Analysis of cornea curvature using radial basis functions – Part II: Fitting to data-set

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ABSTRACT

In part I we discussed the solution of corneal curvature using a 2D meshless method based on radial basis functions (RBFs). In Part II we use these methods to fit a full nonlinear thin membrane model to a measured data-set in order to generate a topological mathematical description of the cornea. In addition, we show how these results can lead to estimations for corneal radius of curvature and certain physical properties of the cornea; namely, tension and elasticity coefficient. Again all calculations and graphics generation were performed using the R language programming environment.

The model describes corneal topology extremely well, and the estimated properties fall well within the expected range of values. The method is straightforward to implement and offers scope for further analysis using more detailed 3D models that include corneal thickness.

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1. Introduction

This paper is a continuation of part I of the study [5], concerning a mathematical model of corneal topography. The importance of the cornea, the eye's most frontal part, cannot be overestimated since it is the place where an incident light beam is refracted directly into the interior part of the eye. Focused by the lens, the signal excites the retina from which is subsequently transmitted, via the optic nerve, into the brain. There, the image is constructed (for a technical treatment see [1]).

The cornea, as a biomechanical structure, can be characterized by both its optical and structural properties. On the one hand, it has to be almost completely transparent in order not to affect the light signal, while on the other, it should be very durable to provide a protective shield from external factors. It is a layered structure consisting of five distinguishable layers: epithelium, Bowman's layer, stroma, Descement's membrane and endothelium (from anterior to posterior). The stroma, which constitutes about 90% of the corneal thickness, is made of an arrangement of collagen fibrils and is responsible for many of its material properties.

The cornea is joined with sclera (the white of the eye) with a junction called corneal limbus. A thorough exposition of the human eye anatomy can found, for example, in [8].

Corneal topography is an important factor in the process of refraction. Although, the cornea is responsible for a majority of the eye's refractive power (about 2/3), its focal point is fixed which makes the lens the main focusing organ. Any disturbance in the shape of the cornea can perturb vision quality. This is why it is important to measure and model corneal topography adequately. In earlier work [13], we have devised a medium-complexity mathematical model of corneal topography. A 1D simplified version was then studied analytically and numerically in [15,16]. Unless stated otherwise, plots were generated using case 3 data using IMQ RBFs.

The description of corneal topography data (collected for example with high-speed videokeratoscopy [23] or Scheimpflug camera [18]) requires very nontrivial mathematical modeling and various approaches have been proposed to encode (and compress) the information. We mention the adaptive modeling (for example [19]), a mixture of parametric surfaces and polynomials (see [12] and references therein), splines [6] and Zernike polynomials [3,20]. Apart from this, there are a number of different modeling approaches which are discussed, for example, in [21]. The Reader will also find a relevant comparison of models in Part I of this paper [5].
In parts I and II of this paper we extend the analysis of our previous work [15] from a 1D corneal model to a full 2D model with no prior assumptions regarding radial symmetry. Part I described a solution methodology that uses a meshless method based on radial basis functions (RBFs). This enabled a full topological solution to be obtained for the cornea. From this solution, radius of curvature values can be calculated over the surface. In addition, it was demonstrated that various corneal physical parameters can also be estimated from the simulation results.

We now use this methodology to fit a 2D model to a measured data-set of corneal elevations obtained from an Oculus Pentacam® device - see Fig. 1.

2. Mathematical model

2.1. Corneal topology

Part I of this paper [5] included a false transient analysis and solution to cornea curvature based on the following normalized thin-plate 2D boundary value PDE curvature model,

\[ \frac{\partial h}{\partial t} = \nabla \left( \frac{\nabla h}{1 + |\nabla h|^2} \right) - ah + \frac{b}{1 + |\nabla h|^2}, \]

\[ h = h(x, t), \quad t \geq 0, \quad x \in \mathbb{R}^2, \]

BCs: \( \partial \Omega h_{\text{ib}} = 0 \), \( r_{\text{ib}} = \left( \frac{x^2}{2} + \frac{y^2}{2} \right) \),

\[ \nabla h_{x=0} = 0, \quad \text{IC: } h(x, 0) = h_0(x), \tag{1} \]

where, \( a = kR^2/T \) and \( b = PR/T \) (refer to Part I) and \( h(x) \) represents the non-dimensional height of the cornea over its surface. The maximum height \( h_{\text{max}} \) is the so-called ocular sagitta or sagittal height/depth, from geometric theory. This model is a result of an equilibrium force balance between membrane tension \( T \), elastic restoring force proportional to the corneal deflection with the constant of proportionality \( k \) and intraocular pressure \( P \) (for derivation see [13] and discussion in Part I [5]). A typical length dimension of the cornea is here designated by \( R \), which we choose to be the semi-major axis of the cornea.

The calculations were performed over the spatial domain \( \Omega \), assumed elliptical, with boundary \( \partial \Omega \). The overall topology solution was obtained using a meshless method based on radial basis functions (RBFs). This same model will be used to provide a fit to a measured data-set of corneal elevation, with tuning achieved by varying parameters \( a \) and \( b \). Once \( a \) and \( b \) are known, we can back calculate estimates for certain physical properties of the cornea, as detailed in part I.

2.2. Radius of curvature

The important corneal property curvature, we define in terms of two invariants: Gaussian curvature

\[ K = \frac{h_{xx}h_{yy} - h_{xy}^2}{(1 + h_{xx}^2 + h_{yy}^2)^2}, \]

and mean curvature

\[ H = \frac{1 + h_{xx}^2 + h_{yy}^2}{2(1 + h_{xx}^2 + h_{yy}^2)^{3/2}}, \]

where we define \( h = h(x, y) \) in Cartesian coordinates, and we have used subscript notation to represent partial derivatives, i.e. \( h_x = \partial h/\partial x \) etc. From above, and using real units, the principal curvatures from differential geometry are given by

\[ k_1 = H + \sqrt{H^2 - K}, \quad \text{maximum curvature}, \]

\[ k_2 = H - \sqrt{H^2 - K}, \quad \text{minimum curvature}, \]

from which it follows that \( K = (k_1k_2)/2 \) and \( H = (k_1 + k_2)/2 \) [2]. Curvatures \( k_1, k_2 \) and \( H \) have units of reciprocal meters, while \( K \) has units of reciprocal meters squared – see also part I.

We calculate the (Gaussian) radius of curvature (ROC) from Gaussian curvature as

\[ \rho = \frac{1}{\sqrt{K}} = \frac{1}{\sqrt{k_1k_2}}. \tag{5} \]

Other ophthalmology refractive measures (in diopters) derived from \( k_1 \) and \( k_2 \) are

\[ \text{MS} = (n_1 - n_0)(k_1 + k_2)/2, \quad (\text{mean power or mean sphere power}), \]

\[ \text{CYL} = (n_1 - n_0)(k_1 - k_2), \quad (\text{cylinder power}), \]

\[ \text{GP} = (n_1 - n_0)\sqrt{k_1k_2}, \quad (\text{Gaussian power}), \tag{6} \]

where \( n_1 = 1.3375 \) is the standard keratometric index (SKI) and \( n_0 = 1.0 \) represents the refractive index of air.

Once the model of Eq. (1) has been tuned to the measured data by varying \( a \) and \( b \), the above curvature values can be calculated.

3. The measured cornea data-set

3.1. Data source

The cornea data-set used to test the mathematical model and the RBF meshless solution method was obtained in January 2007 from an Oculus Pentacam® HR system with software version: 1.16r03. The Pentacam consists primarily of a rotating mirror and a blue LED source. It uses a rotating camera to derive a precision measurement elevation map of the cornea - refer to [11] for discussion on accuracy. The imaging technology is based upon the Scheimpflug principle1 [19].

The data formed part of previous research [17], for which approval from the university human research ethics committee was obtained before commencement of the study. All subjects gave written informed consent to participate, and were treated in accordance with the tenets of the Declaration of Helsinki.

3.2. The data-set

The topography data-set discussed in this paper has been taken from a 25 year old healthy subject with a normal cornea. The data is

\[ ^1 \text{Named after Theodor Scheimpflug, an Austrian army captain (1865–1911).} \]
anonymized and consists of elevation values measured at locations over the anterior surface together with corresponding x–y coordinates for each point. The data also includes the corresponding value for intraocular pressure, as measured by a tonometer. Posterior data was made available but is not discussed here.

The raw data was provided in the form of a spreadsheet with each elevation being assigned corresponding x–y coordinates. The elevations were given in micron units, with zero at the corneal apex and increasing towards the corneal-limbal ring (CLR). For simulation purposes they were inverted such that the peripheral values became zero with the maximum elevation at the apex. The values were converted to millimeters by dividing by 1000. The data-set consisted of 9521 x–y coordinated elevations, uniformly scattered over the cornea anterior surface, see (Fig. 2).

Precision measurements of the corneal sagittal height and CLR were not available for this work. Also, the Pentacam data points at the edge of the analyzed area could not be reliably interpreted as delineating the CLR. However, from an examination of the Pentacam elevation data, the greatest elevation was determined to be 2.631 mm, and this value was used as a reference point for sagittal height. From plotted data, we estimated the horizontal and vertical diameters to be 12.05 and 12.00 mm respectively, giving an eccentricity of 0.091 at the estimated zero datum. Model parameters were varied to obtain the final accepted model results, as described in (Section 4).

4. Results

4.1. Main solution

The overall solution is shown as a surface plot in (Fig. 3), and a summary of the results is given in (Table 1). Three solutions of the problem were obtained with differing numbers of nodes — see (Table 1). The problem converged smoothly and quickly to the final solution — see part I for more details. Plots were generated using case 3 data using IMQ RBFs.

In order to be able to compare simulated to measured elevations, it is necessary to convert model normalized values to the real measurement scale. This was achieved by noting that, from measured data, the horizontal and vertical diameters were found to be 12.05 and 12.00 mm respectively, giving a domain eccentricity of 0.091. These values corresponded to normalized simulation model values $r_x = 1$ and $r_y = 0.9959$, giving a scale factor of 6.025.

Unlike in part I, where we had a target central corneal height to aim for, here our target is to aim for the best overall fit of simulation elevations to measured data taken over the cornea surface. This was achieved with parameters $a = 1.54$ and $b = 2.15$, when the simulation normalized height at the cornea center was equal to 0.4367, equivalent to 2.631 mm.

The values chosen for $a$ and $b$ were determined iteratively by running a series of simulations and selecting the pair that minimized the maximum elevation error between measured and simulated cornea height $h$, taken over the surface. It was found that the values of $a$ and $b$ each affect both the maximum elevation and the radius of curvature of the cornea. Increasing the value of $a$ has the effect of reducing the maximum elevation while at the same time increasing central curvature. On the other hand, increasing $b$ has the opposite effect by increasing the maximum elevation and reducing overall curvature. Thus, $a$ and $b$ tend to have competing

$$a = 1.54$$
$$b = 2.15$$

Fig. 2. Left: 2D surface plot of cornea data-set shown with elevation contours in mm. Right: 3D surface plot of cornea with data points superimposed.

Fig. 3. 3D surface plot of cornea model solution. Dimensions are in mm.

Table 1

<table>
<thead>
<tr>
<th>Case</th>
<th>HVID</th>
<th>VVID</th>
<th>$\rho_0$</th>
<th>$\epsilon$</th>
<th>Total nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MQ</td>
<td>MQ</td>
<td>MQ</td>
<td>MQ</td>
<td>MQ</td>
</tr>
<tr>
<td>1</td>
<td>12.05</td>
<td>12.00</td>
<td>12.05</td>
<td>12.00</td>
<td>8.155</td>
</tr>
<tr>
<td>2</td>
<td>12.05</td>
<td>12.00</td>
<td>12.05</td>
<td>12.00</td>
<td>8.157</td>
</tr>
<tr>
<td>3</td>
<td>12.05</td>
<td>12.00</td>
<td>12.05</td>
<td>12.00</td>
<td>8.158</td>
</tr>
</tbody>
</table>
effects, with $b$ dominating. However, it was necessary to vary both $a$ and $b$ in order to obtain a good fit to measured data, whereby maximum curvature ($\min \rho_0$) occurs in the central region and falls off towards the corneal periphery. Interestingly, the final values arrived at for $a$ and $b$ are close to those found in part I for the generic cornea. Three simulation cases were run to ensure that sufficient collocation points were used to provide adequate resolution of the corneal topography, with results shown in (Table 1).

All three cases give very similar results for corneal central height $h_{\text{max}}$, located at $(0, 0)$, and we take the converged value to be 2.631 mm. However, the corneal central radius of curvature $\rho_0 = \rho(0, 0)$ results differ slightly with the mean being around 8.156 mm. Both results are generally in line with published measured data and this confirms the utility of the methodology outlined in part I for estimating corneal topology.

It was found that solutions were not sensitive to the initial condition and both $h_0(x) = 0$ and $h_0(x) = 1$ gave the same results. However, as $h_0(x) = 0$ resulted in slightly faster convergence, this value was used for all cases.

In order to compare simulated results with the measured data-set, cross-section error plots are provided in (Fig. 4). In addition, errors between model and data-set are shown as a 2D surface plot in (Fig. 5). The contours show clearly the accurate topology representation achieved by the RBF solution to model Eq. (1), particularly around the central region of the cornea. From the information available to the authors, it is not clear whether the difference at the left hand edge in Fig. 5 is due to a departure of the model from cornea measurement data, or if it represents a convex corneo-scleral transition region.

All calculations were performed on the same standard PC as described in part I, with similar timings.

4.2. Radius of curvature

Applying the calculations outlined in Section 2.2 with real units, the above results enable the radius of curvature to be calculated – see Figs. 6 and 7.

In order to confirm the accuracy of the radius of curvature estimate, a circle equal to the calculated apical radius of curvature, $\rho_0 = 8.156$ mm, was overlaid on top of the central area of the corneal measured data-set and the differences calculated along the $x$- and $y$-axes, see (Fig. 8). The small absolute errors indicate a good fit and provides additional confirmation of the utility of the part I calculations.

In addition, the following values for the central cornea are also obtained:
A good fit of the physically-based mode described in (Section 2) to a measured data-set with very small errors, by adjusting just two parameters.

- Reasonable estimates for differential geometry values $k_1$, $k_2$, $K$ and $H$.
- Reasonable estimates for ophthalmology refractive powers MS, CYL, and GP.
- Estimates for $T$ and $k$.

6. Further work

While the modeling work reported here constitutes a useful addition to the tools available for analyzing corneal topography, there remains much work to be done. For example, the analysis can be extended to include:

- Fitting model to additional data-sets for both healthy patients and those with abnormalities such as keratoconus. This will require that $a$ and $b$ are changed from fixed parameters to functions of $x$ and $y$, i.e. $a(x, y)$ and $b(x, y)$. We have done some work in this area, see [14].
- Fitting model to cornea posterior surface.
- Expanding the analysis to a 3D model that includes corneal thickness.
- Consider our model as being a limiting case of the structural mechanical description of a cornea.
- Provide a mathematical treatment of non-axisymmetric perturbations in the corneal data.
- Conduct a statistical analysis of model performance in respect of corneal parameter estimates, in particular where corresponding CLR measurements are available.
- Perform further analysis of parameter estimates for $T$ and $k$, and compare to measured data.

The authors hope to address some of these issues in future papers.

Conflict of interest

None declared.

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Appendix A. Some coding aspects

In this section we provide listings of the main R functions used in the corneal analysis.

A.1. RBF calculations

Listing 1. R code of function $mq()$ used to calculate multi-quadratic RBF values and those of the first and second derivatives with...
Listing 2. R code of function \texttt{imq()} used to calculate inverse multi-
quadratic RBF values and those of the first and second derivatives
with respect to \(r, x, y\).

```r
imq <- function(r, e, dX, dY){
  # inverse multi-quadratic RBF function arguments
  # e = shape parameter value, scalar
  # r = radial distances, matrix
  # X = x locations, matrix
  # Y = y locations, matrix
  #
  phi <- 1/sqrt((epsilon*r)^2+1)
  #
  d1rphi <- -epsilon^2*r/phi^3
  d2rphi <- -epsilon^2*(2*epsilon*r^2-1)/phi^5
  #
  d1xphi <- -epsilon^2*dX/phi^3
  d1yphi <- -epsilon^2*dY/phi^3
  #
  d2xphi <- -epsilon^2*((2*dX^2-dY^2)*epsilon^2-1)*phi^5
  d2yphi <- -epsilon^2*((2*dY^2-dX^2)*epsilon^2-1)*phi^5
  return(list(phi=phi, d1rphi=d1rphi, d2rphi=d2rphi,
               d1xphi=d1xphi, d1yphi=d1yphi,
               d2xphi=d2xphi, d2yphi=d2yphi))
}
```

Listing 3. R code of \texttt{main} program used to calculate corneal to-
pographical solutions.

```r
A.2. Solving the PDE

Listing 3. R code of \texttt{main} program used to calculate corneal to-
pographical solutions.

```
cat("\014") # Clear console
rm(list = ls(all = TRUE)) # Delete workspace
#
require(compiler)
enableJIT(3)
#
library("deSolve") # differential equation solver package
library(rgl) # 3D interactive graphics package
library(pracma) # advanced functions for numerical analysis
library(fields) # provides image plot with legend
source("circCtrs.R") # node data calcs
source("nodeData.R") # node data calcs based on distmesh package
source("img.R") # img RBF calcs
source("mq.R") # mq RBF calcs
source("cornea2D_DerivList.R") # derivative calcs
#
ptm0 <- proc.time()
print(Sys.time())
******************************************************************************
# Note: ra, rb are major and minor axes of ellipse, where
# ra is normalized to 1
paperNo <- 1
******************************************************************************
# Paper part I
if(paperNo == 1){
aDia <- 11.75   # mm
bDia <- 11.50  # mm
ra <- 1      # Normalized major axis
rb <- bDia/aDia # Normalized minor axis
eccentricity <- sqrt(1-(rb/ra)^2)
eMQ <- c(1.5,2.0,3.2)  # MQ shape parameter values
eIMQ <- c(1.2,1.8,3.0) # IMQ shape parameter values
a_i <- 1.48; b <- 2.104 # Best for Part I(h0=2.5)
******************************************************************************
# Paper part II
}else{
aDia <- 12.05  # mm
bDia <- 12.00  # mm
ra <- 1      # Normalized major axis
rb <- bDia/aDia # Normalized minor axis
eccentricity <- sqrt(1-(rb/ra)^2)
eMQ <- c(1.35,2.2,4.0)  # MQ shape parameter values
eIMQ <- c(1.1,2.0,2.8) # IMQ shape parameter values
a_i <- 1.54; b <- 2.15 # Best for Part II
}
******************************************************************************
# Set data for 3 cases
nDat <- c(8,11,15)  # data point index values
caseNo <- 1 # case number
n <- nDat[caseNo]  # number of data point
rbfType <- 0 # RBF type: 0=MQ, 1=IMQ
if(rbfType == 0){
epsilon <- eMQ[caseNo] # MQ epsilon value
}else{
epsilon <- eIMQ[caseNo] # IMQ epsilon value
}
hratio <- (aDia/2)/ra
******************************************************************************
# Calculate spatial domain node data
source("circDatCalc.R") # uses circCtrs() function
#
bn_index <- which(abs(xx)<1e-12 & abs(yy)<1e-12,arr.ind = TRUE) # Neuman BC
simTitle <- "Cornea_2D"
******************************************************************************
# 2D Cornea thin-shell Model:
# del(del_h/sqrt(1 + abs(h.x+h.y)^2 ))+a*h =
# b/sqrt(1 + abs(h.x + h.y)^2 )
# Ref: W. Okrasinski, L Plociniczak (2013). Bessel function model
#
cat(sprintf("%n Boundary Nodes = \%d, Inner Nodes = \%d, Total Nodes = \%d\n", Nb, Ni, N))
#
# Perform RBF calculations
# XX <- outer(xx,rep(1,N),FUN="*")
# YY <- outer(rep(1,N),yy,FUN="*")
# dX <- t(XX)-XX; dY <- t(YY)-YY
# r <- sqrt(dX^2+dY^2)
# if(rbfType == 0){
#   rbfText <- "Multiquadratic (MQ) Globally Supported RBF"
#   rbfDat <- mq(r, epsilon, dX, dY)
# } else if(rbfType == 1){
#   rbfText <- "Inverse Multiquadratic (IMQ) Globally Supported RBF"
#   rbfDat <- imq(r, epsilon, dX, dY)
# }
phi1 <- rbfDat$phi1; dirphi <- rbfDat$dirphi1; dzphi1 <- rbfDat$dzphi1
# d1xphi <- rbfDat$d1xphi; d1yphi <- rbfDat$d1yphi
# d2xphi <- rbfDat$d2xphi; d2yphi <- rbfDat$d2yphi
# dudy <- general applies to all RBFs
# d2xyphi <- (dX*dY/y^2)*dzphi1 - (dX*dY/y^3)*dirphi1 # Fasheur, p443
# pts <- which(abs(r)<1e-16)
# d2xyphi[pts] <- 0
#
# cat(sprintf("%n RBF Type = %s, epsilon = %5.2f\n", rbfText, epsilon))
#
# A <- phi1
invA <- inv(phi1)
Dx <- d1xphi%*%invA
Dy <- d1yphi%*%invA
Dxx <- d2xphi%*%invA
Dyy <- d2yphi%*%invA
Dxy <- d2xyphi%*%invA
#
# t1 <- proc.time()-tm0
cat(sprintf("%n Time to calculate RBFs, 1st and 2nd derivatives, t1 = %f\n", t1[3]))
# Boundary Conditions
Ubcr1 <- rep(0,Nb) # Dirichlet BC, fixed
Ubcr0 <- 0 # Neumann BC, fixed
# Initial condition
Uni <- rep(0,Nt) # Total nodes
Uni[b_index] <- Ubcr1
#
# Set simulation times
# tf <- 5; nout <- S1
t <- seq(from=0,to=tf,length.out=nout)
#
# DDF integration
# ncall<<0
# out<-ode(method="lsodes",y=c(Uni), t=.,
#          func=cornea2D_derivList,
#          ncall=0, method="lsode", ynames = FALSE,
#          rtol=1e-8, atol=1e-8, parms=NULL)
# Listing 4. R code of function cornea2D_derivList() that calculates numerical derivatives used by the lsodes integrator.

cornea2D_derivList <- function(t, u, parms) {
  ncall <- ncall +1
  # u[b_index] <- Ubcr1
  dhdx <- Dz %*% u
  dhdy <- Dy %*% u
  #print(u)
  dhdx[bN_index]<-Ubcro # Neuman BC
  dhdy[bN_index]<-Ubcro # Neuman BC
  d2hdx <- Dz %*% dhdx[,1]
  d2hdy <- Dy %*% dhdy[,1]
  d2hdxu <- Dz %*% dhdx[,1]
  #
  sr <- sqrt(rep(1,N)+d2hdx^2 + d2hdy^2)
  #
  ut <- (d2hdx+d2hdy+d2hdx*(d2hdy)^2+2*d2hdy*(d2hdx)^2 -
        2*d2hdx*d2hdy*d2hdxu)/(sr^3 - a*u + b/sr)
  ut[b_index] <- 0 # Dirichlet BC's
  #
  return(list(ut))
}
A.3. Interpolating scattered data

Listing 5. R code of function `meshInterp()` that interpolates scattered data.

```r
meshInterp2D <- function(inV, ctrs, f, epsilon, X, Y) {
    # Function to interpolate scattered data
    # inV    = inverse of RBF matrix A
    # ctrs   = vector of points used to calculate A
    # f      = vector of known values at ctrs
    # epsilon = shape parameter used to calculate A
    # X, Y   = location matrices at which f is to be interpolated
    # NOTE: One could build full matrix B and obtain the solution
    # B = B %*% c. But for many est points, B would be very large,
    # i.e. size = n[1]*n[2] rows by k cols. Also B would be a
    # column matrix of length n[1]*n[2], which is not so convenient
    # for plotting purposes.
    if (rbfType == 0) { # MQ
        B <- sqrt((r*epsilon)^2 + 1) # length=k
    } else if (rbfType == 1) { # IMQ
        B <- 1/sqrt((r*epsilon)^2 + 1) # length=k
    } else if (rbfType == 2) { # Wend30 RBF
        B <- pmax((1-r*epsilon), 0)^2 # length=k
    } else if (rbfType == 3) { # GMQ
        B <- (1+(r*epsilon)^2)^(5/2) # length=k
    }
    P[i, j] <- B %*% c
}
return(Pf)
}
```
Listing 6. R code of function `interpPlot()` that generates a surface plot \( h(x,y) \) and cross-section plots \( h(x,0) \) and \( h(y,0) \).

```r
source("meshInterp2D.r")
ctrs <- cbind(xy.xx)
# plotOn <- 0
#
# Interpolate scattered data values for \( h \) to facilitate plotting
Np <- 21; Nt <- 121
phi <- seq(0,pi/2,len=Np) # semi-ellipsoid
# phi <- seq(0,pi,len=Np) # ellipsoid
theta <- seq(0,2*pi,len=Nt)
# grd <- meshgrid(phi,theta) # alternative to use of outer() - see below
# Phi2 <- t(grd[2]); Theta2 <- t(grd[1])
Phi <- outer(phi,rep(1,Nt),FUN="*")
Theta <- outer(rep(1,Np),theta,FUN="*")
# Convert to a mesh on an elliptical domain
XXe <- ra*sin(Phi)*cos(Theta)
YYe <- t(rb*sin(Phi)*sin(Theta))
# Estimated/interpolated values - elliptical domain
E <- meshInterp2D(inrv,ctrs,U,epilson,XXe,YYe)
if(plotOn == 1){
  jet.colors <- colorRampPalette(
# c("white","white","white","white","white","black")) # for eye look
# c("white","white","white","white","lightblue","lightblue","lightblue","black")) # for eye look
  c("#000000", "#00007F", "#007F00", "#007FFF", "#7F77FF", "#77FF00", "#77FF00", "#77FF00")
  # Set palette
  pal <- jet.colors(100)
  op3d()
  bg3d("white")
  # brks <- c(seq(-0.1,1,len=100))
  # Set colour indices of each point for pers3d()
  col.ind <- cut(E,100) #breaks=brks)
  # pers3d(XXe,t(YYe),E,color=pal[col.ind], # for real units
  pers3d(XXe,t(YYe),E,color=pal[col.ind], # "lightblue",
     ylab = "", xlab = "", zlab = "", #back="lines",
     ylim = c(-6.6), xlim = c(-6.6),zlim=c(-0.2,7),
     box=FALSE,size=TRUE,axes=TRUE)
  um <- c( 0.9166561, -0.3999206, -0.00597051), 0,
  0.1969353, 0.4394726, 0.87639608, 0,
  -0.3477703, -0.8065896, 0.481548250, 0,
  0.0000000, 0.0000000, 0.0000000, 0.0000000, 1)
  UM <- matrix(data=um, byrow=TRUE,nrow=4,ncol=4)
  rgl.viewpoint(fov=90) # set before userMatrix
  par3d(userMatrix=UM)
  par3d(window=c(20,100,20,900), zoom=1) # use zoom=100 for WebGL
  bg3d(color=c("white","white"))
  grid3d(c("x", "y", "z"),col="black",lty=2,lwd=1)
  #axes3d(c(0,10),c(0,1),c(0,1))
  # place after decorate
  # points3d(XXe,t(YYe),E,color = "green")
  # rgl.snapshot("cornell2D_model1Surface.png", fmt="png") # print to file
}
```

A.4. Plotting surfaces from scattered data
# Central cross-section values for height of cornea
# No easing points (Same as data if fitting to data-set)
# Allocate y values at x=0
# Estimated h(0, y)
# EhOY <- neshinterp2D(invz, crss, U, epsilon, X0,YY0)
# Used for nodes calculated by nodeDatCalc.R (distMesh software)
# yax_index <- which(abs(xx)<0.01, arr.ind=TRUE)
# xax_index <- which(abs(yy)<0.01, arr.ind=TRUE)
# EhOY <- splinefun(yy[yax_index],U[yax_index], method = "natural") (YY0)
# EhOY[c(1,1),] <- 0 # ensure interpolated values are zero at end points
# Allocate x values at y=0
# Estimated h(x, 0)
# EhXO <- neshinterp2D(invz, crss, U, epsilon, XX0,YY0)
# EhXO <- splinefun(xx[xax_index],U[xax_index], method = "natural") (XX0)
if(plotOn == 1){
  # Plot central h cross-sections
  par(mfrow=c(2,1))
  par(mar=c(5, 4, 1, 0)+0.1) # (bottom, left, top, right)
  plot(XY0*(aDia/2),EhOY*(aDia/2),type="l",col="blue",asp=1,xlim=(-6,6),ylim=(-0.3),
       main="Central cross-section, h(0,y)")
  grid(1,1,col="darkgray")
  plot(XY0*(aDia/2),EhXO*(aDia/2),type="l",col="blue",asp=1,xlim=(-6,6),ylim=(-0.3),
       main="Central cross-section, h(x,0)")
  grid(1,1,col="darkgray")
  par(mfrow=c(1,1))
}

References