in the first column and at most $r$ dots are not in the first column. Delete the first column to obtain a Ferrers diagram for an arbitrary partition of $r$. Conversely, given a Ferrers diagram for an arbitrary partition of $r$, add a first column of $r+k$ rows to obtain a Ferrers diagram for a partition of $2r+k$ into $r+k$ parts. Thus $p_{2r+k,r+k}$ is equal to the number of partitions of $r$, independent of $r$.

(b) Given a partition of $r+k$ into $k$ parts delete the first column of its Ferrers diagram to get a diagram for a partition of $r$ with at most $k$ rows. Taking the conjugate yields a diagram for a partition of $r$ with at most $k$ columns. It is straightforward to see that this can be reversed so $p_{r+k,k}$ counts the partitions of $r$ into parts of size at most $k$. 
(c) Use the notation \((γ_1, γ_2, \ldots, γ_k)\) with \(1 ≤ γ_1 ≤ γ_2 ≤ \cdots ≤ γ_k\) and \(\sum γ_i = n\) for a partition of \(n\) into \(k\) parts (with the \(γ_i\) indicating the sizes of the parts). Let \(γ'_i = γ_i + i - 1\). Then the \(γ'_i\) are distinct and \(\sum γ'_i = \sum γ_i + (1 + 2 + \cdots (k - 1)) = n + (k - 1)k/2\). Hence the \(γ'_i\) correspond to a partition of \(n + (k - 1)k/2\) into distinct parts. This can be reversed for \(n ≥ k\), given a partition of \(n + (k - 1)k/2\) into distinct parts (represented by \(γ'_i\) such that \(1 ≤ γ'_1 < γ'_2 < \cdots γ'_k\) with \(\sum γ'_i = n + (k - 1)k/2\)) letting \(γ_i = γ'_i - i + 1\) yields the \(γ_i\) for a partition of \(n\) into \(k\) parts. Hence for \(n ≥ k\), \(p_{n,k}\) counts the number of partitions of \(n + (k - 1)k/2\) into distinct parts. Letting \(n = r + k\) yields the result in the problem.

(d) A partition of \(n\) into \(k\) parts either has a part of size one or it does not. If it does, deleting a 1 leaves a partition of \(n - 1\) into \(k - 1\) parts. If it does not, subtracting 1 from each part leaves a partition of \(n - k\) into \(k\) parts. It is straightforward to check that this can be reversed so that this is indeed a bijection. In terms of Ferrer’s diagrams, we delete the last row if it has size 1 (leaving a diagram with 1 less dot and 1 less row) or delete the first column if the last row has at least 2 dots (leaving a diagram with \(n - k\) dots and \(k\) rows).

4.1.17 For the fall the number of assignments is the number of anagrams of the \(2n\) symbols \(1, 1, 2, 2, \ldots, n, n\). If \(i\) appears in locations \(r\) and \(s\) then professor \(i\) teaches courses \(r\) and \(s\). This number is \((2n)!/2^n\).

Let \(A_i\) denote the event that professor \(i\) teaches the same courses in the spring as in the fall. If \(g_k\) counts the number of ways that a given set of \(k\) professors teach the same course then this is the number of ways of assigning the remaining \(n - k\) professors courses which from above is \((2n - 2k)!/2^{n-k}\). Hence the number of events with none of the \(A_i\) is \(f(∅) = \sum_{k=0}^n (-1)^k \binom{n}{k} g_k = \sum_{k=0}^n (-1)^k \binom{n}{k} (2n - 2k)!/2^{n-k}\). The probability is just this answer divided by the answer in the first paragraph.