Prove that a matching $M$ in a graph $G$ is a maximum matching if and only if $G$ has no $M$-augmenting path.

Prove the marriage theorem: Every non-trivial regular bipartite graph has a perfect matching. (You may use Hall’s Theorem or the Konig-Egevary Theorem.)

(a) State Tutte’s Theorem for the existence of a perfect matching
(b) State the matching duality theorem
(c) The theorems in parts (a) and (b) are equivalent. Pick one implication and prove it. That is, either prove that Tutte’s theorem implies matching duality or prove that matching duality implies Tutte’s theorem.

Assume that a graph $G$ satisfies Tutte’s condition for a perfect matching: $\text{odd}(G - S) \leq |S|$ for all subsets $S$ of the vertex set $V$. Let $S'$ be a maximal set for which equality holds. (It holds for $\emptyset$ so it makes sense to talk about a maximal set. This also implies that the number of vertices is even.) Let $C_i$ be a component of $G - S'$. Prove that $C_i$ is factor critical. That is, prove that $C_i - x$ has a perfect matching for every vertex $x$ in $C_i$. You may use Tutte’s Theorem.

Prove that Christofide’s heuristic for a Hamiltonian in a graph with weights given by a Euclidean distance (in particular the weights satisfy the triangle inequality $w(xz) \leq w(xy) + w(yz)$) finds a Hamiltonian tour that has weight no more than 50% greater than that of a minimum weight tour. Recall that this heuristic first finds a minimum weight spanning tree, then finds a minimum cost perfect matching between the vertices with odd degree in the tree, duplicates the matched edges, finds an Eulerian circuit in the resulting graph and then shortcuts this to a Hamiltonian tour.

Let $G$ be a graph of even order having no $K_{1,r+1}$ as an induced subgraph. Suppose also that $G - S$ is connected whenever $S$ is a set of fewer than $r$ vertices. For $r \geq 1$, prove that $G$ has a perfect matching.

Prove Gallai’s identities: In a graph $G$ without isolated vertices $\alpha(G) + \beta(G) = n(G)$ and $\alpha'(G) + \beta'(G) = n(G)$. 
8: For hereditary systems prove the equivalence of the following matroid axioms:
(a) \((G) \Rightarrow (I)\)
Here \((G)\) is the greedy algorithm axiom: for each nonnegative weight function on \(E\),
the greedy algorithm selects an independent set of maximum total weight and \((I)\) is
augmentation: if \(I_1\) and \(I_2\) are independent with \(|I_1| > |I_2|\), then \(I_1 + e\) is independent
for some \(e \in I_2 - I_1\).

(b) \((J) \Rightarrow (C)\)
Here \((J)\) is the induced circuits axiom: if \(I\) is independent then \(I + e\) contains at most
one circuit and \((C)\) is weak elimination: for distinct circuits \(C_1, C_2\) and \(x \in C_1 \cap C_2\)
there is another circuit contained in \((C_1 \cup C_2) - x\).

9: (a) Prove \(\leq\) in the matroid intersection formula. That is, given two matroids \(M_1, M_2\)
on a set \(E\) with independent sets \(I_1\) and \(I_2\) respectively, the size \(|I|\) of a largest common
independent set satisfies \(\max\{|I| : I \in I_1 \cap I_2\} \leq \min_{X \subseteq E} \{r_1(X) + r_2(X)\}\) where \(r_i\) are
the rank functions.

(b) Prove that the rank function \(r^*\) of the dual of a matroid \(M\) on \(E\) is defined by
\(r^*(X) = |X| - (r(E) - r(X))\).

10: Recall that a rainbow spanning tree in an edge colored graph is a spanning tree
for which every edge has a different color. Prove that an edge colored graph has a
rainbow spanning tree if and only if for every \(k\) and every partition \(V_1, V_2, \ldots, V_k\) of
the vertex set of \(G\) into \(k\) parts the number of colors appearing on edges between
parts is at least \(k - 1\).