Moments for the parabolic Anderson model: on a result by Hu and Nualart.

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Abstract

We consider the Parabolic Anderson Model \( \partial_t u = Lu + u \dot{W}, \) where \( L \) is the generator of a Lévy Process and \( \dot{W} \) is a white noise in time, possibly correlated in space. We present an alternate proof and an extension to a result by Hu and Nualart ([14]) giving explicit expressions for moments of the solution. We do not consider a Feynman-Kac representation, but rather make a recursive use of Itô’s formula.

Keywords and phrases. Stochastic partial differential equations, moment formulae, Parabolic Anderson Model.

1 Introduction

We consider the Parabolic Anderson Model, namely the Stochastic Partial Differential Equation given by

\[
\frac{\partial u}{\partial t}(t,x) = Lu(t,x) + u(t,x)\dot{W}(t,x), \quad t > 0, x \in \mathbb{R}^d,
\]

where \( L \) is the generator of a real-valued Lévy Process \( (X_t)_{t \geq 0} \). The noise \( \dot{W}(t,x) \) is white in time and possibly correlated in space, with covariance function informally given by

\[
\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s)f(x-y),
\]

where \( f \) is a (possibly generalized) non-negative symmetric function on \( \mathbb{R}^d \setminus \{0\} \). We consider a non-random, bounded and measurable initial condition \( u_0 : \mathbb{R}^d \to \mathbb{R} \).

It is well-known that this equation admits a random-field solution \( \{u(t,x) : t > 0, x \in \mathbb{R}^d\} \) such that \( \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E}[u(t,x)^2] < \infty \), provided that

\[
\int_{\mathbb{R}} \frac{\mu(d\xi)}{1 + 2 \Re \Psi(\xi)} < \infty,
\]

where \( \Psi(\xi) = \mathbb{E}[e^{i\xi X_1}] \) is the Lévy exponent of \( X \) and \( \mu(d\xi) \) is the Fourier transform of \( f \), usually called the spectral measure of the noise.

Equation (1.1) arises in different contexts. It is the continuous form of the Parabolic Anderson Model studied by Carmona and Molchanov ([3]). We also would like to mention the major role played by

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equation (1.1) in the study of the so-called KPZ equation of physics ([15]). In particular, intermittency and chaos properties of the solution to (1.1) are of importance. We refer to [12], [6], [7] for more details about these properties.

The purpose of this paper is to obtain explicit expressions for the moments of the solution \( u \) to (1.1). Such results have already been obtained in the particular case where \( L = \Delta \) is the Laplacian; the generator of Brownian Motion and \( f = \delta_0 \) the Dirac measure in [1]. A similar formula has been developed in [14] in the case where \( L = \Delta \) and the noise is fractional in time, white in space. Both [1] and [14] use a Feynman-Kac type representation of the solution to deduce the behavior of the moments. In this paper, we will follow a different approach and obtain results through a direct computation based on an iterative use of Itô’s formula. We will only consider a white noise in time, but we will not restrict our attention to the case where \( L \) is the Laplacian only. We would like to point out that the results of this paper are used in the proofs of results of [7] related to the chaotic behavior of the Parabolic Anderson Model with spatially correlated noise.

It is also believed that this method of proof could lead to moment formulae for SPDEs of parabolic type with a general non-linearity \( \sigma(u) \), if \( \sigma \) is a sufficiently nice function. This is subject to ongoing research.

Section 2 below is a short reminder about the Parabolic Anderson Model, existence, uniqueness and series representation of the solution. We also recall a few results about Lévy processes. The moment formulae of Theorem 3.1 and its Corollary 3.2 constitute the main results of this paper. They are stated and proved in Section 3 in the case where the spatial covariance of the noise is a measurable function. Section 4 is devoted to the case of space-time white noise (Theorem 4.1).

# 2 Parabolic Anderson model

In this section, we are going to remind a few known results about the Parabolic Anderson model and Lévy processes in general. Let’s start by setting the framework in which we are going to work.

We remind that \( L \) denotes the generator of a symmetric Lévy process \((X_t)_{t \geq 0} \) on \( \mathbb{R}^d \). For instance, one can consider \( L = \Delta \), the Laplacian operator : it is the generator of Brownian Motion \((B_t)_{t \geq 0}\).

Let’s assume first that the spatial covariance is a measurable function \( f \) of \((X_t)_{t \geq 0}\). We ask \( f \) to be locally integrable around 0. Let \( D(\mathbb{R}^{d+1}) \) be the space of \( C^\infty \) functions with compact support on \( \mathbb{R}^{d+1} \). Let \( W = \{ W(\phi), \phi \in D(\mathbb{R}^{d+1}) \} \) be a centered Gaussian noise with covariance functional given by

\[
\mathbb{E}[W(\phi)W(\psi)] = \int_0^\infty dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \phi(t,x)f(x-y)\psi(t,y).
\]

This noise can be extended to a worthy martingale measure in the sense of Walsh (see [9] and [16] for details). Hence, by the theory of Dulan [8], we can define stochastic integrals with respect to the noise \( W \).

We notice that the covariance functional can be written as

\[
\mathbb{E}[W(\phi)W(\psi)] = \int_0^\infty dt \int_{\mathbb{R}^d} dx f(x)(\phi(t,\cdot) * \tilde{\psi}(t,\cdot))(x),
\]  

(2.1)

where \( * \) denotes spatial convolution and \( \tilde{\psi}(t,x) = \psi(t,-x) \) for all \( x \in \mathbb{R}^d \). Using the representation (2.1), we can then define the noise \( W \) in the case where \( f \) is a finite measure, using

\[
\mathbb{E}[W(\phi)W(\psi)] = \int_0^\infty dt \int_{\mathbb{R}^d} f(dx)(\phi(t,\cdot) * \tilde{\psi}(t,\cdot))(x).
\]

We will consider a random-field solution to (1.1), i.e. a jointly measurable stochastic process \((u(t,x))_{t \geq 0, x \in \mathbb{R}^d}\) such that

\[
\sup_{0 \leq t \leq T, x \in \mathbb{R}^d} \mathbb{E}[u(t,x)^2] < \infty,
\]

for every fixed \( T > 0 \) and satisfying the mild-form equation

\[
u(t,x) = (\tilde{\mu}_t * u_0)(x) + \int_0^t \int_{\mathbb{R}^d} \mu_{t-s}(y-x)u(s,y)W(ds,dy),
\]

where...
where \( p_t \) is the fundamental solution of the homogeneous equation \( \frac{\partial}{\partial t} p_t(x, t) = L p_t(x, t) \) for all \( x \in \mathbb{R}^d \) and the stochastic integral is taken in the sense of Walsh [16].

Notice that since \( L \) is the generator of a Lévy process \( X_t \), the fundamental solution \( p_t \) corresponds to the law of \( X_t \). We assume that the Lévy process \( X \) admits densities, so that \( p_t \) is a well-defined measurable function. An extension to the case where \( p_t \) is a measure would require to work with integrals in the spectral domain. Such an extension is not discussed here, but an insight can be found in [5, Section 6].

We have the following existence and uniqueness result.

**Proposition 2.1.** Let \( \mu \) be the spectral measure of the noise and \( \Psi \) the Lévy exponent of the process generated by \( L \). If
\[
\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + 2 \Re \Psi(\xi)} < \infty, \tag{2.2}
\]
then there exists a unique random-field solution to (1.1).

For a proof of this result, we refer to [8], [10] and [16].

In the case where \( f \) is a measurable function, condition (2.2) implies that
\[
E^X \left[ \int_0^t ds f(X_s^{(1)} - X_s^{(2)}) \right] = \int_0^t ds \int_{\mathbb{R}^d} p_s(x) f(x - y) p_s(y) < \infty,
\]
where \( X^{(1)} \) and \( X^{(2)} \) are two independent copies of the Lévy process generated by \( L \) and \( E^X \) is the expectation with respect to these processes. This implies that the additive functional \( A_t^f := \int_0^t f(X_s) ds \) (associated to the function \( f \) of the Lévy process \( X := X^{(1)} - X^{(2)} \)) is well-defined. Since the Lévy process \( X \) is symmetric, its Lévy exponent is real given by 2 \Re \Psi(\xi). A direct computation allows to prove that \( A_t^f \in L^p(\Omega) \) for all \( p \geq 1 \).

In the case where \( f = \delta_0 \), we have \( \mu(d\xi) = d\xi \) and (2.2) becomes a standard condition for existence of local times for the Lévy process \( X \). See [2, Theorem 1, p.126]. We denote by \( (L^x_t, t > 0, x \in \mathbb{R}^d) \) the local times of the symmetrized process \( X \). We informally have
\[
L^x_t = \int_0^t \delta_x(X_s^{(1)} - X_s^{(2)}) \, ds.
\]

We also refer to the paper by Foondun, Khoshnevisan and Nualart [13] on the connection between existence of local times for Lévy processes and existence of solutions to SPDE’s driven by space-time white noise.

We would like to point out the fact that the proof of Proposition 2.1 shows that the solution \( u \) is given as the limit of a Picard iteration scheme. Namely, we set \( u_0(t, x) = (\tilde{p}_t * u_0)(x) \) for all \( t \geq 0, x \in \mathbb{R}^d \). Then, for all \( n \geq 1 \), we define recursively a sequence of stochastic process \( (u_n(t, x))_{t \geq 0, x \in \mathbb{R}^d} \) by
\[
u_n(t, x) = u_0(t, x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y - x) u_{n-1}(s, y) W(ds, dy).
\]

Then, \( u(t, x) = \lim_{n \to \infty} u_n(t, x) \) in \( L^2(\Omega) \), uniformly over \( t \in [0, T] \) and \( x \in \mathbb{R}^d \). One can show that the convergence also occurs in \( L^p(\Omega) \) for all \( p > 2 \) (see [8] for details).

We are now going to show that the solution \( u \) to (1.1) can be written as a series of iterated integrals. This expansion, which corresponds to the Wiener-chaos expansion of \( u \), will be the main tool used in order to obtain explicit expressions for the moments of \( u \). This result is a direct consequence of the existence result and already appears in different contexts, among which [11] and [5] and the wide litterature of Malliavin Calculus.

**Proposition 2.2.** Under the assumptions above, we can show that the solution \( u \) to (1.1) is given by
\[
u(t, x) = \sum_{n=0}^{\infty} v_n(t, x), \quad a.s.
\]
where \( v_0 = u_0 \) is deterministic and the processes \( (v_n(t,x) : t \geq 0, x \in \mathbb{R}^d) \) are defined recursively by

\[
v_n(t,x) = \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y-x) v_{n-1}(s,y) W(ds,dy).
\]

(2.3)

The series is convergent in \( L^2(\Omega) \).

**Proof.** Fix \( t \geq 0 \) and \( x \in \mathbb{R}^d \). Using the Picard iteration scheme, we have

\[
u_n(t,x) = u_0(t,x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y-x) u_{n-1}(s,y) W(ds,dy)
\]

As a consequence,

\[
u_{n+1}(t,x) - u_n(t,x) = \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y-x) (u_n(s,y) - u_{n-1}(s,y)) W(ds,dy).
\]

Hence, we set \( v_n(t,x) = u_n(t,x) - u_{n-1}(t,x) \) for all \( n \geq 1 \). Then, (2.3) is satisfied for \( n \geq 2 \) and we have

\[
u_n(t,x) = u_0(t,x) + \sum_{j=1}^n (u_j(t,x) - u_{j-1}(t,x)) = u_0(t,x) + \sum_{j=1}^n v_j(t,x),
\]

provided we set \( v_0 = u_0 \), which implies that (2.3) is satisfied for \( n = 1 \) as well. Finally, taking the limit as \( n \) goes to \( \infty \) establishes the result. \( \blacksquare \)

**Remark 2.3.**

- The process \( v_n \) is a \( n \)-times iterated stochastic integral. As a consequence, 2.2 shows that \( u \) is given as a series of iterated stochastic integrals. This also helps finding moments if one considers a hyperbolic equation, see [5] and [4].

- The expansion obtained in Proposition 2.2 corresponds to the Wiener-chaos expansion of the solution \( u(t,x) \) to (1.1). Indeed, since \( v_n \) is an \( n \)-th iterated stochastic integral, it belongs to the \( n \)-th Wiener chaos.

### 3 Moment formula for a measurable covariance.

In this section, we will consider the covariance \( f : \mathbb{R}^d \setminus \{0\} \) to be a measurable function, locally integrable on \( \mathbb{R}^d \setminus \{0\} \). For instance, one can consider bounded functions, such as \( f(x) = e^{-\|x\|} \) or functions which are unbounded at \( x = 0 \), such as the Riesz kernels, given by \( f(x) = \|x\|^{-\alpha} \) for \( 0 < \alpha < d \wedge 2 \).

We are going to establish explicit formulas for moments of the processes \( v_n \). From this, we will use the series expansion stated in Proposition 2.2 to deduce a formula for the moments of the solution \( u \).

First of all, we would like to point out that it is possible to write explicit expressions for moments of \( v_n \) in terms of the Fourier transform of \( p_t \), since \( \mathcal{F}_{p_t}(\xi) = e^{i\varphi(\xi)} \). This is not precisely the formula that we would like to obtain here. For details, we refer to [4, Lemma 5.2], which mainly concerns the hyperbolic equation but that applies without restrictions to the parabolic case.

We are going to use similar techniques as in the proof of [4, Lemma 5.2] but we will not write the integrals in spectral form and rather express those as functionals of the Lévy process associated to the operator \( L \).

We are now ready to state our results. Theorem 3.1 below states the formula for the expectation of the product of \( u \) taken at different points in space. Corollary 3.2 gives the formula for a moment of order \( p \geq 1 \).

**Theorem 3.1.** Let \( u \) denote the solution of (1.1) with operator \( L \) as given in Proposition 2.2. Let \( t \geq 0 \) and \( x_1, \ldots, x_p \in \mathbb{R}^d \). Then,

\[
E \left[ \prod_{j=1}^p u(t,x_j) \right] = E_{x_1,\ldots,x_p}^X \left[ \prod_{k=1}^p u_0(X_t^{(k)}) \times \exp \left( \sum_{i,k=1}^p \int_0^t dr f(X_r^{(j)} - X_r^{(k)}) \right) \right].
\]

(3.1)
where the processes \((X^{(j)}_t)_{t \geq 0} \) \((j = 1, \ldots, p)\) are \(p\) independent copies of the Lévy process generated by the operator \(L\) and \(E^X_{x_1}, \ldots, x_p\) is the expectation with respect to the law of these processes conditioned such that \(X^{(1)}_0 = x_1, \ldots, X^{(p)}_0 = x_p\).

**Corollary 3.2.** Let \(u\) denote the solution of (1.1) with operator \(L\) as given in Proposition 2.2. Let \(t \geq 0\) and \(x \in \mathbb{R}^d\). Then,

\[
E[u(t,x)^p] = E^X_0 \left[ \prod_{i=1}^p u_0(X^{(i)}_t) \times \exp \left( \sum_{j \neq k} \int_0^t dr \, f(X^{(j)}_r - X^{(k)}_r) \right) \right],
\]

where the processes \((X^{(j)}_t)_{t \geq 0} \) \((j = 1, \ldots, p)\) are \(p\) independent copies of the Lévy process generated by the operator \(L\) and \(E^X_0\) is the expectation with respect to the law of these processes conditioned such that \(X^{(1)}_0 = \cdots = X^{(p)}_0 = x\).

In order to prove Theorem 3.1, we need to go through a series of partial results, the most important of which is Proposition 3.7 below. First of all, let us define the following processes. For \(x \in \mathbb{R}^d\), then, for \(s \neq t\), set

\[
w_0(s; t, x) := v_0(t, x) = u_0(t, x) = (\tilde{p}_t * u_0)(x).
\]

Then, for \(n \geq 1, t \in \mathbb{R}^d\), \(t \geq 0\) and \(s \leq t\), we set

\[
w_n(s; t, x) := \int_0^s \int_{\mathbb{R}^d} p_{t-r}(y-x)v_{n-1}(r,y)W(dr, dy).
\]

Obviously, we have \(v_n(t, x) = w_n(t; t, x)\). The point of this construction is that, by the definition of Walsh stochastic integrals, the processes \(s \mapsto w_n(s; t, x)\) are martingales for each fixed \(t \geq 0, x \in \mathbb{R}^d\) and \(n \in \mathbb{N}\). We start with a lemma.

**Lemma 3.3.** Let \((X_t)_{t \geq 0}\) be the Lévy process with generator \(L\), then

\[
v_0(t, x + y) = E^X_0[u_0(x + X_t)],
\]

where \(E^X_0\) denotes the expectation with respect to the law of \(X\) under the condition \(X_0 = y\).

**Proof.** Recall that we assume that the process \(X\) admits densities. We have

\[
v_0(t, x + y) = \int_{\mathbb{R}^d} dz \, p_t(z - x - y)u_0(z) = \int_{\mathbb{R}^d} dz \, p_t(z - y)u_0(x + z) = E^X_0[u_0(x + X_t)],
\]

as \(p_t\) is the density of the process \(X_t\) starting at 0. \(\blacksquare\)

We remind the Markov property for Lévy processes in the form that we are going to use in this paper.

**Proposition 3.4** (Markov Property). Let \((X_t)_{t \geq 0}\) be a Lévy process with values in \(\mathbb{R}^d\) and \(g : \mathbb{R}^d \to \mathbb{R}\) a bounded continuous function. Let \((F_t)_{t \geq 0}\) be the filtration generated by \(X\). Then, the following Markov property holds

\[
E^X_{X_{t-s}}[g(X_s)] = E^X_0[g(X_t) \mid F_{t-s}],
\]

where \(0 \leq s \leq t\) and \(E^X_0\) denotes the expectation with respect to the law of \(X\) under the condition \(X_0 = y\).

We refer to [2] for more details about the Markov property for Lévy processes. In order to prove Proposition 3.7 below, we will need a Markov property stated in a slightly different way. Namely, we will need to consider a functional of two independent processes conditioned at two different times.

**Lemma 3.5** (Markov Property). Let \((X^{(1)}_t)_{t \geq 0}\) and \((X^{(2)}_t)_{t \geq 0}\) be two independent Lévy processes with values in \(\mathbb{R}^d\). Let \((F^{(i)}_t)_{t \geq 0}\) be the filtration generated by \(X^{(i)}\) \((i = 1, 2)\). Let \(g : (\mathbb{R}^d)^n \times (\mathbb{R}^d)^m \to \mathbb{R}\) be a bounded continuous function. Then, for all \(r_1, r_2, t_1, \ldots, t_n, s_1, \ldots, s_m \in \mathbb{R}_+\), we have

\[
E^{X^{(1)}_{r_1}, X^{(2)}_{r_2}}_0[g(X^{(1)}_{t_1}, \ldots, X^{(1)}_{t_n}, X^{(2)}_{s_1}, \ldots, X^{(2)}_{s_m})] = E^X_{0,0}[g(X^{(1)}_{t_1+r_1}, \ldots, X^{(1)}_{t_n+r_1}, X^{(2)}_{s_1+r_2}, \ldots, X^{(2)}_{s_m+r_2}) \mid F^{(1)}_{r_1} \otimes F^{(2)}_{r_2}],
\]

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where \( E_{x_1,x_2}^X \) denotes the expectation with respect to the joint law of \( X^{(1)}, X^{(2)} \) under the condition \( X_0^{(1)} = x_1, X_0^{(2)} = x_2 \).

**Proof.** Without loss of generality, we can show the result when \( n = m = 1 \). The general case works the same way. As the processes \( X^{(1)} \) and \( X^{(2)} \) are independent,

\[
E^X[Y \mid F_{r_1}^{(1)} \otimes F_{r_2}^{(2)}] = E^{X^{(1)}}[E^{X^{(2)}}[Y \mid F_{r_2}^{(2)}] \mid F_{r_1}^{(1)}].
\]

Hence, by the Markov property for \( X^{(2)} \) first and then for \( X^{(1)} \), we have

\[
E_{0,0}^X \left[ g(X_{t+r_1}^{(1)}, X_{s+r_2}^{(2)}) \mid F_{r_1}^{(1)} \otimes F_{r_2}^{(2)} \right] = E_{0}^{X^{(1)}} \left[ E_{0}^{X^{(2)}} \left[ g(X_{t+r_1}^{(1)}, X_{s+r_2}^{(2)}) \mid F_{r_2}^{(2)} \right] \mid F_{r_1}^{(1)} \right]
\]

Remark 3.6. We notice that the use of the Markov property above is completely formal when the function \( g \) is bounded. In the proofs below, we may want to consider the Markov property for a function that is unbounded at \( x = 0 \). We can then consider the function \( g_n(x) := g(x) \wedge n \), apply the Markov property and then pass to the limit as \( n \to \infty \). This procedure will be valid in the proofs below since \( E{\left[ \int_0^t f(X_s) \, ds \right]}^n < \infty \) for all \( p \geq 1 \) by the assumption of the existence result (Proposition 2.1). We will use this procedure throughout the proofs below without developing the details.

We can now turn to the first intermediate result of this section, a moment formula for the processes \((w_n)_{n \in \mathbb{N}}\).

**Proposition 3.7.** Fix an integer \( m \geq 1 \) and \( n_1, \ldots, n_m \) positive integers. Fix \( x_1, \ldots, x_m \in \mathbb{R}^d \) and \( t_1, \ldots, t_m \in \mathbb{R}_+ \). Let \( s \geq 0 \) such that \( s \leq t_1 \wedge \cdots \wedge t_m \). Set \( N = \sum_{i=1}^m n_i \).

- If \( N \) is odd, then \( E \left[ \prod_{j=1}^m w_{n_j}(s; t_j, x_j) \right] = 0 \).
- If \( N \) is even, set \( n = \frac{N}{2} \). Then,

\[
E \left[ \prod_{j=1}^m w_{n_j}(s; t_j, x_j) \right] = \sum_{\mathcal{P}(n_1, \ldots, n_m)} E^X \left[ \prod_{j=1}^m u_0(x_j + X_t^{(j)}) \right] 
\]

\[
\times \int_0^s dr_1 \cdots \int_0^{r_{n-1}} dr_n \prod_{i=1}^n f(x_{p_i} + X_{t_{p_i} - r_i}^{(p_i)} - x_{q_i} - X_{t_{q_i} - r_i}^{(q_i)}),
\]

where \( \mathcal{P}(n_1, \ldots, n_m) \) is the set of all orderings in pairs of different integers of the set \( \mathcal{N}(n_1, \ldots, n_m) = \{1, \ldots , 1, 2, \ldots , 2, \ldots , m, \ldots , m\} \) with \( n_i \) occurrences of the integer \( i \). More precisely,

\[
\mathcal{P}(n_1, \ldots, n_m) = \left\{ \{(p_1, q_1), \ldots, (p_n, q_n)\} : \{p_1, q_1, \ldots, p_n, q_n\} = \mathcal{N}(n_1, \ldots, n_m) \text{ and } p_j \neq q_j, \forall j = 1, \ldots, n \right\}.
\]

The processes \( (X_t^{(j)})_{t \geq 0} (j = 1, \ldots, m) \) are \( m \) independent copies of the Lévy process generated by \( L \), starting at 0 and \( E^X \) denotes expectation with respect to the joint law of these processes. If the set \( \mathcal{P}(n_1, \ldots, n_m) = \emptyset \), then the expectation vanishes.
Remark 3.8. Proposition 3.7 is the main ingredient in order to obtain explicit expression for moments. The proof looks technical, but the idea is rather simple. It will be done by induction, first on the number $m$ of terms in the product and, second, on the total order $N$ of the terms. A recursive use of Itô’s formula allows to reduce the order of the terms in the product. Then, properties of Walsh integrals and the Markov property for Lévy processes (Proposition 3.4) allow to compute explicitly the moments considered. The recursive use of Itô’s formula is similar to [5] (see also [4]), where the spectral representation of $p$ is used. Here, we use the Lévy process $X$ to express the integrals as expectations of additive functionals of the symmetrized Lévy process $X$.

**Proof.** Let us start the proof with the case $m = 1$ which is handled as a particular case. In that case, by the martingale property of the stochastic integral, we have $E[w_{n_1}(s; t, x)] = 0$ for all $x \in \mathbb{R}^d$, $0 \leq s \leq t$ and this proves the result if $n_1$ is odd. If $n_1$ is even, the left-hand side of (3.4) vanishes. Moreover, for $n_1 \in \mathbb{N}$, the set $\mathcal{N}(n_1)$ is formed with $n_1$ occurrences of the integer 1. It is not possible to find any pairing of different integers from this set and, hence, $\mathcal{P}(n_1) = 0$ and the right-hand side of (3.4) vanishes as well.

We are now going to prove the result by induction on $m$. We first consider the case $m = 2$, for which we are going to obtain the result by induction on $N = \sum_{i=1}^2 n_i$. The smallest possible value of $N$ is $N = 2$, with $n_1 = 1, n_2 = 1$. In that case, using the definition of $w_1$ and the properties of Walsh stochastic integrals, we have

$$E[w_1(s; t_1, x_1)w_1(s; t_2, x_2)] = \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \, p_{t_1-r}(y-x_1)f(y-z)p_{t_2-r}(z-x_2)v_0(r, y)v_0(r, z).$$

Using the fact that $p_1$ is the density of the Lévy process $(X_t)_{t \geq 0}$ starting at 0 and Lemma 3.3, we have

$$E[w_1(s; t_1, x_1)w_1(s; t_2, x_2)] = \int_0^s dr E^X[f(x_1 + X^{(1)}_{t_1-r} - x_2 - X^{(2)}_{t_2-r})v_0(r, x_1 + X^{(1)}_{t_1-r})v_0(r, x_2 + X^{(2)}_{t_2-r})] = \int_0^s dr E^X[f(x_1 + X^{(1)}_{t_1-r} - x_2 - X^{(2)}_{t_2-r})E^X[u_0(x_1 + X^{(1)}_{t_1-r})]E^X[u_0(x_2 + X^{(2)}_{t_2-r})].$$

Now, let $(\mathcal{F}^{(i)}_t)_{t \geq 0}$ denote the filtration generated by $X^{(i)}$. By the Markov properties (Theorem 3.4) for the processes $X^{(1)}$ and $X^{(2)}$ and their independence, we have

$$E[w_1(s; t_1, x_1)w_1(s; t_2, x_2)] = \int_0^s dr E^X[f(x_1 + X^{(1)}_{t_1-r} - x_2 - X^{(2)}_{t_2-r})E^X[u_0(x_1 + X^{(1)}_{t_1-r})]E^X[u_0(x_2 + X^{(2)}_{t_2-r})] = \int_0^s dr E^X[f(x_1 + X^{(1)}_{t_1-r} - x_2 - X^{(2)}_{t_2-r})u_0(x_1 + X^{(1)}_{t_1-r})u_0(x_2 + X^{(2)}_{t_2-r})].$$

because $f(x_1 + X^{(1)}_{t_1-r} - x_2 - X^{(2)}_{t_2-r})$ is $\mathcal{F}^{(1)}_{t_1-r} \otimes \mathcal{F}^{(2)}_{t_2-r}$ measurable. Finally, by Fubini’s theorem,

$$E[w_1(s; t_1, x_1)w_1(s; t_2, x_2)] = E^X[u_0(x_1 + X^{(1)}_{t_1-r})u_0(x_2 + X^{(2)}_{t_2-r})] \int_0^s dr f(x_1 + X^{(1)}_{t_1-r} - x_2 - X^{(2)}_{t_2-r}).$$

As $\mathcal{N}(1, 1) = \{1, 2\}$, $\mathcal{P}(1, 1) = \{(1, 2)\}$ and the result is proved for $m = 2, N = 2$.

Let us consider the case $N = 3$. We have $n_1 = 2, n_2 = 1$ (or the other way around). In that case, using the definitions of $w_1$ and $w_2$ and the properties of Walsh stochastic integrals, we have

$$E[w_1(s; t_1, x_1)w_1(s; t_2, x_2)] = \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \, p_{t_1-r}(y-x_1)f(y-z)p_{t_2-r}(z-x_2)E[w_1(r, y)]v_0(r, z) = 0,$$

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as $E[v_1(r, y)] = 0$ for all $r \geq 0$, $y \in \mathbb{R}^d$. The result is proved for $m = 2$, $N = 3$.

Now, let us assume that the result is proved for $m = 2$ and all $N \leq M - 1$. Let $n_1, n_2 \in \mathbb{N}$ such that $n_1 + n_2 = M$. Then, again by the definition of $w_n$ and the properties of Walsh stochastic integrals, we have

$$E[w_{n_1}(s; t_1, x_1)w_{n_2}(s; t_2, x_2)]$$

$$= \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \, p_{r-t}(y-x)f(y-z)p_{r-t}(z-x)E[v_{n_1-1}(r, y)v_{n_2-1}(r, z)].$$

If $M$ is odd, then $(n_1 - 1) + (n_2 - 1) = M - 2$ is odd as well and $E[v_{n_1-1}(r, y)v_{n_2-1}(r, z)] = 0$ for all $r \geq 0$, $y, z \in \mathbb{R}^d$ by the induction assumption applied with $t_1 = t_2 = r$, $x_1 = y$, $x_2 = z$.

If $M$ is even, then $(n_1 - 1) + (n_2 - 1) = M - 2$ is even, and we can write $E[v_{n_1-1}(r, y)v_{n_2-1}(r, z)]$ using the induction assumption with $t_1 = t_2 = r$, $x_1 = y$, $x_2 = z$. First, using the fact that $p_r$ is the density of $X$, we have

$$E[w_{n_1}(s; t_1, x_1)w_{n_2}(s; t_2, x_2)]$$

$$= \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \, p_{r-t}(y-x)f(y-z)p_{r-t}(z-x)E[v_{n_1-1}(r, y)v_{n_2-1}(r, z)]$$

$$= \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \, p_{r-t}(y-x)f(y-z)p_{r-t}(z-x)E[v_{n_1-1}(r, y)v_{n_2-1}(r, z)]$$

$$= \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \, p_{r-t}(y-x)f(y-z)p_{r-t}(z-x)E[v_{n_1-1}(r, y)v_{n_2-1}(r, z)]$$

(3.5)

$$\times \left( E[w_{n_1-1}(r; r, x_1 + y)w_{n_2-1}(r; r, x_2 + z)] \right)_{y = X^{(1)}_{t_1-r}, z = X^{(2)}_{t_2-r}}$$

But, as $N(n_1 - 1, n_2 - 1)$ is composed of $n_1 - 1$ occurrences of 1 and $n_2 - 1$ occurrences of 2, the only element of $P(n_1 - 1, n_2 - 1)$ is $(1, 2, \ldots, (1, 2))$. As a consequence, by the induction assumption,

$$E[w_{n_1-1}(r; r, x_1 + y)w_{n_2-1}(r; r, x_2 + z)]$$

$$= E^X \left[ u_0(x_1 + y + X^{(1)}_r)u_0(x_2 + z + X^{(2)}_r) \right]$$

$$\times \int_0^r dr_1 \cdots \int_0^{r_{n_2-2}} dr_{n_1-1} \prod_{j=1}^{n_1-1} f(x_1 + y + X^{(1)}_r - X^{(2)}_r)$$

$$= E^X_{y, z} \left[ u_0(x_1 + X^{(1)}_r)u_0(x_2 + X^{(2)}_r) \right]$$

$$\times \int_0^r dr_1 \cdots \int_0^{r_{n_2-2}} dr_{n_1-1} \prod_{j=1}^{n_1-1} f(x_1 + X^{(1)}_r - X^{(2)}_r)$$

$$= \int_0^r dr_1 \cdots \int_0^{r_{n_2-2}} dr_{n_1-1} E^X_{y, z} \left[ u_0(x_1 + X^{(1)}_r)u_0(x_2 + X^{(2)}_r) \right]$$

$$\times \prod_{j=1}^{n_1-1} f(x_1 + X^{(1)}_r - X^{(2)}_r)$$

$$\times \prod_{j=1}^{n_1-1} f(x_1 + X^{(1)}_r - X^{(2)}_r)$$

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where \( n = M/2 \). Further, by Lemma 3.5,

\[
E[w_{n_1-1}(r; r, x_1 + y) w_{n_2-1}(r; r, x_2 + z)] \bigg|_{y = X_{t_1-r}^{(1)}, z = X_{t_2-r}^{(2)}} = \int_0^r dr_1 \cdots \int_0^{n-2} dr_{n-1} E[X_{t_1-r}^{(1)}, X_{t_2-r}^{(2)}] \left[ u_0(x_1 + X_t^{(1)}) u_0(x_2 + X_t^{(2)}) \times \prod_{j=1}^{n-1} f(x_1 + X_{t_1-r_j}^{(1)} - x_2 - X_{t_2-r_j}^{(2)}) \right]
\]

Replacing (3.6) in (3.5), we obtain

\[
E[w_{n_1}(s; t_1, x_1) w_{n_2}(s; t_2, x_2)]
= \int_0^s dr E[X_{t_1-r}^{(1)} - x_2 - X_{t_2-r}^{(2)}] \int_0^r dr_1 \cdots \int_0^{n-2} dr_{n-1} E[X_{t_1-r}^{(1)}, X_{t_2-r}^{(2)}] \left[ u_0(x_1 + X_t^{(1)}) u_0(x_2 + X_t^{(2)}) \times \prod_{j=1}^{n-1} f(x_1 + X_{t_1-r_j}^{(1)} - x_2 - X_{t_2-r_j}^{(2)}) \right].
\]

Using Fubini’s theorem, the fact that \( f(x_1 + X_{t_1-r_j}^{(1)} - x_2 - X_{t_2-r_j}^{(2)}) \) is \( \mathcal{F}_{t_1-r}^{(1)} \otimes \mathcal{F}_{t_2-r}^{(2)} \)-measurable and renumbering the integration variables, we obtain

\[
E[w_{n_1}(s; t_1, x_1) w_{n_2}(s; t_2, x_2)]
= \int_0^s \int_0^t dr_1 \cdots \int_0^{n-1} dr_n E^{X_{t_1-r}^{(1)}, X_{t_2-r}^{(2)}} \left[ u_0(x_1 + X_t^{(1)}) u_0(x_2 + X_t^{(2)}) \prod_{j=1}^n f(x_1 + X_{t_1-r_j}^{(1)} - x_2 - X_{t_2-r_j}^{(2)}) \right]
= E^{X_{t_1-r}^{(1)}, X_{t_2-r}^{(2)}} \left[ u_0(x_1 + X_t^{(1)}) u_0(x_2 + X_t^{(2)}) \prod_{j=1}^n f(x_1 + X_{t_1-r_j}^{(1)} - x_2 - X_{t_2-r_j}^{(2)}) \right].
\]

As the only element in the set \( \mathcal{P}(n_1, n_2) \) is \((1, 2), \ldots, (1, 2)\), this proves the result in the case where \( m = 2 \).

We will now suppose that the result is proved for any number of terms in the product, up to \( m - 1 \), and consider the case where we have \( m \) terms. Again, we are going to prove the result by induction on \( N \). The smallest possible value is \( N = m \) with \( n_1 = \cdots = n_m = 1 \). In that case, the process \( s \mapsto w_1(s; t, x) \) is a martingale for all \( t \geq 0, x \in \mathbb{R}^d \). We apply Itô’s formula with the function \( h(x_1, \ldots, x_m) = \prod_{j=1}^m x_j \), then take an expectation, which cancels the martingale term. We are only left with the expectation of
the quadratic variation term. (For details, one can refer to [5, Proof of Lemma 6.2].) We finally obtain

\[
E \left[ \prod_{j=1}^{m} w_1(s; t_j, x_j) \right]
= \sum_{i=1}^{m} \sum_{j=1}^{i-1} \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \, p_{t_i-r}(y-x_i)f(y-z)\rho_{t_j-r}(z-x_j)\nu_0(r, y)\nu_0(r, z)
\times E \left[ \prod_{k=1 \atop k \neq i, j}^{m} w_1(r; t_k, x_k) \right]
= \sum_{i=1}^{m} \sum_{j=1}^{i-1} \int_0^s dr \, E^{X_i} \left[ f(x_i + X_{t_i-r}^{(i)} - x_j - X_{t_j-r}^{(j)}) \nu_0(r, x_i + X_{t_i-r}^{(i)})\nu_0(r, x_j + X_{t_j-r}^{(j)}) \right]
\times E \left[ \prod_{k=1 \atop k \neq i, j}^{m} w_1(r; t_k, x_k) \right].
\]

We now handle the first term in the time integral as in the case \( m = 2 \) using Lemma 3.3. The second term is the expectation of a product of \( m-2 \) terms. Hence, we can use the induction assumption on \( m \) to express it. If \( m \) is odd, then the second expectation in the integral vanishes and the whole expression as well. If \( m \) is even, we can use (3.4). Let \( \tilde{P}_{i,j} \) denote the set of ordered pairs of different integers of the set \( \mathcal{N}(n_1, \ldots, n_m) \), from which we delete the occurrence of each \( i \) and \( j \). We finally obtain

\[
E \left[ \prod_{j=1}^{m} w_1(s; t_j, x_j) \right]
= \sum_{i=1}^{m} \sum_{j=1}^{i-1} \int_0^s dr \, E^{X_i} \left[ u_0(x_i + X_{t_i}^{(i)})u_0(x_j + X_{t_j}^{(j)})f(x_i + X_{t_i-r}^{(i)} - x_j - X_{t_j-r}^{(j)}) \right]
\times \sum_{p \in \tilde{P}_{i,j}} E^{X_i} \left[ \prod_{k=1 \atop k \neq i, j}^{m} u_0(x_k + X_{t_k}^{(k)}) \int_0^r dr_1 \cdots \int_0^{r-n-1} dr_{n-1} \prod_{\ell=1}^{n-1} f(x_{p_\ell} + X_{t_\ell-r_\ell}^{(p_\ell)} - x_{q_\ell} - X_{t_{q_\ell-r_\ell}}^{(q_\ell)}) \right],
\]

where \( n = m/2 \). By Fubini’s theorem, a renumbering of the integration variables and the fact that}

\[
\bigcup_{i=1}^{m} \bigcup_{j=1}^{i-1} \bigcup_{p \in \tilde{P}_{i,j}} \{(i, j), p\} = P(1, \ldots, 1), \quad \text{m times}
\]

the result is proved for \( N = m \).

Now consider the case where \( N = m + 1 \). Then, without loss of generality, we can suppose that \( n_1 = 2 \)
and \( n_2 = \cdots = n_m = 1 \). In that case, Itô’s formula shows that

\[
E \left[ w_2(s; t_1, x_1) \prod_{j=2}^m w_1(s; t_j, x_j) \right] \\
= \sum_{i=2}^{m-1} \sum_{j=2}^{i-1} \int_0^s \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dz \, p_{t_i,r}(y - x_i) f(y - z) p_{t_j,r}(z - x_j) v_0(r, y) v_0(r, z) \\
\times E \left[ w_2(r; t_1, x_1) \prod_{k:\geq 2, k \neq i,j}^m w_1(r; t_k, x_k) \right] \\
+ \sum_{j=2}^m \int_0^s \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dz \, p_{t_1,r}(y - x_1) f(y - z) p_{t_j,r}(z - x_j) v_0(r, z) \\
\times E \left[ w_1(r; r, y) \prod_{k:\geq 2, k \neq j}^m w_1(r; t_k, x_k) \right],
\]

\[
(3.7)
\]

First, we can see that if \( m \) is even, then \( N = m + 1 \) is odd. In that case, the last expectation in the first term in (3.7) corresponds to the case \( m = 2 \), \( N = m - 1 \) and hence vanishes by induction. The last expectation in brackets in the second term of (3.7) corresponds to the case \( m - 1, N = m - 1 \) and vanishes as well. Hence, the result is true if \( m \) is even. Now, if \( m \) is odd, we can handle the first term above with Lemma 3.3 and the induction assumption, since the second expectation does not depend on \( X^{(i)} \) and \( X^{(j)} \). For the second term, we first use the induction assumption, then Lemma 3.5 (Markov property) and Fubini’s theorem. The arguments are analogous to those in the case \( m = 2 \) and we skip the details. This proves the result for \( N = m + 1 \).

Now, suppose that the result is true for all \( N \leq M - 1 \) and pick \( n_1, \ldots, n_m \in \mathbb{N} \) such that \( \sum_{i=1}^m n_i = M \).
By Itô’s formula, we have

\[
E \left[ \prod_{j=1}^{m} w_{n_j} (s; t_j, x_j) \right] = \sum_{i=1}^{m} \sum_{j=1}^{i-1} \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz p_{t_i-r} (y-x_i) f(y-z) p_{t_j-r} (z-x_j) \times E \left[ w_{n_i-1}(r; r, y) w_{n_j-1}(r; r, z) \prod_{k=1, k \neq i,j}^{n_k} w_{n_k}(r; t_k, x_k) \right].
\]

(3.8)

Now, if \( N = \sum_{j=1}^{m} n_j \) is odd, then \( n_i - 1 + n_j - 1 + \sum_{k=1, k \neq i,j}^{n_k} n_k = N - 2 \) is odd as well and the expectation above vanishes by induction. The result is proved for \( N \) odd. If \( N \) is even, then \( N - 2 \) is even as well and we can use the induction assumption to compute the expectation in (3.8). We obtain

\[
E \left[ \prod_{j=1}^{m} w_{n_j} (s; t_j, x_j) \right] = \sum_{i=1}^{m} \sum_{j=1}^{i-1} \int_0^s dr E[X \left[ f(x_i + X^{(i)}_{t_i-r} - x_j - X^{(j)}_{t_j-r}) \right] \times \left( E \left[ w_{n_i-1}(r; r, x_i + y) w_{n_j-1}(r; r, x_j + z) \prod_{k=1, k \neq i,j}^{n_k} w_{n_k}(r; t_k, x_k) \right] \right) | y=X^{(i)}_{t_i-r}, z=X^{(j)}_{t_j-r}].
\]

(3.9)

Let

\[
\tilde{x}_k = \begin{cases} 
  x_k & \text{if } k \neq i, j \\
  x_i + y & \text{if } k = i \\
  x_j + z & \text{if } k = j
\end{cases}
\quad \text{and} \quad
\tilde{t}_k = \begin{cases} 
  t_k & \text{if } k \neq i, j \\
  r & \text{if } k = i, j.
\end{cases}
\]

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Then, by the induction assumption with \((\tilde{t}_1, \ldots, \tilde{t}_m)\) and \((\tilde{x}_1, \ldots, \tilde{x}_m)\),

\[
E \left[ w_{n_1-1}(r; r, x_1 + y) w_{n_2-1}(r; r, x_2 + z) \left( \prod_{k=1}^m w_{n_k}(r; t_k, x_k) \right) \right]
\]

\[
= \sum_{P(n_1, n_2, \ldots, n_m)} E^{X} \left[ u_0(x_1 + y + X^{(i)}_t) u_0(x_2 + z + X^{(j)}_t) \prod_{k=1}^m u_0(x_k + X^{(k)}_{t_k}) \right. \\
\times \int_0^r dr_1 \cdots \int_0^{r_{n-2}} dr_{n-1} \prod_{\ell=1}^{n-1} \int_0^{r_{n-2}} f(\tilde{x}_{p_\ell} + X^{(p_\ell)}_{t_{p_\ell} - r_\ell} - \tilde{x}_{q_\ell} - X^{(q_\ell)}_{t_{q_\ell} - r_\ell}) \bigg),
\]

\[
= \sum_{P(n_1, n_2, \ldots, n_m)} E^{Y, Z} \left[ u_0(x_1 + X^{(i)}_t) u_0(x_2 + X^{(j)}_t) \prod_{k=1}^m u_0(x_k + X^{(k)}_{t_k}) \right. \\
\times \int_0^r dr_1 \cdots \int_0^{r_{n-2}} dr_{n-1} E^{X}_{Y, Z} \left[ u_0(x_1 + X^{(i)}_t) u_0(x_2 + X^{(j)}_t) \prod_{k=1}^m u_0(x_k + X^{(k)}_{t_k}) \right. \\
\times \int_0^{r_{n-2}} \int_0^{r_{n-2}} f(x_{p_\ell} + X^{(p_\ell)}_{t_{p_\ell} - r_\ell} - x_{q_\ell} - X^{(q_\ell)}_{t_{q_\ell} - r_\ell}) \bigg] .
\]

Further, by Lemma 3.5,

\[
E \left[ w_{n_1-1}(r; r, x_1 + y) w_{n_2-1}(r; r, x_2 + z) \left( \prod_{k=1}^m w_{n_k}(r; t_k, x_k) \right) \bigg| y = X^{(i)}_{t_i}, z = X^{(j)}_{t_j} \right]
\]

\[
= \sum_{P(n_1, n_2, \ldots, n_m)} \int_0^r dr_1 \cdots \int_0^{r_{n-2}} dr_{n-1} E^{X}_{X^{(i)}_{t_i}, X^{(j)}_{t_j}} \left[ u_0(x_1 + X^{(i)}_t) u_0(x_2 + X^{(j)}_t) \prod_{k=1}^m u_0(x_k + X^{(k)}_{t_k}) \right. \\
\times \int_0^{r_{n-2}} \int_0^{r_{n-2}} f(x_{p_\ell} + X^{(p_\ell)}_{t_{p_\ell} - r_\ell} - x_{q_\ell} - X^{(q_\ell)}_{t_{q_\ell} - r_\ell}) \bigg] .
\]

\[
(3.10)
\]
because \( t_{pi} - r_x + t_{pi} - r = t_{pi} - r_k \), whenever \( p_i = i \) or \( j \). Replacing (3.10) in (3.9), we obtain

\[
E \left[ \prod_{j=1}^{m} w_{n_j}(s; t_j, x_j) \right] \\
= \sum_{\mathcal{P}} \int_0^s \! dr_1 \cdots \int_{r_{n-1}}^{r_n} \! dr_n E^X \left[ \prod_{k=1}^{m} u_0(x_k + X^{(k)}_{t_k}) \times \prod_{l=1}^{n} f(x_{pi} + X^{(pl)}_{t_{pi} - r_l} - x_{qi} - X^{(ql)}_{t_{qi} - r_l}) \right] \\
\times \prod_{k=1}^{m} u_0(x_k + X^{(k)}_{t_k}) \times \int_0^s \! dr_1 \cdots \int_{r_{n-1}}^{r_n} \! dr_n \prod_{l=1}^{n} f(x_{pi} + X^{(pl)}_{t_{pi} - r_l} - x_{qi} - X^{(ql)}_{t_{qi} - r_l}) \\
\times E^X \left[ \prod_{k=1}^{m} u_0(x_k + X^{(k)}_{t_k}) \times \prod_{l=1}^{n} f(x_{pi} + X^{(pl)}_{t_{pi} - r_l} - x_{qi} - X^{(ql)}_{t_{qi} - r_l}) \right].
\]

Using Fubini's theorem, the fact that \( f(x_i + X^{(i)}_{t_i - r} - x_j - X^{(j)}_{t_j - r}) \) is \( \mathcal{F}^{(i)}_{t_i - r} \otimes \mathcal{F}^{(j)}_{t_j - r} \)-measurable, renumbering the integration variables and the fact that

\[
\bigcup_{i,j} \bigcup_{p \in \mathcal{P}} \{ (i,j), p \} = \mathcal{P}(1, \ldots, m),
\]

we obtain

\[
E \left[ \prod_{j=1}^{m} w_{n_j}(s; t_j, x_j) \right] \\
= \sum_{\mathcal{P}} \int_0^s \! dr_1 \cdots \int_{r_{n-1}}^{r_n} \! dr_n E^X \left[ \prod_{k=1}^{m} u_0(x_k + X^{(k)}_{t_k}) \times \prod_{l=1}^{n} f(x_{pi} + X^{(pl)}_{t_{pi} - r_l} - x_{qi} - X^{(ql)}_{t_{qi} - r_l}) \right].
\]

One should notice that in the case where one of the \( n_j \)'s is equal to 1, then the argument is similar but slightly different. Indeed, \( w_{n_j - 1} = w_0 \) and, hence, \( w_{n_j - 1} \) comes out of the expectation. We can still apply the induction assumption for the expectation, but we also have to apply Lemma 3.3 for the additional \( w_0 \) outside the expectation. The result is proved.

In Proposition 3.7, we assumed that the \( n_j \)'s were all positive. But, in the general case, there might be some of the \( n_j \)'s being equal to 0. In Proposition 3.9, we show that the same expression is valid in that case.

**Proposition 3.9.** Fix an integer \( M \geq 1 \) and \( n_1, \ldots, n_M \) non-negative integers. Fix \( x_1, \ldots, x_M \in \mathbb{R}^d \) and \( t_1, \ldots, t_M \in \mathbb{R}_+ \). Let \( s \geq 0 \) such that \( s \leq t_1 \wedge \cdots \wedge t_M \). Let \( N = \sum_{i=1}^{M} n_i \).

- If \( N \) is odd, then \( E \left[ \prod_{j=1}^{M} w_{n_j}(s; t_j, x_j) \right] = 0 \).
- If \( N \) is even, set \( n = \frac{N}{2} \). Then,

\[
E \left[ \prod_{j=1}^{M} w_{n_j}(s; t_j, x_j) \right] = \sum_{\mathcal{P}} E^X \left[ \prod_{j=1}^{M} u_0(x_j + X^{(j)}_{t_j}) \right] \\
\times \int_0^s \! dr_1 \cdots \int_{r_{n-1}}^{r_n} \! dr_n \prod_{i=1}^{n} f(x_{p_i} + X^{(p_i)}_{t_{p_i} - r_i} - x_{q_i} - X^{(q_i)}_{t_{q_i} - r_i}),
\]

where the notations are those of Proposition 3.7. If the set \( \mathcal{P}(1, \ldots, n_M) = \emptyset \), then the expectation vanishes.
suppose that there exists \( m \) such that \( n_j = 0 \) for all \( m + 1 \leq j \leq M \) and \( n_j > 0 \) for \( j \leq m \). Then, as \( w_0 \) is deterministic,

\[
E \left[ \prod_{j=1}^{M} w_{n_j}(s; t_j, x_j) \right] = \left( \prod_{k=m+1}^{M} u_0(s; t_k, x_k) \right) E \left[ \prod_{j=1}^{m} w_{n_j}(s; t_j, x_j) \right]
\]

Now, we can use Proposition 3.7 to compute the expectation and Lemma 3.3 to compute the product outside the expectation. We obtain

\[
E \left[ \prod_{j=1}^{M} w_{n_j}(s; t_j, x_j) \right] = \left( \prod_{k=m+1}^{M} E\left[ u_0(x_k + X_k^{(k)}) \right] \right) \sum_{\mathcal{P}(n_1, \ldots, n_m)} E\left[ \prod_{j=1}^{m} u_0(x_j + X_j^{(j)}) \right] \times \int_0^s dr_1 \cdots \int_0^{r_{n-1}} dr_n \prod_{i=1}^{n} f(x_{p_i} + X_{t_i - r_i}^{(p_i)} - x_q - X_{t_i - r_i}^{(q_i)})
\]

\[
= \sum_{\mathcal{P}(n_1, \ldots, n_m)} E\left[ \prod_{j=1}^{M} u_0(x_j + X_j^{(j)}) \right] \times \int_0^s dr_1 \cdots \int_0^{r_{n-1}} dr_n \prod_{i=1}^{n} f(x_{p_i} + X_{t_i - r_i}^{(p_i)} - x_q - X_{t_i - r_i}^{(q_i)})
\]

The result is proved since \( \mathcal{P}(n_1, \ldots, n_M) = \mathcal{P}(n_1, \ldots, n_m) \) when \( n_{m+1} = \cdots = n_M = 0 \).

Now, we can simplify our expression by coming back to the processes \( v_n \) rather than \( w_n \).

**Corollary 3.10.** Fix an integer \( m \geq 1 \) and \( n_1, \ldots, n_m \) non-negative integers. Fix \( x_1, \ldots, x_m \in \mathbb{R}^d \) and \( t \in \mathbb{R}_+ \). Set \( N = \sum_{i=1}^{m} n_i \).

- If \( N \) is odd, then \( E \left[ \prod_{j=1}^{m} v_{n_j}(t, x_j) \right] = 0 \).
- If \( N \) is even, set \( n = \frac{N}{2} \). Then,

\[
E \left[ \prod_{j=1}^{m} v_{n_j}(t, x_j) \right] = \sum_{\mathcal{P}(n_1, \ldots, n_m)} E\left[ \prod_{j=1}^{m} u_0(x_i + X_t^{(i)}) \right] \times \int_0^t dr_1 \cdots \int_0^{r_{n-1}} dr_n \prod_{i=1}^{n} f(x_{p_i} + X_{t_i - r_i}^{(p_i)} - x_q - X_{t_i - r_i}^{(q_i)})
\]

where the notations are those of Proposition 3.7. If the set \( \mathcal{P}(n_1, \ldots, n_M) = \emptyset \), then the expectation vanishes.

**Proof.** Set \( s = t_1 = \cdots = t_m = t \) in Proposition 3.9.

**Remark 3.11.** We would like to point out that another way to write down (3.12) is:

\[
E \left[ \prod_{j=1}^{m} v_{n_j}(t, x_j) \right] = \sum_{\mathcal{P}(n_1, \ldots, n_m)} E\left[ \prod_{i=1}^{m} u_0(X_t^{(i)}) \int_0^t dr_1 \cdots \int_0^{r_{n-1}} dr_n \prod_{i=1}^{n} f(X_{t_i - r_i}^{(p_i)} - X_{t_i - r_i}^{(q_i)}) \right]
\]
where \( E^X \) denotes the expectation with respect to the joint law of the processes \( X^{(j)} (j = 1, \ldots, n) \) under the conditions \( X_0^{(1)} = x_1, \ldots, X_0^{(m)} = x_m \).

We also recall the following result from Real Analysis. The proof is not very difficult, hence we leave it to the reader, since it is beyond the scope of this paper.

**Lemma 3.12.** Let \( g_1, \ldots, g_n \) be integrable functions on \( \mathbb{R}_+ \). Suppose that the \( n \) functions are divided in \( \ell \) groups of respectively \( k_1, \ldots, k_\ell \) identical functions \((k_1 + \cdots + k_\ell = n)\). Then, for all \( t \geq 0 \), the following result holds

\[
\sum_{\pi \in S_n(k_1, \ldots, k_\ell)} \int_0^t dr_1 \cdots \int_0^{r_{n-1}} dr_n g_{\pi(1)}(r_1) \cdots g_{\pi(n)}(r_n) = \frac{1}{k_1! \cdots k_\ell!} \prod_{i=1}^n \int_0^t dr \ g_i(r),
\]

where \( S_n(k_1, \ldots, k_\ell) \) is the set of all permutations of \( n \) objects divided in \( \ell \) groups of respectively \( k_1, \ldots, k_\ell \) identical objects. Namely, \(|S_n(k_1, \ldots, k_\ell)| = \frac{n!}{k_1! \cdots k_\ell!} \).

We are now ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** By Proposition 2.2, we know that \( u(t, x) = \sum_{n=0}^{\infty} v_n(t, x) \) and the series converges in \( L^p(\Omega) \), for all \( p \geq 2 \). Hence,

\[
E \left[ \prod_{j=1}^p u(t, x_j) \right] = \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} \mathbb{E} \left[ \prod_{j=1}^p v_{n_j}(t, x_j) \right]
\]

Further, by Corollary 3.10 and distinguishing the vectors \((n_1, \ldots, n_p)\) depending on the sum of their components,

\[
E \left[ \prod_{j=1}^p u(t, x_j) \right] = \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} \mathbb{E} \left[ \prod_{j=1}^p u_{n_j}(x_j + X_t^{(j)}) \right]
\]

\[
= \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} \mathbb{E} \left[ \prod_{j=1}^p u_{n_j}(x_j + X_t^{(j)}) \right]
\]

\[
= \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} \mathbb{E} \left[ \prod_{j=1}^p u_{n_j}(x_j + X_t^{(j)}) \right]
\]

Now, let \( d\mathcal{P}(n_1, \ldots, n_p) \) denote the set of pairings in \( \mathcal{P}(n_1, \ldots, n_p) \) but without taking the order into account. (For instance, \((1,2),(1,3)) \) and \((1,3),(1,2)) \) are different pairings in \( \mathcal{P}(2,1,1) \), but are the same.
in $dP(2,1,1)$. We have

$$E \left[ \prod_{j=1}^{p} u(t,x_j) \right]$$

$$= E^X \left[ \prod_{j=1}^{p} u_0(x_j + X^{(j)}_t) \times \sum_{n=0}^{\infty} \sum_{n_1,\ldots,n_p=0}^{\infty} \prod_{j=1}^{p} \eta_{n_1,\ldots,n_2} \int_0^t \cdots \int_0^{\tau_{n-1}} \int_0^{\tau_n} f(x_{p_i} + X^{(p_i)}_{\tau_{r_i}} - x_{q_i} - X^{(q_i)}_{\tau_{r_i}}) \right].$$

Then, by Lemma 3.12, we have

$$E \left[ \prod_{j=1}^{p} u(t,x_j) \right]$$

$$= E^X \left[ \prod_{j=1}^{p} u_0(x_j + X^{(j)}_t) \times \sum_{n=0}^{\infty} \sum_{n_1,\ldots,n_p=0}^{\infty} \prod_{j=1}^{p} \eta_{n_1,\ldots,n_2} \int_0^t \cdots \int_0^{\tau_{n-1}} \int_0^{\tau_n} f(x_{p_i} + X^{(p_i)}_{\tau_{r_i}} - x_{q_i} - X^{(q_i)}_{\tau_{r_i}}) \right],$$

where $k_1,\ldots,k_\ell$ are the numbers of identical pairs coming in the pairing considered. Now, using the fact that

$$\bigcup_{n_1,\ldots,n_p=0}^{\infty} \sum_{\eta_{1,\ldots,2n}} dP(n_1,\ldots,n_p)$$

$$= \left\{ ((p_1,q_1),\ldots,(p_n,q_n)) : p_j \neq q_j; p_1,\ldots,p_n,q_1,\ldots,q_n \leq p, \text{ without taking the order into account} \right\}$$

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and that there are \( \frac{n!}{x_1! \cdot \ldots \cdot x_n!} \) ways to order a particular pairing, we have

\[
E \left[ \prod_{j=1}^{p} u(t, x_j) \right] = E^X \left[ \prod_{j=1}^{p} u_0(x_j + X_t^{(j)}) \times \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{p_{1}, q_{1}=1}^{p} \ldots \sum_{p_{n}, q_{n}=1}^{p} \prod_{j,k \neq 1}^{n} \int_0^t df(x_{p_{j}} + X_t^{(p_{j})} - x_{q_{j}} - X_t^{(q_{j})}) \right],
\]

\[
= E^X \left[ \prod_{j=1}^{p} u_0(x_j + X_t^{(j)}) \times \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{p_{1}, q_{1}=1}^{p} \int_0^t df(x_{p_{1}} + X_t^{(p_{1})} - x_{q_{1}} - X_t^{(q_{1})}) \right)^n \right],
\]

\[
= E^X \left[ \prod_{j=1}^{p} u_0(x_j + X_t^{(j)}) \times \exp \left( \sum_{p_{1}, q_{1}=1}^{p} \int_0^t df(x_{p_{1}} + X_t^{(p_{1})} - x_{q_{1}} - X_t^{(q_{1})}) \right) \right].
\]

After conditioning on \( X_0^{(1)} = x_1, \ldots, X_0^{(p)} = x_p \), the result is proved.

**Proof of Corollary 3.2.** Take \( x_1 = \cdots = x_p = x \) in Theorem 3.1.

\[ \square \]

## 4 Space-time white noise and Lévy local times.

In this section, we will consider the case where the covariance \( f = \delta_0 \) is the Dirac measure at 0. Hence, the noise \( W \) is a space-time white noise and is defined as a centered Gaussian noise \( \{W(\phi) : \phi \in D(\mathbb{R}^{1+1})\} \) with covariance functional given by

\[
E[W(\phi)W(\psi)] = \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} \delta_0(dx) (\phi(t, \cdot) * \tilde{\psi}(t, \cdot))(x)
= \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} d\xi \mathcal{F} \phi(t, \xi) \mathcal{F} \psi(t, \xi).
\]

We notice that no solution exists in dimension \( d \geq 2 \) for space-time white noise, hence we restrict our attention to the case \( d = 1 \). The moment formula is given by Theorem 4.1 below.

**Theorem 4.1.** Let \( u \) denote the solution of (1.1) with operator \( L \), driven by a space-time white-noise. Let \( t \geq 0 \) and \( x \in \mathbb{R} \). Let \( (X_t^{(j)})_{j \geq 0} \) \( (j = 1, \ldots, p) \) be \( p \) independent copies of the Lévy process generated by \( L \). Let \( L_t^0(j, k) \) be the local time at 0 of the process \( X_t^{(j)} - X_t^{(k)} \). Informally,

\[
L_t^0(j, k) = \int_0^t dr \delta_0(X_r^{(j)} - X_t^{(k)}).
\]

Then,

\[
E[u(t, x)^p] = E^X \left[ \prod_{j=1}^{p} u_0(x + X_t^{(j)}) \times \exp \left( \sum_{j \neq k} L_t^0(j, k) \right) \right]. \tag{4.1}
\]

In order to prove Theorem 4.1, we would like to apply the moment formulae of Section 3. Therefore, we need to smooth the covariance of the noise through a convolution procedure. Namely, let \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) be the standard gaussian kernel. Namely,

\[
\varphi(x) = (2\pi)^{-1/2} \exp(-x^2/2).
\]
Let \( \varphi_\varepsilon(x) := \varepsilon^{-1/2} \varphi(x/\sqrt{\varepsilon}) \). Notice that the noise \( W \) can be extended to the space of rapidly decreasing \( C^\infty \)-functions. Hence, we can define a noise \( W_\varepsilon := \{ W_\varepsilon(\phi) : \phi \in \mathcal{D}(\mathbb{R}^{1+1}) \} \) by

\[
W_\varepsilon(\phi) := W(\phi \ast \varphi_\varepsilon),
\]

where \( \ast \) stands for spatial convolution. The noise \( W_\varepsilon \) is well-defined on the same probability space as the noise \( W \). Now, notice that

\[
E[W_\varepsilon(\phi)W_\varepsilon(\psi)] = E[W(\phi \ast \varphi_\varepsilon)W(\psi \ast \varphi_\varepsilon)]
\]

\[
= \int_0^{\infty} dt \int_{\mathbb{R}} \delta_0(dx)((\phi(t,\cdot) \ast \varphi_\varepsilon) \ast (\psi(t,\cdot) \ast \varphi_\varepsilon))(x)
\]

\[
= \int_0^{\infty} dt \int_{\mathbb{R}} \delta_0(dx)((\phi(t,\cdot) \ast \tilde{\psi}(t,\cdot) \ast \varphi_\varepsilon)(x)
\]

\[
= \int_0^{\infty} dt \int_{\mathbb{R}} \delta_0(dx)\varphi_\varepsilon(\phi(t,\cdot) \ast \tilde{\psi}(t,\cdot) \ast \varphi_\varepsilon)(x).
\]

Hence, the noise \( W_\varepsilon \) is a Gaussian noise with covariance informally given by

\[
E[W_\varepsilon(t,x)W_\varepsilon(s,y)] = \delta_0(t-s)\varphi_\varepsilon(x-y).
\]

Since \( \varphi_\varepsilon \) is a bounded covariance function, we can apply the results of Section 3 to the solution \( u^\varepsilon \) of (1.1) with the noise \( W_\varepsilon \).

Informally, \( \varphi_\varepsilon \) converges to \( \delta_0 \) as \( \varepsilon \to 0 \). Hence, \( W_\varepsilon \) is actually an approximation of the noise \( W \), in which we have turned the covariance measure into a bounded function. We state two results which make this convergence formal.

**Proposition 4.2.** Let \( Z = \{ Z(t,x), t > 0, x \in \mathbb{R}^d \} \) be a Walsh-integrable stochastic process with respect to space-time white noise \( W \). Then, for all \( p \geq 1 \), \( \int_0^{\infty} \int_{\mathbb{R}} Z(s,y)W_\varepsilon(ds,dy) \) converges in \( L^p(\Omega) \) to \( \int_0^{\infty} \int_{\mathbb{R}} Z(s,y)W(ds,dy) \) uniformly in \( t \in [0,T] \) as \( \varepsilon \to 0 \).

**Proof.** For a noise with covariance \( f \), let \( Q_W([0,t] \times A \times B) := t \int_{\mathbb{R}} f(dx)(1_A \ast \tilde{1}_B)(x) \) be the covariance measure of the extension of \( W \) as a martingale measure in the sense of Walsh [16]. Let \( \phi, \psi \in \mathcal{D}(\mathbb{R}) \). We have

\[
\left| \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(x)\psi(y)Q_W([0,t] \times dx \times dy) - \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(x)\psi(y)Q_W([0,t] \times dx \times dy) \right|
\]

\[
= t \left| \int_{\mathbb{R}} dx \varphi_\varepsilon(x)(\phi \ast \tilde{\psi})(x) - \int_{\mathbb{R}} \delta_0(dx)(\phi \ast \tilde{\psi})(x) \right|
\]

\[
= t \left| \int_{\mathbb{R}} d\xi \mathcal{F}\varphi_\varepsilon(\xi)\mathcal{F}\phi(\xi)\overline{\mathcal{F}\tilde{\psi}(\xi)} - \int_{\mathbb{R}} d\xi \mathcal{F}\phi(\xi)\overline{\mathcal{F}\tilde{\psi}(\xi)} \right|
\]

\[
\leq t \int_{\mathbb{R}} d\xi |\mathcal{F}\phi(\xi)||\mathcal{F}\tilde{\psi}(\xi)||1 - e^{-\frac{4\xi^2}{\varepsilon}}| \to 0,
\]

as \( \varepsilon \to 0 \). This shows that the covariance measure of the martingale measure \( W_\varepsilon \) converges to the covariance measure of \( W \) as \( \varepsilon \to 0 \). Using the results of Walsh [16] and Dalang [8], this is enough to ensure the convergence of the stochastic integral in \( L^p(\Omega) \) for all \( p \geq 2 \).

**Proposition 4.3.** The solution to (1.1) with noise \( W_\varepsilon \) converges to the solution to (1.1) with noise \( W \) as \( \varepsilon \to 0 \) in \( L^p(\Omega) \), uniformly on \( t \in [0,T] \) and \( x \in \mathbb{R} \).

**Proof.** This is a consequence of Proposition 4.2 and of the existence result (Proposition 2.1) using the Picard iteration scheme.
ensures that
\[ \int_{\mathbb{R}^d} \frac{d\xi}{1 + 2 \Re \Psi(\xi)} < \infty. \]
This implies that the symmetrized Lévy process \((\bar{X}_t)_{t \geq 0}\) admits local times, where the process \(\bar{X}\) is defined by \(X_t := X^{(1)}_t - X^{(2)}_t\) for all \(t \geq 0\), for \(X^{(1)}\) and \(X^{(2)}\) two independent copies of the process generated by \(L\) (see [2, Theorem 1, p.126]). Let \(L^\varepsilon_t\) denote the local time at \(x\) of \(\bar{X}\). Then, for any non-negative bounded function \(g\), we have
\[
\int_0^t g(\bar{X}_s) \, ds = \int_{\mathbb{R}^d} g(x) L^\varepsilon_t \, dx, \quad \text{a.s.} \tag{4.3}
\]
([2, p.126]). We have the following approximation result.

**Proposition 4.4.** Let \((X_t)_{t \geq 0}\) be a Lévy process which admits local times \((L^\varepsilon_t : t > 0, x \in \mathbb{R})\). Then, for all \(p \geq 1\),
\[
\mathbb{E} \left[ \left( \exp \left( \int_0^t \varphi_{2\varepsilon}(X_s) \, ds \right) - \exp(L^\varepsilon_t) \right)^p \right] \to 0,
\]
as \(\varepsilon \to 0\).

**Proof.** We know by [2, Prop.4, p.130], that the local time \(L^0_t\) has finite exponential moments. Hence, it suffices to prove that
\[
\mathbb{E} \left[ \left( \exp \left( \int_0^t \varphi_{2\varepsilon}(X_s) ds - L^\varepsilon_t \right) - 1 \right)^p \right] \to 0,
\]
as \(\varepsilon \to 0\). By (4.3) applied to \(g \equiv 1\), we know that \(x \mapsto L^\varepsilon_t\) is in \(L^1(\mathbb{R})\) a.s. and it admits a Fourier transform \(\hat{L}_\varepsilon\). Using (4.3) again,
\[
\left| \int_0^t \varphi_{2\varepsilon}(X_s) ds - L^\varepsilon_t \right| = \left| \int_\mathbb{R} \varphi_{2\varepsilon}(x) L^\varepsilon_t \, dx - \int_\mathbb{R} L^\varepsilon_t \, d\delta_0(dx) \right|
\[
= \int_\mathbb{R} |\mathcal{F}\varphi_{2\varepsilon}(\xi) - 1| \hat{L}_\varepsilon(\xi) \, d\xi
\]
\[
\leq 2 \int_\mathbb{R} \hat{L}_\varepsilon(\xi) \, d\xi = 2L^0_t, \tag{4.4}
\]
Now, since \(|\mathcal{F}\varphi_{2\varepsilon}(\xi) - 1|\) converges to 0 pointwise as \(\varepsilon \to 0\), the result follows from the bound (4.4) and the Dominated Convergence Theorem, since \(L^0_t\) has finite exponential moments.

We are now ready to prove the moment formula in the case where \(f = \delta_0\).

**Proof of Theorem 4.1.** Let \(u_\varepsilon\) denote the solution to (1.1) with noise \(W_\varepsilon\) defined in (4.2). By Corollary 3.2, we know that
\[
E[u_\varepsilon(t,x)^p] = E^X \left[ \prod_{i=1}^p u_0(x + X^{(i)}_t) \times \exp \left( \sum_{j,k} \int_0^t dr \varphi_{2\varepsilon}(X^{(j)}_s - X^{(k)}_s) \right) \right]. \tag{4.5}
\]
Moreover, we know by Proposition 4.3 that \(E[u(t,x)^p] = \lim_{\varepsilon \to 0} E[u_\varepsilon(t,x)^p]\). Also, by Proposition 4.4, we know that for any choice of \(j,k \in \{1, \ldots, p\}\), \(\exp(\int_0^t \varphi_{2\varepsilon}(X^{(j)}_s - X^{(k)}_s) \, ds)\) converges to \(\exp(L^\varepsilon_t(j,k))\) in \(L^p(\Omega)\) for any \(p \geq 1\). This implies that the right-hand side of (4.5) converges to the right-hand side of (4.1) as \(\varepsilon \to 0\). The result is proved by taking the limit as \(\varepsilon \to 0\) in (4.5).
Remark 4.5. In the case where $L = \Delta$ and $X$ is a Brownian Motion, Theorem 4.1 corresponds to the first part of Theorem 5.3 of [14]. Theorem 3.1 doesn’t require the use of a Feynman-Kac type representation for the solution, but uses the Wiener-Chaos expansion instead. Theorem 4.1 is also slightly more general in the sense that it considers generators of Lévy processes and not only $\Delta$, although it only handles noises which are white in time. We would like to notice that the approach used in [14] can also lead to Theorem 3.1 in the case of Lévy processes.

References


