37. This is clearly a hereditary system: if $S$ has size at most $k$ and is not a subset of any of the $A_i$ then any $S' \subset S$ also has these properties. To see that it is non-empty we need an extra condition, that $|E| \geq k$. Then if $|A_i| < k$ for all $i$ all size $k$ subsets of $E$ are bases. If $|A_i| \geq k$ for some $i$, choose $x \not\in A_i$ (which exists as we assumed $A_i \neq E$) and $X \subseteq A_i$ with $|X| = k - 1$. Then $X \cup \{x\}$ is a base. It has size $k$ and if it was contained in any $A_j$ then $|A_j \cap A_i| \geq k - 2$.

Let $B_1$ and $B_2$ be two bases and $e \in B_1 - B_2$. Let $B_2 - B_1 = \{f_1, f_2, \ldots, f_i\}$. If $B - e + f_i$ is not a base then, since its size is $k$, for some $A_{\sigma(i)}$ we have $B_1 - e + f_i \subseteq A_{\sigma(i)}$. Assume that $B - e + f_i$ is a base for all $f_i \in B_2 - B_1$. For $f_i$ and $f_j$ in $B_2 - B_1$ we get sets $A_{\sigma(i)}$ and $A_{\sigma(j)}$ with the size $k - 1$ set $B_1 - e$ a subset of both. The intersection condition then implies $A_{\sigma(i)} = A_{\sigma(j)}$ (if they were distinct their intersection would be too large). So all of the $A_{\sigma(i)}$ are the same set, call it $A$. Now $B_3 \subseteq \bigcup_{i=1}^{f} (B_1 - e + f_i) \subseteq \bigcup_{i=1}^{f} A_{\sigma(i)} = A$ contradicting $B_2$ a base. Thus $B_1 - e + f_i$ is a base for some $i$ and the base exchange axiom holds.

38. Let $\{e_1, e_2, \ldots, e_t\} = B_1 - B_2$ and $\{f_1, f_2, \ldots, f_i\} = B_2 - B_1$. Construct a bipartite graph $H$ with parts $B_1 - B_2$ and $B_2 - B_1$. Put an edge in $H$ between $e_i$ and $f_j$ if $B_1 - e_i + f_j$ is a base. A perfect matching in $H$ gives the bijection. Use Hall’s Theorem to show that $H$ has a perfect matching. Consider $S \subseteq B_1 - B_2$. The $B_2$ is an independent set in $(B_1 \cup B_2) - S$. Thus we can augment the independent set $B_1 - S$ to an independent set $(B_1 - S) \cup T$ of size $B_2$ in $(B_1 \cup B_2) - S$. Note that $T \subseteq B_2 - B_1$ and has size $|S|$. Consider $f_j \in T$. $B_1 + f_j$ contains a circuit and this circuit must intersect $S$ as otherwise it is a subset of the independent set $(B_1 - S) \cup T$. So for some $e_i \in B_1 - B_2$ we have $B_1 - e_i + f_j$ a base. This implies that $T$ is contained in the neighborhood of $S$ in $H$. So Hall’s condition holds in $H$ and we can find a perfect matching.

39. 18.1.23(a) By monotonicity of the rank function $r(X) \leq r(X + e + f)$ we need to show $r(X) \geq r(X + e + f)$ from $r(X + e) = r(X + f)$ and submodularity. By submodularity $r((X + e) \cap (X + f)) + r((X + e) \cup (X + f)) \leq r(X + e) + r(X + f)$ which implies $r(X) + r(X + e + f) \leq r(X) + r(X)$ and cancelling gives $r(X + e + f) \leq r(X)$ as needed.

18.1.23(d) Assume that $x \in C_1 \cap C_2$. If $(C_1 \cup C_2) - x$ does not contain a circuit and is thus independent. By uniqueness $((C_1 \cup C_2) - x) + x = C_1 \cup C_2$ contains at most one circuit, a contradiction.
40. (18.2.2(a)): Note that \( \overline{S} \) is independent in both dual matroids \( M_1^* \) and \( M_2^* \). By the intersection formula there is some set \( Y \) attaining the minimum \( |\overline{S}| = r_1^*(Y) + r_2^*(\overline{Y}) \). Putting \( \overline{Y} \) into the intersection formula for the original matroids \( M_1 \) and \( M_2 \) we get \( |I| \leq r_1(\overline{Y}) + r_2(Y) \). With \( |S| = |E| - |\overline{S}| \) and \( |E| = |Y| + |\overline{Y}| \) we get

\[
|I| + |S| \leq r_1(\overline{Y}) + r_2(Y) + |E| - (r_1^*(Y) + r_2^*(\overline{Y}))
\]

\[
= (|Y| - r_1^*(Y) + r_1(\overline{Y})) + (|\overline{Y}| - r_2^*(\overline{Y}) + r_2(Y))
\]

\[
= r_1(E) + r_2(E)
\]

where the last equality follows from the dual rank formula.

Similarly let \( X \) attain the minim in the intersection formula to get \( |I| = r_1(X) + r_2(\overline{X}) \) and in the duals we have \( |\overline{S}| \leq r_1^*(\overline{X}) + r_2^*(X) \). With \( |S| = |E| - |\overline{S}| \) this gives a lower bound on \( |S| \). Then

\[
|I| + |S| \geq r_1(X) + r_2(\overline{X}) + |E| - (r_1^*(\overline{X}) + r_2^*(X))
\]

\[
= (|X| - r_2^*(X) + r_2(\overline{X})) + (|\overline{X}| - r_1^*(\overline{X}) + r_1(X))
\]

\[
= r_2(E) + r_1(E)
\]

where the last equality follows from the dual rank formula. Combining we get \( |I| + |S| = r_1(E) + r_2(E) \).

(b) Let \( G \) be a bipartite graph with bipartition \( U_1, U_2 \) and let \( M_{U_i} \) be the partition matroid with a set of edge independent if and only if the endpoints in \( U_i \) are distinct. Thus the ranks of the matroids are \( |U_i| \). A set of edges is independent in both \( M_{U_1} \) and \( M_{U_2} \) if and only if the edges have distinct ends in both \( U_1 \) and \( U_2 \). That is, if they form a matching. A set of edges is spanning in \( U_i \) if every vertex in \( U_i \) is the end of at least one of the edges (if not then edges incident to such a vertex are not in the span as adding them increases the rank). Thus a set of edges spanning both matroids is an edge cover of the vertices. For both of these the correspondence goes both ways so we have \( \alpha'(G) = |I| \) and \( \beta'(G) = |S| \). Then from part (a) we get \( \alpha'(G) + \beta'(G) = |I| + |S| = r_{U_1}(E) + r_{U_2}(E) = |U_1| + |U_2| = n(G) \). Note that the set of vertices not covered by a set of edges independent in both matroids must form an independent set as otherwise we could add an edge. Thus \( \alpha(G) \geq n(G) - \alpha'(G) = n(G) - (n(G) - \beta'(G)) = \beta'(G) \). Since independent vertices must be covered by distinct edges we also have \( \alpha(G) \leq \beta(G) \). Thus \( \alpha(G) = \beta'(G) \).

41. Do 18.2.3: Let \( k \) denote the number of paths in a minimum disjoint path partition, \( n \) the number of vertices in \( G \) and use \( \alpha \) for \( \alpha(G) \) and \( \beta \) for \( \beta(G) \). A set of edges is independent in both the head and tail partition matroids \( M_H \) and \( M_T \) if and only if all indegrees and outdegrees are at most one. That is, if and only if the edges form disjoint paths. In any forest with \( t \) components (possibly including some isolated
vertices, which will correspond to trivial paths in the path partition) the number of edges is \( n - t \). Thus \( k = n - |I| \) where \( I \) is a maximum size set independent in both matroids. Recall also Gallai’s identity \( \alpha + \beta = n \). Then using the matroid intersection formula \( k = n - |I| = \alpha + \beta - |I| = \alpha + \beta - \min_{X \subseteq E} \{r_H(X) + r_T(\overline{X})\} \). From this \( k \leq \alpha \) will follow if we show \( \beta \leq \min_{X \subseteq E} \{r_H(X) + r_T(\overline{X})\} \). An independent set in the head partition matroid corresponds to a set of edges that induce a graph where every indegree is at most 1. That is, an inforest. Similarly, an independent set in the tail partition matroid corresponds to an outforest. For a given set of edges \( X \) let \( S \) be a maximal independent set in \( X \). We have \( r_H(X) = |S| \). Let \( R_H \) be the set of vertices with indegree 1 in the graph induced by \( S \). Note that \( |R_H| = |S| = r_H(X) \). These vertices cover the edges of \( X \). Adding an edge not covered by \( R_H \) to \( S \) would increase the indegree of a vertex with indegree 0 and hence we would still have an independent set, contradicting maximality of \( S \). In a similar manner we get a set \( R_T \) of \( r_T(X) \) vertices covering the edges of \( \overline{X} \) by looking at the tail partition matroid. Then \( R_H \cup R_T \) is a vertex cover of the edges and \( \beta \leq |R_H \cup R_T| \leq |R_H| + |R_T| = r_H(X) + r_T(\overline{X}) \) and the result follows as this holds for all \( X \).

42. Given a rainbow spanning tree and a partition \( V_1, V_2, \ldots, V_k \) there are at least \( k - 1 \) edges joint the parts as the tree is spanning. Since the edges must have different colors we have at least \( k - 1 \) colors. It remains to show that if this color condition holds there is a rainbow spanning tree.

Let \( M_1 \) be the cycle matroid on the edges of \( G \) and let \( M_2 \) be the partition matroid on the edges with a set independent if and only if the edges have different colors. Then a set that is independent in both matroids is a rainbow forest. A common independent set of size \( |V| - 1 \) would be a rainbow spanning tree. Thus the result follows from matroid intersection if \( \min_{X \subseteq E} \{r_1(X) + r_2(\overline{X})\} \geq |V| - 1 \) as this would imply a common independent set of size \( |V| - 1 \). We will show that \( r_1(X) + r_2(\overline{X}) \geq |V| - 1 \) for all \( X \). The graph \( G' \) induced by a set \( X \) of edges yields a partition of the vertices \( V_1, V_2, \ldots, V_k \) with two vertices in the same part if and only if they are in the same component of \( G' \). Since each component has a spanning tree there is a forest with \( |V| - k \) edges among the edges of \( X \). That is, \( r_1(X) \geq |V| - k \) (in fact it is equal). The set \( \overline{X} \) contains the set of edges between the parts of the partition and thus by assumption contains edges of at least \( k - 1 \) colors. Thus \( r_2(\overline{X}) \geq k - 1 \). So \( r_1(X) + r_2(\overline{X}) \geq (|V| - k) + (k - 1) = |V| - 1 \) as needed.