# Path Homotopy Invariants and their Application to Optimal Trajectory Planning

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### Abstract

We consider the problem of optimal path planning in different homotopy classes in a given environment. Though important in applications to robotics, homotopy path-planning in applications usually focuses on subsets of the Euclidean plane. The problem of finding optimal trajectories in different homotopy classes in more general configuration spaces (or even characterizing the homotopy classes of such trajectories) can be difficult. In this paper we propose automated solutions to this problem in several general classes of configuration spaces by constructing presentations of fundamental groups and giving algorithms for solving the *word problem* in such groups. We present explicit results that apply to knot and link complements in 3-space, and also discuss how to extend to cylindrically-deleted coordination spaces of arbitrary dimension.

## 1. Introduction

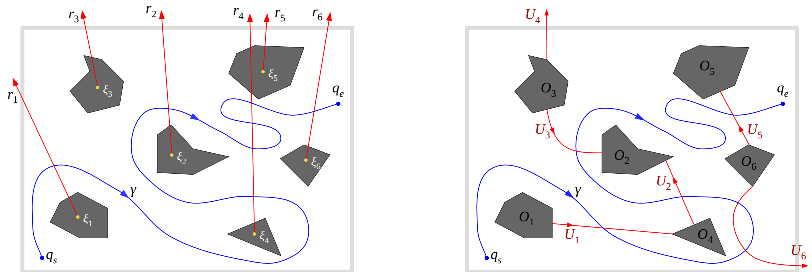
In the context of robot motion planning, one often encounters problems requiring optimal trajectories (paths) in different homotopy classes. For example, consideration of homotopy classes is vital in planning trajectories for robot teams separating/caging and transporting objects using a flexible cable (Bhattacharya et al. 2015), or in planning optimal trajectories for robots that are tethered to a base using a fixed-length flexible cable (Kim et al. 2014). This paper addresses the problem of optimal path planning with homotopy class as the optimization constraint.

There is, certainly, a large literature on minimal path-planning in computational geometry (for a brief sampling and overview, see (Mitchell and Sharir 2004)) Of course, since the problem of computing shortest paths (even for a 3-d simply-connected polygonal domain) is NP-hard (Canny and Reif 1987), we must restrict attention to subclasses of spaces, even when using homotopy path constraints. This paper focuses on two interesting and completely different types: (1) knot and link complements in 3-d; and (2) cylindrically-deleted coordination spaces (Ghrist and LaValle 2006).

## 2. Configuration Spaces with Free Fundamental Groups

### 2.1. *Motivation: Homotopy Invariant in* $(\mathbb{R}^2 - \mathcal{O})$

We are interested in constructing computable *homotopy invariants* for trajectories in a configuration space that are amenable to graph search-based path planning. To that end there is a very simple construction for configuration spaces of the form $\mathbb{R}^2 - \mathcal{O}$ (Euclidean plane punctured by obstacles) (Grigoriev and Slissenko 1998; Hershberger and Snoeyink 1991; Tovar et al. 2008; Bhattacharya et al. 2015; Kim et al. 2014): We start by placing *representative points*, $\zeta_i$, inside the $i^{th}$ connected component of the obstacles, $O_i \subset \mathcal{O}$. We then construct non-intersecting rays, $r_1, r_2, \cdots, r_m$, emanating from the representative points (this is always possible, for example, by choosing the rays to be parallel to each other). Now, given a curve $\gamma$ in $\mathbb{R}^2 - \mathcal{O}$, we construct a *word* by tracing the curve, and every time we cross a ray $r_i$ from its right to left, we insert the letter "$r_i$" into the word, and every time we cross it from left to right, we insert a letter "$r_i^{-1}$" into the word, with consecutive $r_j$ and $r_j^{-1}$ canceling each other. The word thus constructed is written as $h(\gamma)$. For example, in Figure 1(a), $h(\gamma) =$ "$r_1^{-1} r_4\, r_2^{-1} r_4^{-1} r_4\, r_4^{-1} r_6^{-1}$" $=$ "$r_1^{-1} r_4\, r_2^{-1} r_4^{-1} r_6^{-1}$". This word, called the *reduced word* for the trajectory $\gamma$, is a complete homotopy invariant




(a) With *rays* as the $U_i$'s: $h(\gamma)$ = "$r_1^{-1} r_4\, r_2^{-1} r_4^{-1} r_6^{-1}$".

(b) With a different choice of the $U_i$'s: $h(\gamma) =$ "$u_1^{-1} u_6\, u_6^{-1} u_2\, u_3\, u_5^{-1}$" = "$u_1^{-1} u_2\, u_3\, u_5^{-1}$".

FIGURE 1. Homotopy invariants of curves such as $\gamma$ in $(\mathbb{R}^2 - \mathcal{O})$ are words constructed by tracing $\gamma$ and inserting letters in the word for every crossing of the chosen oriented sub-manifolds, $U_i$ (in red).

for trajectories connecting the same set of points. That is, $\gamma_1, \gamma_2 : [0,1] \rightarrow (\mathbb{R}^2 - \mathcal{O})$, with $\gamma_i(0) = q_s, \gamma_i(1) = q_e$ are homotopic if and only if $h(\gamma_1) = h(\gamma_2)$.

2.2. *Words as Homotopy Invariants in Spaces with Freely Generated Fundamental Groups* In a more general setting the aforesaid construction can be generalized as follows:

CONSTRUCTION 1. *Given a $D$-dimensional manifold (possibly with boundary), $X$, suppose $U_1, U_2, \cdots U_n$ are $(D-1)$-dimensional oriented sub-manifolds (not necessarily smooth and possibly with boundaries) such that $\partial U_i \subseteq \partial X$. Then, for any curve, $\gamma$ (connecting fixed start and end points, $x_s, x_e \in X$), which is in general position (transverse) w.r.t. the $U_i$'s, one can construct a word by tracing the curve and inserting into the word a letter, $u_i$ or $u_i^{-1}$, whenever the curve intersects $U_i$ with a positive or negative orientation respectively.*

The proposition below is a direct consequence of a simple version of the Van Kampen's Theorem (of which several different generalizations are available in the literature).

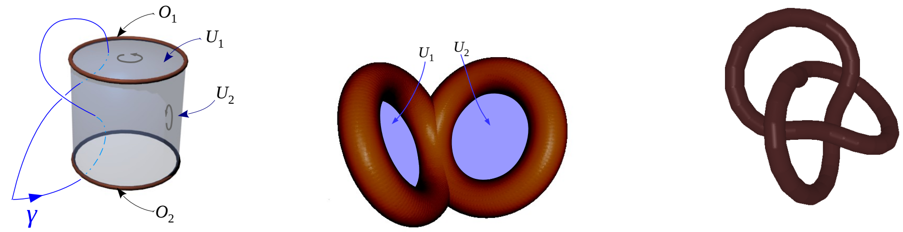
PROPOSITION 1. *Words constructed as described in Construction 1 are complete homotopy invariants for curves in $X$ joining the given start and end points if the following conditions hold:*
*(a)* $U_i \cap U_j = \emptyset,\ \forall i \neq j$.
*(b)* $X - \bigcup_{i=1}^n U_i$ *is simply-connected, and,*
*(c)* $\pi_1(X - \bigcup_{i=1, i\neq j}^n U_i) \simeq \mathbb{Z},\ \forall j = 1, 2, \cdots, n,$

*Proof.* Consider the spaces $X_0 = X - \bigcup_{i=1}^n U_i$ and $X_j = X - \bigcup_{i=1, i\neq j}^n U_i,\ j = 1, 2, \cdots, n$. Due to the aforesaid properties of the $U_i$'s the set $\mathcal{C}_X = \{X_0, X_1, \cdots, X_n\}$ constitutes an open cover of $X$, is closed under intersection, the pairwise intersections $X_i \cap X_j = X_0, i \neq j$ are simply-connected (and hence path connected), and so are $X_i \cap X_j \cap X_k = X_0, i \neq j \neq k$.

The proof, when $\gamma$ is a closed loop (*i.e.* $x_s = x_e$), then follows directly from the Seifert-van Kampen theorem (Hatcher 2001; Crowell 1959) by observing that $\pi_1(X) \simeq \pi_1(X_0) * \pi_1(X_1) * \pi_1(X_2) * \cdots * \pi_1(X_n) \simeq *_{i=1}^n \mathbb{Z}$, the free product of $n$ copies of $\mathbb{Z}$, each $\mathbb{Z}$ generated due to the restriction of the curve to $X_i,\ i = 1, 2, \cdots, n$.

When $\gamma_1$ and $\gamma_2$ are curves (not necessarily closed) joining points $x_s$ and $x_e$, they are in the same homotopy class iff $\gamma_1 \cup -\gamma_2$ is null-homotopic — that is, $h(\gamma_1) \circ h(-\gamma_2) =$ " " $\Leftrightarrow h(\gamma_1) = h(\gamma_2)$ (where by "$\circ$" we mean word concatenation). $\square$

The Construction 1 gives a *presentation* (Epstein 1992) of the fundamental group of $X$ (which, in this case, is a free group due to the Van Kampen's theorem) as the group generated by the set of letters $\mathsf{U} = \{u_1, u_2, \cdots, u_q\}$, and is written as $G = \pi_1(X) = < u_1, u_2, \cdots, u_q > = < \mathsf{U} >$. In our earlier construction with the rays, $X = \mathbb{R}^2 - \mathcal{O}$ was the configuration space, and $U_i = X \cap r_i$ were the support of the rays in the configuration space. It is easy to check that the conditions in the above proposition are satisfied with these choices. However such choices of rays is not




(a) The surfaces, $U_i$'s, satisfy the conditions of Proposition 1. $h(\gamma) =$ "$u_2\ u_1^{-1} u_2$"

(b) With a genus-2 unknotted obstacle, one can still find, $U_i$'s satisfying the conditions of Proposition 1.

(c) When the obstacle is a trefoil knot, it's not possible to find $U_i$'s satisfying the conditions of Proposition 1.

FIGURE 2. The fundamental groups of configuration spaces, $\mathbb{R}^3 - \mathcal{O}$, may or may not be freely generated.

the only possible construction of the $U_i$'s satisfying the conditions of Proposition 1. Figure 1(b) shows a different choice of the $U_i$'s that satisfy all the conditions.

### 2.3. *Simple Extension to* $(\mathbb{R}^3 - \mathcal{O})$ *with Unlinked Unknotted Obstacles*

The construction described in Section 2.1 can be easily extended to the 3-dimensional Euclidean space punctured by a finite number of un-knotted and un-linked toroidal (possibly of multi-genus) obstacles. The role of the "rays" in here is played by 2-dimensional sub-manifolds, $U_i$, that satisfy the conditions in Proposition 1, with a letter, $u_i$ (or $u_i^{-1}$), being inserted in $h(\gamma)$ every time the curve, $\gamma$, crosses/intersects a sub-manifold $U_i$ (Figures 2(a), 2(b)).

However, a little investigation makes it obvious that such 2-dimensional sub-manifolds cannot always be constructed when the obstacle are knotted or linked (Figure 2(c)). One can indeed construct surfaces (*e.g.* Seifert surfaces) satisfying some of the properties, but not all.

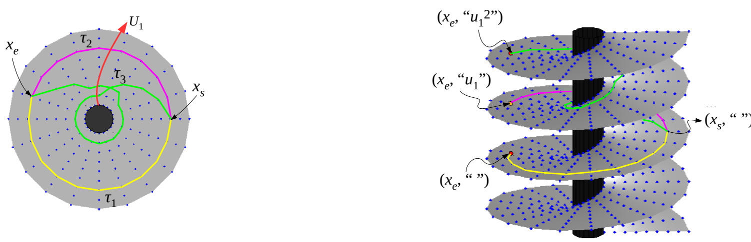
### 2.4. *Application to Graph Search-based Path Planning*

Using the homotopy invariants described in the previous sub-section, we describe a graph construction for use in search-based path planning for computing optimal (in the graph) trajectories in different homotopy classes. We first fix the set of sub-manifolds $\{U_1, U_2, \cdots, U_n\}$ as described earlier. Now, given a discrete graph representation of the configuration space, $G = (V, E)$ (*i.e.*, the vertex set, $V$, consists of points in $X$, and the edge set, $E$, contains edges that connect *neighboring* vertices) such that $x_s \in V$, we construct an $h$-augmented graph, $G_h = (V_h, E_h)$, which is essentially a *lift* of $G$ into the *universal covering space* of $X$ (Hatcher 2001). The construction of such augmented graphs has been described in our prior work (Bhattacharya et al. 2015; Kim et al. 2014), and the explicit construction of $G_h$ can be described as follows:

i. Vertices in $V_h$ are tuples of the form $(x, w)$, where $x \in V$ and $w$ is a word made out of letters $u_i$ and $u_i^{-1}$.
ii. $(x_s, \text{`` ''}) \in V_h$.
iii. For every edge $[x_1, x_2] \in E$ and every vertex $(x_1, w) \in V_h$, there exists an edge $(x_2, w \circ h(\overrightarrow{x_1 x_2})) \in E_h$, where $\overrightarrow{x_1 x_2}$ denotes the directed curve that constitutes the edge $[x_1, x_2]$.
iv. The length/cost of an edge in $G_h$ is same as its projection in $G$: $C_{G_h}([(x_1, w_1), (x_2, w_2)]) = C_G([x_1, x_2])$.

The item 'i.' is just a qualitative description of the vertices in $G_h$. Item 'ii.' describes one particular vertex in $G_h$, and using that, item 'iii.' describes an incremental construction of the entire graph $G_h$. The topology of $G_h$ can be described as a lift of $G$ into the universal covering space, $\widetilde{X}$, of $X$, and is illustrated in Figure 3 for a uniform cylindrically discretized space with a single disk-shaped obstacle.

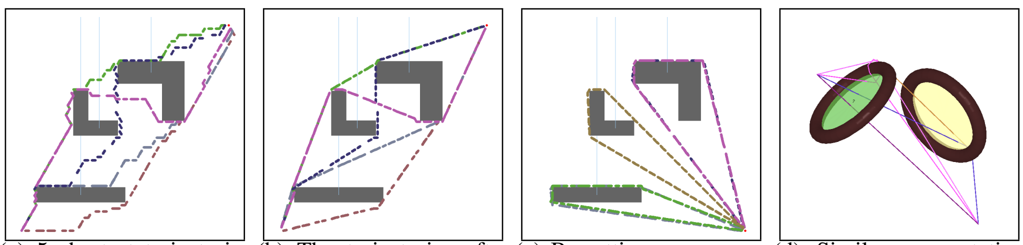
Such an incremental construction is well-suited for use in graph search algorithms such as Dijkstra's or A* (Cormen et al. 2001), in which one initiates an *open set* using the start vertex (in item ii.), and then gradually *expands* vertices, generating only the neighbors at every expansion (the recipe for which is given by item 'iii.'). Executing a search (Dijkstra's/A*) in $G_h$ from




(a) The configuration space, $X$ (light gray), and the vertex set, $V$ (blue dots).

(b) The universal covering space, $\widetilde{X}$, and the vertex set, $V_h$. Note how the trajectories lift to have different end points.

FIGURE 3. The $h$-augmented graph, $G_h$, is a lift of $G$ into $\widetilde{X}$.



(a) 5 shortest trajectories in $G$ belonging to different homotopy classes. (obstacles in gray)

(b) The trajectories after being *shortened*, but belonging to same homotopy classes.

(c) By setting $x_s = x_e = x_0$, the same method can be used to find shortest loops passing through $x_0$.

(d) Similar computation in $\mathbb{R}^3 - \mathcal{O}$, when its fundamental group is freely generated.

FIGURE 4. Simple results in configuration spaces that have freely generated fundamental groups. The dot/dash pattern and colors are shown to distinguishing between the trajectories.

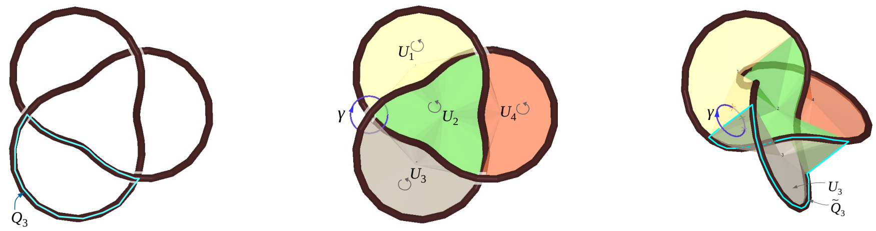
$(x_s, \text{“ ”})$ to vertices of the form $(x_e, *)$ (where '$*$' denotes any word), and projecting it back to $G$, gives us optimal trajectories in $G$ that belong to different homotopy classes. Figure 4(a) shows 5 such optimal trajectories in the graph, connecting a given start and goal vertex, where $G$ was constructed by an uniform hexagonal discretization of the planar configuration space. One can then employ a simple curve shortening algorithm (Kim et al. 2014) to obtain ones more optimal than the ones restricted to $G$ (Figure 4(b)). Similarly, shortest trajectories connecting $x_s$ and $x_e$ can be obtained in 3-dimensional configuration spaces (Figure 4(d)) with freely generated fundamental group.

## 3. Knot and Link Complements

As described earlier, when the obstacle set in $\mathbb{R}^3$ consists of knots and links, it is in general not possible to find the sub-manifolds $U_i \subset (\mathbb{R}^3 - \mathcal{O})$ as required by Proposition 1. However, thankfully we have more generalized versions of the Van Kampen theorem at our disposal that lets us extend the proposed methodology to such spaces. We first illustrate the generalization in $\mathbb{R}^3 - \mathcal{O}$ using knot/link diagrams.

### 3.1. *Dehn Presentation of Fundamental Group of Knot/Link Complements*

For simplicity we consider knots and links in $\mathbb{R}^3$ as obstacles. We assume that the knots/links are described by polygons, all of which together constitute $O \subset \mathbb{R}^3$. The *thickened* obstacles (the knots/links with the tubular neighborhoods) will be referred to as $\mathcal{O}$. We consider a knot/link diagram (Lickorish 1997) of the obstacles: Given a projection map, $p : \mathbb{R}^3 \to \mathbb{R}^2$, the knot/link diagram is the projection of the knot/link, $p(O)$, along with additional information about the $z$-*ordering* at the self-intersections of $p(O)$. We assume that in this diagram the self-intersections are all transverse (which can always be achieved through infinitesimal perturbations) and that the diagram divides the plane into simply-connected regions (say $q$ counts of them) each bounded by segments of the projected obstacles, and one unbounded exterior region. The *boundary* (the boundary of the closure) of each of the bounded regions is itself a polygon, $Q_i \subseteq p(O)$, $i =$




(a) Knot diagram, showing one of the polygons, $Q_3$ (in cyan).

(b) The surfaces, $U_i$, shown in different colors. The closed loop $\gamma$ is null-homotopuc, but $h(\gamma) =$ "$u_1^{-1} u_2\, u_3^{-1}$". Top and side views.

FIGURE 5. Constructing the surfaces, $U_i$, from polygonal knot/link diagrams (polygon segments shown as thickened cylinders for easy visualization). Null-homotopic loops as $\gamma$ have non-empty words.

$1, 2, \cdots, q$ (Figure 5(a)). Clearly $p^{-1}(Q_i) \cap O$ (the preimage of $Q_i$ in the original obstacle) will be a discontinuous polygon, with discontinuities at the preimages of the self-intersection points on the knot diagram. But these discontinuities can be removed simply by "connecting" the preimages at each self-intersection point, resulting into a spatial polygon, $\widetilde{Q}_i$ with the property that $p(\widetilde{Q}_i) = Q_i$. A simple triangulation can then be employed to construct a surface, $U_i$, in $\mathbb{R}^3 - O$, such that its boundary is $\widetilde{Q}_i$ and $p(U_i)$ is the simply-connected region bounded by $Q_i$ (this can be achieved by first triangulating the planar region, $p(U_i)$, and then *lifting* the triangulation to $\mathbb{R}^3$) – see Figure 5(a).

The $U_i$'s thus constructed satisfy properties ($b$) and ($c$) of Proposition 1, but not property ($a$), nor do they satisfy the property $\partial U_i \subseteq \partial X$. The consequence of this is that near the regions where the $U_i$'s intersect, there can be closed loops in $\mathbb{R}^3 - O$ which are null-homotopic, but words constructed simply by tracing the loop and inserting letters corresponding to intersections with the $U_i$'s, as we did earlier, may not be the empty word (identity element). An example is illustrated in Figure 5(b)). Due to our construction, such intersection of the $U_i$'s happen only along lines passing through the pre-image of the self-intersections in the knot diagram, for each of which we end up getting a null-homotopic closed loop with non-empty word.

The Dehn presentation (Weinbaum 1971) uses surfaces as constructed to describe the fundamental group of knot/link complements. We consider the free group, $G = < u_1, u_2, \cdots, u_q > = < \mathsf{U} >$. In general, for every self-intersection in the knot/link diagram, there are four adjacent surfaces, $U_{i_1}, U_{i_2}, U_{i_3}$ and $U_{i_4}$ in the order as shown in Figure 6 (when the self-intersection is adjacent to the unbounded region in the knot diagram, there are only three). Correspondingly, there is a closed null-homotopic loop, $\gamma_i$, that has a word $\rho_i =$ "$u_{i_1}\, u_{i_2}^{-1} u_{i_3}\, u_{i_4}^{-1}$". Thus we have such words $\rho_1, \rho_2, \cdots, \rho_m$ (assuming there are $m$ counts of self-intersections) that represent null-homotopic loops. These words are called *relations* and we call the set $\mathsf{R} = \{\rho_1, \rho_2, \cdots, \rho_m\}$ the *relation set*. It can be easily noted that inverses and cyclic permutations of each $\rho_i$ also corresponds to null-homotopic loops. We thus define the *symmetricized relation set*, $\overline{\mathsf{R}}$, as the set containing all the words in $\mathsf{R}$, all their inverses, and all cyclic permutation of each of those.

Let the normal subgroup of $G$ generated by $\overline{\mathsf{R}}$ be $N = \{$"$\alpha_1 \rho_{i_1} \alpha_1^{-1}\ \alpha_2 \rho_{i_2} \alpha_2^{-1} \cdots \alpha_\kappa^{-1}\ \alpha_\kappa \rho_{i_\kappa} \cdots$" $\mid \alpha_k \in G,\ \rho_{i_k} \in \overline{\mathsf{R}}\} = < \overline{\mathsf{R}}^G >$ (normal closure of $\overline{\mathsf{R}}$ in $G$). It is easy to observe that a closed loop, $\gamma$, in $X = \mathbb{R}^3 - \mathcal{O}$, has a word that is an element of $N$ iff it is null-homotopic. Due to a more general version of the Van Kampen's theorem (Hatcher 2001), the fundamental group of $X$ is the quotient group, $\pi_1(X) = G/N = < \mathsf{U} \mid \mathsf{R} >$ — the group in which, under the quotient map, elements of $N$ are mapped to the identity element.

### 3.2. *The Word Problem and Dehn Algorithm*

Due to the discussion above, two trajectories, $\gamma_1, \gamma_2$, connecting $x_s$ and $x_e$ in the knot/link complement, $X$, belong to the same homotopy class iff the word $h(\gamma_1 \cup -\gamma_2) = h(\gamma_1) \circ h(\gamma_2)^{-1}$ belongs to $N = < \overline{\mathsf{R}}^G >$. This problem in group theory is known as the *word problem* (Epstein 1992), and there are various algorithms, each suitable for specific types of groups, for solving the

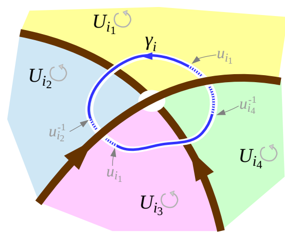



FIGURE 6. A "self-intersection" in a knot/link diagram, with a null-homotopic loop, $\gamma_i$, intersecting the surfaces adjacent to the intersection. $h(\gamma_i) =$ "$u_{i_1}\, u_{i_2}^{-1} u_{i_3}\, u_{i_4}^{-1}$". One or more of the surfaces can be non-existent, in which case the corresponding letters are simply absent from the word. These words constitute $\mathsf{R}$, and should map to the identity element in $\pi_1(X) =<\mathsf{U} \mid \mathsf{R}>$

word problem. We, in particular, will focus on a very simple algorithm due to Max Dehn (Lyndon and Schupp 2001; Greendlinger and Greendlinger 1986), which is applicable to a wide class of groups and their presentations.

*Dehn's metric algorithm:* Given a presentation of a group, $\pi_1 =< \mathsf{U} \mid \mathsf{R} >$, we construct the *symmetricized* relation set $\overline{\mathsf{R}}$ as described earlier. Given a cyclically reduced word, $w$, made up of letters (and their inverses) from $\mathsf{U}$, one checks for every element $\rho \in \overline{\mathsf{R}}$ if $w$ and $\rho$ share a common sub-words that is of length greater than $|\rho|/2$ ($|\rho|$ being the length of $\rho$). If they do (say, $\rho = \alpha\beta\gamma$, with $\beta$ being a sub-word appearing in $w$, and $|\beta| > |\rho|/2$), we replace the sub-word with the shorter equivalent that one obtains by setting $\rho$ to the identity element (*i.e.*, replace $\beta$ by $\alpha^{-1}\gamma^{-1}$ in $w$). This process is repeated, and the algorithm terminates when no more such sub-words are found. The final word at which the algorithm terminates indicates if the initial word, $w$, is in $N$ (whether it maps to the identity element in $\pi_1$).

This algorithm can be used in conjunction with search in $G_h$ as before for finding optimal trajectories in different homotopy classes, with two vertices $(x, w_1), (x, w_2) \in V_h$ being the same iff $h(w_1) \circ h(-w_2)$ reduces to the empty word upon applying the Dehn's metric algorithm.

### 3.3. *Guarantees of Dehn Algorithm*

It's well known (Greendlinger and Greendlinger 1986) that if Dehn algorithm terminates at the empty word, then $w \in N$. However, the converse is not necessarily true. One can derive several sufficient (and often highly restrictive) conditions on the presentation $<\mathsf{U} \mid \mathsf{R}>$ under which the converse holds (Lyndon and Schupp 2001). If, for a given presentation of a group the converse holds, we say that Dehn algorithm is *complete* for that presentation (or that the presentation is complete with respect to the Dehn algorithm, or that the word problem is solvable using the specific presentation and Dehn algorithm).

Due to the result of (Weinbaum 1971), the Dehn presentation of the fundamental groups of the complement of a tame, alternating, prime knot is complete with respect to the Dehn algorithm. It is also known (Epstein 1992) that automatic groups (including hyperbolic groups) have presentations that are complete with respect to Dehn algorithm.

## 4. Cylindrically-deleted Configuration Spaces

The previous results are limited to 3-dimensional spaces: one suspects that higher dimensions are more difficult. However, there are some classes of spaces for which optimal path-planning with homotopy constraints is still computable via a Dehn algorithm, independent of dimension. The following class of examples is inspired by robot coordination problems, in which individual agents with predetermined motion paths have to coordinate their motions so as to avoid collision.

Consider a collection of $n$ graphs $(\Gamma_i)_1^n$, each embedded in a common workspace (usually $\mathbb{R}^2$ or $\mathbb{R}^3$) with intersections permitted. In the simplest case, each $\Gamma_i$ will be homeomorphic to a closed interval, but more general graphs are permitted, such as roadmap approximations to a configuration space. On each $\Gamma_i$, a robot $R_i$ with some particular fixed size/shape is free to move. Such motion may be Euclidean (by translation/rotation); more general motions are possible, so long as the region occupied by the robot $R_i$ in the workspace is purely a function of location on $\Gamma_i$. A point in the product space $\prod_i \Gamma_i$ determines the locations of the $n$ robots in the common workspace. Certain configurations are illegal, due to collisions. For example, if the robots are point-like, and each $\Gamma_i = \Gamma$ is identical, then the configuration space of $n$ points on $\Gamma$ is the cross product $\prod_i \Gamma_i$ minus the pairwise diagonal $\Delta$. If the robots are given finite extent, then this system has a configuration space obtained by the graph product $\prod_i \Gamma_i$ minus an $\epsilon$-neighborhood of the pairwise diagonal. However, more general types of collisions can be defined, say, if the robots are irregularly shaped and the graphs $\Gamma_i$ are all different. In this most general case, the natural analogue of a configuration space is the coordination spaces of (Ghrist and LaValle 2006).

The *coordination space* of this system is defined to be the space of all configurations in $\prod_i \Gamma_i$ for which there are no collisions – the geometric robots $R_i$ have no overlaps in the workspace. Under the assumption that collisions between robots are pairwise-defined, the coordination space is *cylindrically deleted* and of the form

$$X = \left(\prod_{i=1}^{n} \Gamma_i\right) - \mathcal{O} \quad \text{where} \quad \mathcal{O} = \bigcup_{i<j} \left\{ (x_k)_1^N \in \prod_{k=1}^{N} \Gamma_k \ : \ (x_i, x_j) \in \Delta_{i,j} \right\},$$

for some (open, "collision") sets $\Delta_{i,j} \subset \Gamma_i \times \Gamma_j$ where $1 \leq i < j \leq N$. In what follows, we assume that the $\Delta_{i,j}$ are sufficiently tame (e.g., semialgebraic) so as to avoid issues of non-finitely-generated $\pi_1$. Given (internal, path-) metrics on each $\Gamma_i$, the coordination space $X$ inherits a locally-Euclidean metric on products of edges in the graphs. Such $X$ are complete path-spaces and thus the problem of geodesics is well-posed. Their fundamental groups can be (highly) non-trivial, depending on the obstacle set $\mathcal{O}$. However, finding optimal paths subject to homotopy classes is still computable. To that end one can construct the subspaces $U_i \subset X$ of co-dimension 1, and the relation set $\mathsf{R}$, and use them to design complete homotopy invariants as before. We do not discuss the explicit construction of the $U_i$'s for cylindrically-deleted coordination spaces in this paper, but provide the following theorem on solvability of the word problem in such spaces.

THEOREM 1. *Any compact cylindrically-deleted coordination space $X$ admits a Dehn algorithm for $\pi_1$.*

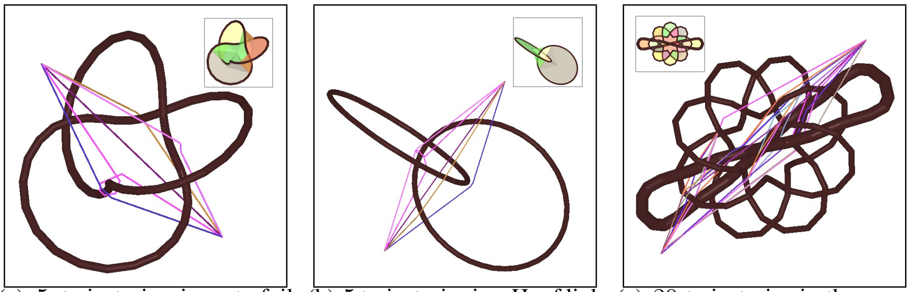
*Proof.* Any such $X$ is realized as a Hausdorff limit of cubcial complexes which were shown in (Ghrist and LaValle 2006, Thm 4.4) to be nonpositively-curved and to stabilize in $\pi_1$ by tameness. All nonpositively-curved piecewise-Euclidean cube complexes have fundamental groups which are, by a famous result of Niblo-Reeves (Niblo and Reeves 1998), *biautomatic*. Biautomatic groups all admit a Dehn algorithm (specifically, there is a quadratic isoperimetric inequality) (Epstein 1992). □

It is worth noting that $\ell_2$-shortest paths are perhaps not the most natural optimization for coordination spaces. It would be interesting to consider other ($\ell_1$, $\ell_\infty$) pointwise norms.

## 5. Simulation Results in Knot and Link Complements

Given obstacles $\mathcal{O} \subset \mathbb{R}^3$, and their "skeletons" (1-dimensional homotopy equivalents), $O \subseteq \mathcal{O}$ as polygons in $\mathbb{R}^3$, we first choose a projection map, $p : \mathbb{R}^3 \to \mathbb{R}^2$, for the knot/link diagram. With this information, we implemented the automated construction of the surfaces, $U_i$, for the Dehn presentation of the knot/link complement, and the symmetricized relation set, $\overline{\mathsf{R}}$, by computing the self-intersections in $p(O)$. We then used a uniform cubical discretization of $\mathbb{R}^3 - \mathcal{O}$ to construct the graph $G$ as a discrete representation of the free space, and in the $h$-augmented graph, $G_h$, we find trajectories from $(x_s, \text{" "})$ to $(x_g, *)$. We then employ a curve shortening algorithm to shorten the obtained trajectories. All our implementations were done in C++ programming language and visualization were done using OpenGL. The program ran on a laptop running on a Intel i7-4500U processor @ 1.80GHz with 8 GB memory.

Figure 7(a) shows results in the complement of a trefoil knot. The inset figure shows the surfaces, $U_i$, used for Dehn presentation. The graph $G$ was constructed out of uniform $100 \times 100 \times 100$ cubical discretization of the environment. The entire computation (computation of



(a) 5 trajectories in a trefoil knot complement.
(b) 5 trajectories in a Hopf link complement.
(c) 20 trajectories in the complement of a $(3, 8)$ torus knot linked to a genus-2 torus.

FIGURE 7. Optimal trajectories (in discrete graph representation, followed by curve shortening) in different homotopy classes in complements of knots and links. Insets show the surfaces, $U_i$.

the surfaces, the symmetricized relation set $\mathsf{R}$, and computation of the 5 shortest trajectories) required about $8.1$ s. Likewise, Figure 7(b) shows results in the complement of a simple Hopf link, with the same discretization of the environment, and total computation time of about $8.2$ s. Figure 7(c) shows a much more complex obstacle involving a torus knot linked to a genus-2 obstacle, and the entire computation of 20 trajectories took about $2.6$ s.

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