Jordan Canonical Form

Theory and Practice

Synthesis Lectures on Mathematics and Statistics

Editor
Steven G. Krantz, ***Washington University, St. Louis***

Jordan Canonical Form: Theory and Practice
Steven H. Weintraub
2009

The Geometry of Walker Manifolds
Miguel Brozos-Vázquez, Eduardo García-Río, Peter Gilkey, Stana Nikcevic, Rámon Vázquez-Lorenzo
2009

An Introduction to Multivariable Mathematics
Leon Simon
2008

Jordan Canonical Form: Application to Differential Equations
Steven H. Weintraub
2008

Statistics is Easy!
Dennis Shasha, Manda Wilson
2008

A Gyrovector Space Approach to Hyperbolic Geometry
Abraham Albert Ungar
2008

Jordan Canonical Form: Theory and Practice

Steven H. Weintraub

www.morganclaypool.com

ISBN: 9781608452507 paperback
ISBN: 9781608452514 ebook

DOI 10.2200/S00218ED1V01Y200908MAS006

A Publication in the Morgan & Claypool Publishers series
SYNTHESIS LECTURES ON MATHEMATICS AND STATISTICS

Lecture #6
Series Editor: Steven G. Krantz, *Washington University, St. Louis*
Series ISSN
Synthesis Lectures on Mathematics and Statistics
Print 1938-1743 Electronic 1938-1751

Jordan Canonical Form

Theory and Practice

Steven H. Weintraub
Lehigh University

SYNTHESIS LECTURES ON MATHEMATICS AND STATISTICS #6

MORGAN & CLAYPOOL PUBLISHERS

ABSTRACT

Jordan Canonical Form (*JCF*) is one of the most important, and useful, concepts in linear algebra. The *JCF* of a linear transformation, or of a matrix, encodes all of the structural information about that linear transformation, or matrix. This book is a careful development of *JCF*. After beginning with background material, we introduce Jordan Canonical Form and related notions: eigenvalues, (generalized) eigenvectors, and the characteristic and minimum polynomials. We decide the question of diagonalizability, and prove the Cayley–Hamilton theorem. Then we present a careful and complete proof of the fundamental theorem: *Let V be a finite-dimensional vector space over the field of complex numbers $\mathbf{C}$, and let $\mathcal{T} : V \longrightarrow V$ be a linear transformation. Then T has a Jordan Canonical Form.* This theorem has an equivalent statement in terms of matrices: *Let A be a square matrix with complex entries. Then A is similar to a matrix J in Jordan Canonical Form, i.e., there is an invertible matrix P and a matrix J in Jordan Canonical Form with $A = PJP^{-1}$.* We further present an algorithm to find P and J, assuming that one can factor the characteristic polynomial of A. In developing this algorithm we introduce the eigenstructure picture (*ESP*) of a matrix, a pictorial representation that makes *JCF* clear. The *ESP* of A determines J, and a refinement, the labelled eigenstructure picture (ℓ*ESP*) of A, determines P as well. We illustrate this algorithm with copious examples, and provide numerous exercises for the reader.

KEYWORDS

Jordan Canonical Form, characteristic polynomial, minimum polynomial, eigenvalues, eigenvectors, generalized eigenvectors, diagonalizability, Cayley–Hamilton theorem, eigenstructure picture

To my brother Jeffrey and my sister Sharon

Contents

Preface ... xi

1 Fundamentals on Vector Spaces and Linear Transformations ... 1

1.1 Bases and Coordinates ... 1
1.2 Linear Transformations and Matrices ... 4
1.3 Some Special Matrices ... 8
1.4 Polynomials in $\mathcal{T}$ and A ... 11
1.5 Subspaces, Complements, and Invariant Subspaces ... 13

2 The Structure of a Linear Transformation ... 17

2.5 Eigenvalues, Eigenvectors, and Generalized Eigenvectors ... 17
2.2 The Minimum Polynomial ... 23
2.3 Reduction to *BDBUTCD* Form ... 26
2.4 The Diagonalizable Case ... 31
2.5 Reduction to Jordan Canonical Form ... 33
2.6 Exercises ... 40

3 An Algorithm for Jordan Canonical Form and Jordan Basis ... 43

3.1 The *ESP* of a Linear Transformation ... 43
3.2 The Algorithm for Jordan Canonical Form ... 48
3.3 The Algorithm for a Jordan Basis ... 53
3.4 Examples ... 56
3.5 Exercises ... 75

A Answers to odd-numbered exercises ... 83

A.1 Answers to Exercises–Chapter 2 ... 83
A.2 Answers to Exercises–Chapter 3 ... 88

Notation 93

Index 95

Preface

Jordan Canonical Form (*JCF*) is one of the most important, and useful, concepts in linear algebra. The *JCF* of a linear transformation, or of a matrix, encodes all of the structural information about that linear transformation, or matrix. This book is a careful development of *JCF*.

In Chapter 1 of this book we present necessary background material. We expect that most, though not all, of the material in this chapter will be familiar to the reader.

In Chapter 2 we define Jordan Canonical Form and prove the following fundamental theorem: *Let V be a finite-dimensional vector space over the field of complex numbers $\mathbb{C}$, and let $\mathcal{T} : V \longrightarrow V$ be a linear transformation. Then $\mathcal{T}$ has a Jordan Canonical Form.* This theorem has an equivalent statement in terms of matrices: *Let A be a square matrix with complex entries. Then A is similar to a matrix J in Jordan Canonical Form.* Along the way to the proof we introduce eigenvalues and (generalized) eigenvectors, and the characteristic and minimum polynomials of a linear transformation (or matrix), all of which play a key role. We also examine the special case of diagonalizability and prove the Cayley–Hamilton theorem.

The main result of Chapter 2 may be restated as: *Let A be a square matrix with complex entries. Then there is an invertible matrix P and a matrix J in Jordan Canonical Form with $A = PJP^{-1}$.* In Chapter 3 we present an algorithm to find P and J, assuming that one can factor the characteristic polynomial of A. In developing this algorithm we introduce the idea of the eigenspace picture (*ESP*) of A, which determines J, and a refinement, the labelled eigenspace picture (ℓ*ESP*) of A, which determines P as well. We illustrate this algorithm with copious examples.

Our numbering system in this text is fairly standard. Theorem 1.2.3 is the third numbered result in Section 2 of Chapter 1.

We provide many exercises for the reader to gain facility in applying these concepts and in particular in finding the *JCF* of matrices. As is customary in texts, we provide answers to the odd-numbered exercises here. *Instructors* may contact me at shw2@lehigh.edu and I will supply answers to all of the exercises.

Steven H. Weintraub
Department of Mathematics, Lehigh University
Bethlehem, PA 18015 USA
July 2009