1. Note - there are at least two ways to take care of the ‘at most one of the systems has a solution’ part of the statements. While it is a bit redundant we will show both ways below. First we show it directly and we also show it by the equivalent systems. If the ‘at most one system holds’ is shown first then only the ⇐’s are needed for the equivalent systems.

First we note that for each it is easy to show that at most one of the systems holds.

If both IA and IIA hold then

\[0 = 00 \leq (yA)x = y(Ax) \leq yb < 0\]

a contradiction. We have used \(y \geq 0\) in the second \(\leq\).

If both IB and IIB hold then

\[0 = 00 \leq (yA)x = y(Ax) = yb < 0\]

a contradiction.

So for the remainder we will seek to show at least one of the following holds.

\((A \Rightarrow B)\): Note the following equivalences.

\[(IB) \quad Ax = b \quad x \geq 0 \quad \iff \quad \begin{cases} Ax \leq b \\ -Ax \leq -b \end{cases} \quad x \geq 0 \quad \iff \quad \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix} \quad (IB')\]

and

\[(IIB) \quad yA \geq 0 \quad yb < 0 \quad \iff \quad \begin{cases} (u - v)A \geq 0 \\ (u - v)b < 0 \end{cases} \quad \iff \quad \begin{bmatrix} u & v \\ -v & -u \end{bmatrix} \begin{bmatrix} A \\ -A \end{bmatrix} \geq 0 \quad (IIB')\]

In the second line, given \(y\) one can easily pick non-negative \(u, v\) such that \(y = u - v\) so the first ⇐ in the second line does hold.

Applying A, we get that exactly one of (IB') and (IIB') has a solution. The equivalences then show that exactly one of (IB) and (IIB) has a solution.

\((B \Rightarrow A)\): Note following equivalences.

\[(IA) \quad Ax \leq b \quad x \geq 0 \quad \iff \quad Ax + Is = b \quad x, s \geq 0 \quad \iff \quad \begin{bmatrix} A \\ I \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = b \quad (IA')\]
and

\[
\begin{align*}
\text{(IIA)} \quad yA &\geq 0 \quad \iff \quad y(A \mid I) &\geq 0 \\
y &\geq 0 \quad \iff \quad y(A \mid I) &\geq 0 \\
yb &< 0 \quad \iff \quad yb &< 0
\end{align*}
\]

Applying B, we get that exactly one of (IA’) and (IIA’) has a solution. The equivalences then show that exactly one of (IA) and (IIA) has a solution.

2. (i) We start with

\[
\sum_{j=1}^{n} a_{ij}x_j \leq b_i \quad \text{for} \quad i = 1, 2, \ldots, m. \tag{1}
\]

Let \( U = \{i \mid a_{in} > 0\}, \ L = \{i \mid a_{in} < 0\} \) and \( N = \{i \mid a_{in} = 0\} \). Then

\[
\frac{1}{a_{rn}} \left( b_r - \sum_{j=1}^{n-1} a_{rj}x_j \right) \leq x_n \quad \text{for} \quad r \in L
\]

\[
x_n \leq \frac{1}{a_{sn}} \left( b_s - \sum_{j=1}^{n-1} a_{sj}x_j \right) \quad \text{for} \quad s \in U
\]

\[
\sum_{j=1}^{n-1} a_{tj}x_j \leq b_t \quad \text{for} \quad t \in N
\]

is just a rearrangement of (1). Note that the direction of the inequality changes when \( r \in L \) since \( a_{rn} < 0 \) and we multiply by this. We pair each upper bound with each lower bound and carry along the inequalities not involving \( x_n \) to get

\[
\frac{1}{a_{rn}} \left( b_r - \sum_{j=1}^{n-1} a_{rj}x_j \right) \leq \frac{1}{a_{sn}} \left( b_s - \sum_{j=1}^{n-1} a_{sj}x_j \right) \quad \text{for} \quad r \in L, s \in U
\]

\[
\sum_{j=1}^{n-1} a_{tj}x_j \leq b_t \quad \text{for} \quad t \in N
\]

Due to the construction (1) has a solution if and only if (3) does. (This will also follow from parts (iv) and (v) below.)

(ii) With one variable we have three types of inequalities as above using \( n = 1 \) for \( L, U, N \): \( a_{s1}x_1 \leq b_s \) for \( s \in U \); \( a_{r1}x_1 \leq b_r \) for \( r \in L \) and inequalities with no variable \((t \in N)\) of the form \( 0 \leq b_t \). If we have \( 0 \leq b_t \) for some \( b_t < 0 \) then the system is inconsistent, there is clearly no solution to the system. Set all multiplies \( u_{i1} = 0 \) except \( u_{11} = 1 \) to get a certificate of inconsistency. We can drop inequalities \( 0 \leq b_t \) for \( b_t \geq 0 \) so we can assume now that all \( a_{i1} \) are nonzero. Rewriting the system we have \( x_1 \leq b_{s1}/a_{s1} \) for \( s \in U \) and \( b_r/a_{r1} \leq x_1 \) for \( r \in L \). There is a solution if and only if \( \max_{r \in L} b_r/a_{r1} \leq \min_{s \in U} b_s/a_{s1} \). If this holds then any \( x_1 \in [\max_{r \in L} b_r/a_{r1}, \min_{s \in U} b_s/a_{s1}] \) satisfies all inequalities. If not then for some \( r^* \), \( s^* \) we have \( b_{s^*}/a_{s^*1} < b_{r^*}/a_{r^*1} \). For a certificate of inconsistency take all \( u_{i1} = 0 \) except \( u_{r^*1} = a_{r^*1} > 0 \) and \( u_{s^*1} = -a_{s^*1} > 0 \). Multiplying we have \( a_{s^*1}(a_{r^*1}x_1 \leq b_{r^*}) \) and \( -a_{r^*1}(a_{s^*1}x_1 \leq b_{s^*}) \). Combining these we get \( 0 \leq (b_{r^*}a_{s^*1} - b_{s^*}a_{r^*1}) < 0 \). The last \( < 0 \) follows from \( b_{r^*}/a_{s^*1} < b_{r^*}/a_{r^*1} \) (again noting that the direction of the inequality changes as \( s^* \leq 0 \)).
(iii) If $L \cup N$ is empty the original system is $x_n \leq \frac{1}{a_{sn}} \left( b_s - \sum_{j=1}^{n-1} a_{sj} x_j \right)$ for $s \in U$. For a solution take $x_1 = x_2 = \cdots x_{n-1} = 0$ and $x_n = \min_{s \in U} (b_s/a_{sn})$. It is straightforward to check that this is a solution.

If $U \cup N$ is empty the original system is $\frac{1}{a_{rn}} \left( b_r - \sum_{j=1}^{n-1} a_{rj} x_j \right) \leq x_n$ for $r \in L$. For a solution take $x_1 = x_2 = \cdots x_{n-1} = 0$ and $x_n = \max_{r \in L} (b_r/a_{rn})$. It is straightforward to check that this is a solution.

(iv) If (3) is inconsistent with multipliers $u_{rs}$ for $r \in L$, $s \in U$ and $u_t$ for $t \in N$ construct a certificate $y$ of inconsistency for (1): For $t \in N$ let $y_t = u_t$. For $r \in L$ let $y_r = -\frac{1}{a_{rn}} \sum_{s \in U} u_{rs}$ and for $s \in U$ let $y_s = \frac{1}{a_{sn}} \sum_{r \in L} u_{rs}$. Since we started with a certificate for (3) we have that combining the inequalities

$$u_{rs} \left( \frac{1}{a_{rn}} \left( b_r - \sum_{j=1}^{n-1} a_{rj} x_j \right) \right) \leq \frac{1}{a_{sn}} \left( b_s - \sum_{j=1}^{n-1} a_{sj} x_j \right)$$

for $r \in L$, $s \in U$

$$u_t \left( \sum_{j=1}^{n-1} a_{tj} x_j \leq b_t \right) \text{ for } t \in N$$

yields $0 < b$ for some $b < 0$. For (2) which is a rearrangement of (1), combining

$$\left( -\frac{1}{a_{rn}} \sum_{s \in U} u_{rs} \right) \left( \frac{1}{a_{rn}} \left( b_r - \sum_{j=1}^{n-1} a_{rj} x_j \right) \right) \leq x_n$$

for $r \in L$

$$\left( \frac{1}{a_{sn}} \sum_{r \in L} u_{rs} \right) \left( x_n \leq \frac{1}{a_{sn}} \left( b_s - \sum_{j=1}^{n-1} a_{sj} x_j \right) \right)$$

for $s \in U$

$$(u_t) \sum_{j=1}^{n-1} a_{tj} x_j \leq b_t \text{ for } t \in N$$

(v) Given a solution $(x_1^*, x_2^*, \ldots, x_{n-1}^*)$ to (3) take $x_n^*$ to be any value in the interval

$$\left[ \max r \in L \frac{1}{a_{rn}} \left( b_r - \sum_{j=1}^{n-1} a_{rj} x_j \right), \min s \in U \frac{1}{a_{sn}} \left( b_s - \sum_{j=1}^{n-1} a_{sj} x_j \right) \right]$$

3. We will eliminate the variable $x_1$. Rewriting we get the following ‘equivalent’ systems (where equivalent means either both have solutions or both do not):

For (a)

$$3x_1 + 6x_2 \leq 12 \quad x_1 \leq 4 - 2x_2 \quad \Leftrightarrow \quad -2 + x_2 \leq x_1 \quad \Leftrightarrow \quad -2 + x_2 \leq 4 - 2x_2 \quad \Leftrightarrow \quad 3x_2 \leq 6$$

$$-x_1 + x_2 \leq 2 \quad \Leftrightarrow \quad -2 + x_2 \leq x_1 \quad \Leftrightarrow \quad 5 - 3x_2 \leq 4 - 2x_2 \quad \Leftrightarrow \quad -x_2 \leq -1$$

$$-2x_1 - 6x_2 \leq -10 \quad 5 - 3x_2 \leq x_1 \quad \Leftrightarrow \quad -2 + x_2 \leq x_1 \quad \Leftrightarrow \quad 5 - 3x_2 \leq 4 - 2x_2 \quad \Leftrightarrow \quad -x_2 \leq -1$$
The last system is $1 \leq x_2 \leq 2$. We can pick $x_2$ in this interval and use it to get a solution for $x_1$. Take $x_2 = 3/2$. (The arithmetic for taking $x_2 = 1$ or $x_2 = 2$ is simpler but we happen get only one possibility for $x_1$ so we take $x_2 = 3/2$ for better illustration.) Putting $x_2 = 3/2$ into the original system gives $x_1 \leq 4 - 2(3/2) = 1$, $-1/2 = -2 + 3/2 \leq x_1$, $1/2 = 5 - 3(3/2) \leq x_1$. So any $x_1$ in the interval $[1/2, 1]$ along with $x_2 = 3/2$ is a solution. In particular a solution is $x_1 = 1/2, x_2 = 3/2$.

For (b)

\[
\begin{align*}
3x_1 + 6x_2 & \leq 12 \\
x_1 & \leq 4 - 2x_2 \\
-x_1 + x_2 & \leq 2 \\
7 - 3x_2 & \leq x_1 \\
-2x_1 - 6x_2 & \leq -14 \\
4 - 2x_2 & \leq -3
\end{align*}
\]

The last system is inconsistent as seen by the multipliers $(1, 3)$. (Multiply the first inequality by 1 and the second by 3 and combine to get $0 \leq -3$.) This correspond to multipliers $(1, 3)$ in the third system, multipliers $(4, 1, 3)$ in the second system and $(4/3, 1, 3/2)$ in the original, yielding the same inconsistency $0 \leq -3$.

4. Use induction on the size of the submatrix. A 1 by 1 submatrix is an entry of $A$ which is in $\{0, -1, +1\}$ and the determinant is trivially the same. Given a $k$ by $k$ submatrix $B$, consider three cases: (a) Some column has all zeroes; (b) some column has exactly one +1 or exactly one −1; (c) not (a) or (b).

Then, if:

(a) $\det(B) = 0$ as it has a column of all zeroes.

(b) Say column $s$ has exactly one non-zero and the non-zero is in row $r$. Then expanding the determinant we get $\det(B) = \sum_{i=1}^{k} a_{is} C(ij) = a_{rs} C(rs)$. This follows since $a_{is} = 0$ except when $i = r$. Finally, $a_{rs} = \pm 1$ and $C(rs)$ is $\pm 1$ times a smaller submatrix of $A$, which by induction is in $\{0, -1, +1\}$. Thus $\det(B)$ is a product of three terms all in $\{0, -1, +1\}$ and hence $\det(B) \in \{0, -1, +1\}$.

(c) In this case all columns of $B$ have exactly one +1 and exactly one −1. Summing all rows gives a row of all zeroes and hence $\det(B) = 0$.

5. (See also the hand drawn diagrams.) The digraph $D'$ has the following arcs with weight $-\epsilon$:

$\{l_v r_u, l_w r_u, l_y r_u, l_w r_v, l_w r_x, l_y r_x, l_w r_y\}$ and the following two sets each with arc weights $0$: $\{r_x l_u, r_u l_x, r_x l_v, r_v l_x, r_v l_y, r_v l_w\}$ and $\{r_u l_u, r_v l_w, r_w l_x, r_x l_y\}$. A potential (found by inspection) is $p(r_u) = p(l_u) = p(l_x) = -3\epsilon$, $p(l_v) = p(r_x) = -2\epsilon, p(r_y) = p(l_y) = p(r_v) = -\epsilon, p(w) = p(w) = 0$. After scaling by $\epsilon$ this gives the interval representation $I_u = [-3, -3], I_v = [-2, -1], I_w = [0, 0], I_x = [-3, -2], I_y = [-1, -1]$.

6. (See also the attached hand drawn diagrams.) We get the a digraph with vertices $\{0, 1, \ldots, 7\}$ and arcs $ij$ for all $1 \leq i < j \leq 7$ with lower bound 0 and upper bound 1 and the following arcs and bounds: 01 with $l(01) = u(01) = 01 - 1 + 1 = 6, 02 with l(02) = u(02) = 5 - 2 + 1 = 4, 03 with l(03) = u(03) = 5 - 3 + 1 = 3, 04 with l(04) = u(04) = 3 - 4 + 1 = 0, 05 with l(05) = u(05) = 1 - 5 + 1 = -3, 06 with l(06) = u(06) = 1 - 6 + 1 = -4, 07 with l(07) = u(07) = 0 - 7 + 1 = -6. The set $\{4, 5, 6, 7\}$ has sum of lower bounds on arcs leaving equal to 0 (in fact it has no arcs leaving) and sum of entering upper bounds equal to $12 + (0 + (-3) + (-4) + (-6)) = -1$. (The 12 comes from upper bounds of 1 on the 12 arcs from $\{1, 2, 3\}$ to $\{4, 5, 6, 7\}$ and the remaining terms are from upper bounds on the arcs 04, 05, 06, 07. Thus at most −1 unit of flow can enter $\{4, 5, 6, 7\}$ and
at least 0 units leave. This violates the necessary condition for flows. Then, in the score sequence $s_4, s_5, s_6, s_7$ violate the necessary condition with $5 = 3 + 1 + 1 + 0 = s_4 + s_5 + s_6 + s_7 < 6 = \binom{4}{2}$.
(Note \{5, 6, 7\} also violate the flow condition and $s_5, s_6, s_7$ violate the score condition.)