Example: Cola

- Entire cola industry produces two types of colas.
- Given: last purchased was cola1, there is 90% chance next purchase is cola1.
- Given: last purchase was cola2, there is 80% chance next purchase is cola2.
- Draw the transition diagram.

Cola continued ...

- If Hermione is currently a cola2 purchaser, what is the probability that she will purchase cola1 two purchases from now?
- If Harry is currently a cola1 purchaser, what is the probability that he will purchase cola1 three purchases from now?
n-Step Transition Probabilities

- States = \{1,2,...,s\}
- 2-Step Transition Probabilities
  \[
  p_{ij}(2) = \sum_{k=1}^{s} p_{ik} p_{kj}
  \]
- By extension,
  \[
  p_{ij}(n) = [P^n]_{ij}
  \]

Cola continued ...

Currently, 60% of Lehigh students drink cola1, and 40% cola2.

What fraction of purchasers will be drinking cola1 three purchases from now?
n-Step Transition Probabilities

- Given initial probabilities
  \[ q = [q_1, q_2, \ldots, q_s] \]
- Probability of being in state \( j \) at time \( n \)
  \[
  \sum_{i=1}^{s} q_i P_{ij}(n) = \sum_{i=1}^{s} q_i [P^n]_{ij} = q[P^n]_j
  \]

Cola example: 1-Step Tran Prob

\[
P = \begin{bmatrix}
0.90 & 0.10 \\
0.20 & 0.80 
\end{bmatrix}
\]
Cola example: 2-Step Tran Prob

\[ P^2 = \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix} \]

Cola example: 3-Step Tran Prob

\[ P^3 = \begin{bmatrix} 0.78 & 0.22 \\ 0.44 & 0.56 \end{bmatrix} \]
Cola example: 4-Step Tran Prob

\[ P^4 = \begin{bmatrix} 0.75 & 0.25 \\ 0.51 & 0.49 \end{bmatrix} \]

Cola example: 5-Step Tran Prob

\[ P^5 = \begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix} \]
Cola example: 10-step tran prob

\[ P^{10} = \begin{bmatrix} 0.68 & 0.32 \\ 0.65 & 0.35 \end{bmatrix} \]

Cola example: 20-step tran prob

\[ P^{20} = \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix} \]
Cola example: 30-step tran prob

\[ P^{30} = \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix} \]

Cola example: 40-step tran prob

\[ P^{40} = \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix} \]
Cola example: steady state

Steady State Distribution
a.k.a. Equilibrium Distribution

\[
\lim_{n \to \infty} P^n = \begin{bmatrix}
0.67 & 0.33 \\
0.67 & 0.33
\end{bmatrix}
\]

\[\pi_1 = 0.67, \pi_2 = 0.33\]

Steady State Probabilities

- Let P be the transition matrix of s-state ergodic Markov Chain
- There exists a vector \( \pi = [\pi_1, \pi_2, \ldots, \pi_s] \) such that

\[
\lim_{n \to \infty} P^n = \begin{bmatrix}
\pi_1 & \pi_2 & \cdots & \pi_s \\
\pi_1 & \pi_2 & \cdots & \pi_s \\
\vdots & \vdots & \ddots & \vdots \\
\pi_1 & \pi_2 & \cdots & \pi_s
\end{bmatrix}
\]
Cola continued ...

Write the system of equations to solve for the steady-state probabilities
- Solve for $[\pi_1, \pi_2]$ 

Solving for Steady State
- To find the steady state probabilities, solve the following system of equations:

\[ \pi = \pi \cdot P \]

\[ \sum_{i=1}^{s} \pi_i = 1 \]
More on the Cola Example

- 1 cola purchase/customer/week.
- 100 million cola customers.
- Production cost $1 per unit.
- Selling price $2 per unit.
- For $500M per year, an advertising firm guarantees to decrease from 10% to 5% the fraction of cola1 customers who switch to cola2 after one purchase.
- Should cola1 hire the firm?