On A Theorem of Bendixon

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Abstract. The behavior of solutions of analytic autonomous differential equations is explored as the solutions approach the origin.

In this note we prove a stronger form of a theorem due to Bendixson [BEND]. Our theorem is the following:

Theorem. Let $s(t) = (x(t), y(t))$ be a real solution of the pair of equations

$$\frac{dy}{dt} = A(x,y), \quad \frac{dx}{dt} = B(x,y)(1) \tag{1}$$

defined on a domain of the form $t \in [t_0, \infty)$, approaching $(0,0)$ as a limit as $t \to \infty$, A, B real analytic in a neighborhood of $(0,0)$ in R^2, $A(0,0) = B(0,0) = 0$. Then either the solution spirals endlessly toward the origin, or it approaches the origin with a definite limiting *tangent* direction: that is, at least one of $\lim(y'/x'), \lim(x'/y')$ exists as $t \to +\infty$. (We omit from consideration the trivial solution for which $s(t)$ is at the origin for all t.)

[Recall that a solution curve $(x(t), y(t))$ is said to "spiral endlessly toward the origin" if $(x(t), y(t)) \to (0,0)$, and that if the curve is described in polar coordinates $(r(t), \theta(t))$ with $r(t)$ and $\theta(t)$ continuous, then $r(t) \to 0$ and the function $\theta(t)$ has limit $+\infty$ or limit $-\infty$ as $t \to +\infty$.]

Since the parameterization of the solution is determined (to within an arbitrary additive constant) by A and B individually, not just by the ration A/B, no solution can reach at a finite time t any point (x, y) at which A and

B have a common zero, unless it is already there for all t. It follows that every solution curve s is real analytic – that is, that the two real functions $x(t)$ and $y(t)$ are real analytic.

Bendixson only showed that, in the absence of endless spiralling, there was a limiting *secant* direction: that is, the $\lim(y/x)$ or $\lim(x/y)$ exists. Of course our form of the conclusion implies Bendixson's, but not vice versa: if

$$x = t^{-1}, \quad y = t^{-2}\sin(t^{10})(2) \tag{2}$$

could be a solution of such a differential equation, it would satisfy the conclusion of the Bendixon theorem (since $\lim y/x = 0$), but not ours (since dy/dx oscillates with ever growing amplitude as t approaches $+\infty$).

(Actually, that the curve in the above example (2) couldn't be a solution of any differential equation of the form (1) follows directly from Bendixson's theorem: for if there were such a differential equation (1), replacing y in it by xy throughout – "a quadratic transformation" – and then clearing of fractions, would result in another differential equation of the form (1), namely

$$\frac{dy}{dt} = A(x, xy) - yB(x, xy), \ \frac{dx}{dt} = xB(x, xy).$$

But that is impossible, since for the corresponding solution the value of y/x would oscillate between -1 and 1 as x approaches 0, violating the conclusions of the original Bendixson's theorem. Unfortunately, the rather similar example

$$x = t^{-1}, \quad y = e^{-t^2}\sin(t^{10})(3) \tag{3}$$

can't be disposed of as easily.)

Theorems similar to ours can be found in the literature, but only with added assumptions. A usual assumption is that the origin is an *isolated* singularity, that is, that it's the only point in some neighborhood of the origin where A and B are both zero. Our discussion does not require this assumption.

Then again, in [ANDR] the same result is obtained, but only under the hypothesis that the determinant of the matrix formed from the coefficients of the linear terms in A and B is nonzero. (See [ANDR] [pp. 165–166.) We make no such demand. Novel features of our presentations include the introduction of the *inflection set* of the differential equation (1), and the use of the basic properties of analytic sets.

We alert the reader that we have adopted the convention that when we speak of two parameterized curves intersecting, say, at (x, y), we mean that some point (x, y) lies on both curves, but not necessarily for equal values of the parameters for the two curves. If it is desirable to emphasize this interpretation, we may say that the *point sets* of the two curves intersect.

We will find the following definition useful: Let $s(t) = (x(t), y(t))$ be a pair of differentiable functions for which it is only known that

$$\frac{dy}{dt}B(x,y) = \frac{dx}{dt}A(x,y).(4) \tag{4}$$

Then we call $s(t)$ a *free-parameter solution* of the original problem, since the parameter can be freely changed subject to obvious conditions. To emphasize the difference, we may refer to solutions of (1) as *bound-parameter solutions.*

For example consider

$$\frac{dy}{dt} = -y, \quad \frac{dx}{dt} = -x.$$

Then the bound-parameter solutions all take the form

$$x = x_0 e^{-t}, \quad y = y_0 e^{-t},$$

which do not reach $(0, 0)$; but among the possible free-parameter solutions are those of the form $\{x = t, y = ct\}$, which have no trouble reaching the origin.

Before beginning the proof proper, we note for reference a few obvious or well-known facts. Proofs are given only when considered necessary. For background on the properties of analytic sets, see [LOJA64], [LOJA65], [MILN],and [WHIT].

Lemma 1. For any positive radius r, let D_r denote the closed circular disk $\{(x, y)|x^2+y^2 \leq r^2\}$. Let $K(x, y)$ be a real valued function which is analytic in an open neighborhood N of $(0, 0)$, and let $K_r = \{(x, y) \in N \cap D_r | K(x, y) = 0\}$ denote the set of zeros of $K(x, y)$ within distance r from the origin. Then there exists $R > 0$ such that D_R is contained in N, and such that for any r with $0 < r \leq R$ the set K_r has one of the following forms:

(1) K_r is empty,

(2) $K_r = (0,0)$,

(3) $K_r = D_r$, or

(4) K_r is compact, and is the union of finitely many branches $W_i = \{z_i(u)|z_i(u) = (x_i(u), y_i(u)) \text{ for } u \in [-1,1]\}$ where each z_i is a one-to-one real analytic function on $[-1,1]$ with $z_i(0) = (0,0)$. Each z_i is a homeomorphism of W_i with the closed interval $[-1,1]$. For each branch W_i at least one of $x_i'(u)/y_i'(u)$ and $y_i'(u)/x_i'(u)$ has a limit as $u \to 0$. Any two distinct branches intersect only at the origin.

We will use the term "semibranch" to refer to the part of a branch corresponding to u non-negative or to u non-positive. Note that the origin, corresponding to $u = 0$, is included in both.

Lemma 2. Let $K(x,y), R, r,$ and K_r be as in the nontrivial case, case (4), of Lemma 1. Let $s(t)$ be a solution, not identically $(0,0)$, defined on an interval $[t_0, \infty)$, with $\lim s(t) = (0,0)$ as $t \to +\infty$. We may suppose t_0 chosen large enough that $s(t)$ is in $D_{r/2}$ for all t in the interval. If $s(t)$ meets K_r tangentially at infinitely many points t_j with $t_j > t_0$, $t_j \to \infty$, then the solution lies along one of the semibranches of K_r.

Proof: Let Z be the set of points of tangential meeting described in the hypotheses of the Lemma. Since K_r is the union of only finitely many semibranches, we can choose from Z a sequence of distinct points $\{s(t_i)\}$, with $t_i \to \infty$ monotonically, all of which belong to the same semibranch: say the semibranch $\{z(u) = (x(u), y(u)) \text{ for } u \in [0,1]\}$. Call the entire branch W, the semibranch W_+.

Since the points $s(t_i)$ all lie on W_+, the points also have coordinates, say u_i, when regarded as lying on W_+. Thus we have a sequence $\{u_i\}$, with $\{z(u_i)\} = \{s(t_i)\}$, $i = 1, 2, 3, ...$, $u_i \to 0$, and $z(u_i) \to (0,0)$.

Since $s(t)$ is a solution meeting W tangentially at each $z(u_i)$, we have that the cross product $z'(u_i) \times (B(z(u_i)), A(z(u_i)))$ is zero for $i = 1, 2, 3, ...$ Then the analyticity of z, B, and A implies that $z'(u) \times (B(z(u)), A(z(u)))$ is zero for all $u \in [-1,1]$. Hence $z(u)$ is a free-parameter solution of our differential equation.

Let us examine the situation in the vicinity of any one of the points $s(t_i)$, say $P = s(t_1)$. Since this point is on a solution curve, the point can not be a singular point: that is A and B are not both zero there. Therefore the curves $s(t)$ and $z(u)$ being a solution and a free-parameter solution passing through the same point, must agree in some neighborhood of P. This neighborhood can be extended until one of the following occurs:

(1) the solution $s(t)$ reaches a singularity of (B, A) at some finite value for t and therefore can't go any further. But we know that a *bound-parameter solution* (as contrasted to a *free-parameter* solution) for given analytic functions A and B can not reach a common zero of A and B at a finite time, unless it is already there for all time. So this case can not occur.

(2) the solution reaches the boundary of the closed disk D_r in which we are operating. But this case can't occur, since t_0 was so chosen that the solution remains in $D_{r/2}$.

(3) the solution curve lies on the branch for all sufficiently large t. All that remains to be shown is that it lies on the semibranch. But since the solution remains on the branch, and can't pass through (or even reach in finite time) the origin, where both A and B are zero, it must stay within one of the two components into which removing the origin splits the branch.

This concludes the proof of Lemma 2.

Lemma 3. ("The geometric lemma"). Let $(x(t), y(t))$ be a simple (i.e., one-to-one) smooth arc in the plane for $(a, 0)$ to $(b, 0)$, $a < b$, differentiable everywhere including the endpoints, with $(x(0), y(0)) = (a, 0)$, $(x(T), y(T)) = (b, 0)$. Assume that $y(t) > 0$ everywhere except at the endpoints. Let $\psi(t)$ be a function everywhere expressing the angular direction of the tangent to the path, chosen to be continuous (therefore differentiable) on the closed interval $t \in [0, T]$. Then ψ' can't be positive everywhere on that arc.

Proof.Consider the closed path consisting of that arc from $(a, 0)$ to $(b, 0)$, followed by the straight line segment from $(b, 0)$ back to $(a, 0)$. If we assume that $\psi' > 0$ everywhere on the arc, then this is a simple closed curve, circumscribing a compact region with a clockwise orientation. Therefore, by the well-known Umlaufsatz, (see, for example, [HART] or [MILL]), the total change in the angular direction of the tangent to the curve is -2π, in the sense of the Riemann-Stieltjes integral. But this total change, for our supposed closed path, consists of the sum of:

–the rotation angle at $(b, 0)$, which has a value between $-\pi$ and 0;

–the rotation angle at $(a, 0)$, which has a value between $-\pi$ and 0; and

–the integral from $t = 0$ to $t = T$ of the supposedly strictly positive function $\psi'(t)$.

Since this integral is strictly positive, the total rotation can not be -2π.

We now begin the proof of our theorem. For convenience in cross-referencing we list here the several cases and subcases used in the proof, even though much of the notation used in describing the cases is yet to be introduced.

(1) For all t sufficiently large, $s(t)$ does not lie on the inflection set K.

(1.1) $\zeta(t) \leq 0$ for all $t > a$.

(1.2) $\zeta(c) > 0$ for some $c > a$. [We will show that this subcase can not occur.]

(1.2.1) $y(t) \geq y_L$ for all $t \in (a, c)$. [Shown impossible.]

(1.2.2) $y(t_0) < y_L$ for some $t_0 < c$; and, the last time t that $t < c$ and $s(t)$ lies on the line $y = y_L$, the point $s(t)$ is to the left of $s(c)$. [Shown impossible.]

(1.2.3) $y(t_0) < y_L$ for some $t_0 < c$; and, the last time t that $t < c$ and $s(t)$ lies on the line $y = y_L$, the point $s(t)$ is to the right of $s(c)$. [Shown impossible.]

(2) $s(t)$ is tangent to K infinitely often as $t \to +\infty$.

(3) The solution $s(t)$ crosses some semibranch $k(u)$ of K infinitely often, with infinitely many reversals of orientation as $u \to 0$. [Shown impossible.]

(4) The solution $s(t)$ crosses some semibranch $k(u)$ of K infinitely often as $t \to \infty$; and for all crossings with u close enough to 0, the orientation of the crossing is the same.

If the solution curve $s(t)$ spirals endlessly, i.e. $\lim \theta(t) = +\infty$ or $-\infty$, we are done (trivially). Hence in what follows we can assume that is not the case.

Associated with any smooth curve, in particular with any solution curve $s(t) = (x(t), y(t))$, are continuous functions $\psi(t)$ expressing the angular direction of the [directed] tangent vector; that is, such that

$$\cos\psi = x'/\sqrt{(x')^2 + (y')^2}, \quad \sin\psi = y'/\sqrt{(x')^2 + (y')^2}.$$

Choose one such $\psi(t)$. (Different choices differ by constant integer multiples of 2π). Obviously, $\lim y'/x'$, or $\lim x'/y'$ exists if and only if $\lim \psi(t)$ does.

The curve is a solution of the differential equation, so

$$\frac{d\psi}{dt} = (B^2A_x + ABA_y - ABB_x - A^2B_y)/(A^2 + B^2).$$

Abbreviate the numerator as $h(x, y)$ and the denominator as $c(x, y)$. Hence

$$\frac{d\psi}{dt} = h(x, y)/c(x, y).$$

Note that $h(x, y)$ is analytic, and $c(x, y)$ is not only analytic but is everywhere non-negative. Note also that no attempt is to be made to remove common factors of $h(x, y)$ and $c(x, y)$.

Let $K(x, y) = h(x, y)c(x, y)$, and let K be the set of points in a neighborhood N of the origin where $K(x, y) = 0$ holds. We call K the *inflection set* of the differential equation. We may certainly assume that $c(x, y)$ is not identically zero on N, since otherwise every solution of (1) would be stationary. Since $c(x, y)$ cannot vanish at a point of a solution $s(t)$, see that last statement in case (3) of the proof of Lemma 2, the only way $K(x, y)$ can be zero along $s(t)$ is for $h(x, y)$ to the zero along $s(t)$. But in this case $d\psi/dt$ will be identically zero along $s(t)$, and $\psi(t)$ will be constant, forcing s(t) to lie along a straight line. So there is nothing to prove in this case. Thus in what follows we may assume that $K(x, y)$ is not identically zero on N.

Since K is analytic, by Lemma 1 we know that K, in a sufficiently small neighborhood of the origin, is either just $(0, 0)$, or the union of a finite number of branches. For any single branch under discussion, we shall use the notation $\eta(u) = (\eta_1(u), \eta_2(u))$. From that Lemma, we have that $\eta(0) = (0, 0)$, and η is defined, analytic, and one-to-one for the real parameter u in a neighborhood of 0.

Now let $s(t)$ be a solution of the differential equation, with $s(t) \to (0, 0)$ as $t \to +\infty$, not spiraling endlessly about the origin. We distinguish four cases:

(1) For all t sufficiently large, $s(t)$ does not lie on the inflection set.

In this case, $K(s(t))$ is non-zero and of constant sign when t is sufficiently large. Hence $\psi(t)$ is ultimately strictly monotonic. For the remainder of our discussion of case (1), we will assume it to be ultimately monotonic increasing. The argument for the monotonic decreasing case is analogous.

Let the parameter value $t = a$ be one such that for all $t > a$ we have $|s(t)| < |s(a)|$ and $d\psi/dt > 0$. Then $s(t)$ comes into the circle of radius $|s(a)|$ at $t = a$, and never leaves its interior. That there are such parameter values can be seen as follows: Choose any number $P > 0$ such that $s(t)$ is defined and $d\psi/dt > 0$ for all $t \geq P$. Since $s(t) \to (0, 0)$, there exists a number $Q > P$ such that $|s(t)| < |s(P)|/4$ for all $t > Q$. For some value of t between P and Q, we must have $|s(t)| = |s(P)|/2$. Then for a we can take the maximum of the nonempty compact set of reals $\{t|\ |s(t)| = |s(P)|/2\} \cap [P, Q]$.

Let $\zeta(t) = \sin(\theta - \psi)$, where θ is the polar coordinate angle function of $s(t)$, and ψ is the function giving the angular direction of the [directed] tangent to the curve at that point. $\zeta(t)$ differs by a positive factor (namely $|s||s'|$) from $x'y - xy'$. We have two subcases.

(1.1) $\zeta(t) \leq 0$ for all $t > a$.

This implies that the difference between θ and ψ doesn't vary outside an interval of lenght π, since the sine of their difference remains non-positive, so that $(\theta - \psi)$ is always in the third or fourth quadrant. So since θ does not have $+\infty$ as a limit, and ψ is monotone increasing, the latter approaches a finite limit.

(1.2) $\zeta(c) > 0$ for some $c > a$. [We will show that this subcase can not occur.]

Let L be the tangent to the curve at this time c. More precisely, let L be the ray (half-line) starting at the point of tangency $s(c)$ and extending in the direction which continues the motion of s.

For simplicity of notation, assume the coordinate system rotated so that $\psi(c) = 0$: that is, that at this time the tangent line is horizontal and that $s'(c)$ is pointing to the right. The assumption that $\zeta(c) > 0$ implies that L lies above the origin: that is, that its y coordinate, in the rotated coordinate system, is positive. (This is because $\sin(\theta - \psi) > 0$; the rotation has set $\psi = 0$; so θ is in the first or second quadrant.) Let Y_L denote that y coordinate.

Note that in this coordinate system the curve, at time c, is moving horizontally from left to right and is concave up (since $\psi' > 0$). Immediately to the left and right of that point, $y(t) > y_L$. (Recall that $y(t)$ is the y coordinate of $s(t)$.) We will refer to these observations, or any part of them, as "the concavity at c."

We shall show that $y(t) \geq y_L$ for all $> c$. Hence the curve can't approach the origin, showing that subcase (1.2) can not occur.

(1.2.1) Assume that $y(t) \geq y_L$ for all $t \in (a, c)$. If we also had $y(t) \geq y_L$ for all $t > c$, then the curve would evidently not have the origin as limit. So let b be the first time after $t = c$ that $y(b) = y_L$. (That there is a *first* such time follows from the concavity at c.) We may also use b_i or d to indicate such times.

Consider the region bounded by the simple closed curve C formed from the three arcs C_1, C_2, C_3:

– C_1: the arc of the solution curve from $s(a)$ to $s(c)$;

– C_2: the segment of the straight line L going left-to-right from $s(c)$ to the point W where the line meets the circle of radius $|s(a)|$ about the origin; and

– C_3: the segment of that circle running counterclockwise from W to $s(a)$.

We must examine the behavior of the solution for $t \geq c$.

It follows from the concavity at c that immediately after $t = c$, $s(t)$ is above the line segment L, so is inside the region bounded by C. If the solution curve ever leaves this region after $t > c$, it must be by crossing C_2, that is, L. For the part C_1 of the boundary is part of the solution curve, which can't cross itself; and a was so chosen that the circle of which C_3 is part is not crossed by the solution curve $s(t)$ for any $t > a$.

So if the curve eventually leaves the region, there exists a first parameter value d, $d > c$, where the solution curve meets C_2. The point $s(d)$ must be to the right of $s(c)$, since all of C_2 is to the right. Then, inside the region bounded by the simple closed curve C, we have another, which has clockwise orientation:

– the solution curve from $s(c)$ to $s(d)$;

– the segment of the straight line L back from $s(d)$ to $s(c)$.

But by our geometric Lemma 3, no such simple closed curve is possible.

(1.2.2) Assume that $y(t_0) < y_L$ for some $t_0 < c$. Then $s(t)$ has to cross the line $y = y_L$ at least once for $t \in [t_0, c]$. For this sub-subcase, assume further that, at the last time t that $t < c$ and $s(t)$ lies on the line $y = y_L$, the point $s(t)$ is to the left of $s(c)$. Denote this time by b_1. (That a *last* such tine exists follows from the concavity at c.)

Consider the arc of the solution curve for the parameter values $t \in [b_1, c]$. By the concavity at c, the arc is above the line $y = y_L$, and tangent to it, near the $t = c$ end. Since b_1 is the last prior meeting point, the arc (except for the two endpoints) must be strictly above the line. Therefore the following closed curve is simple, and has clockwise orientation:

– the arc of the solution curve from $s(b_1)$ to $s(c)$;

– a horizontal line segment moving right-to-left from $s(c)$ to the $s(b_1)$.

But by our geometric Lemma 3, no such simple closed curve is possible.

(1.2.3) Now assume, as before, that $y(t_0) < y_L$ for some $t_0 < c$, but that, at the last time t that $t < c$ and $s(t)$ lies on the line $y = y_L$, the point $s(t)$ is to the right of $s(c)$. Denote this time by b_2

The proof for this case is rather similar to that given for the case (1.2.1):

Consider the arc of the solution curve for the parameter values $t \in [b_2, c]$. This arc must be strictly above the line $y = y_L$ everywhere except at the endpoints, and so the following closed curve is simple and has counterclockwise orientation:

– the arc of the solution curve from $s(b_2)$ to $s(c)$;

– the segment of the straight line going left-to-right from $s(c)$ to $s(b_2)$.

Calls this curve C. To find a contradiction we must examine the behavior of the solution for $t \geq c$.

It follows from the concavity at c that immediately after $t = c$, $s(t)$ is above the line segment L, so is inside the region bounded by C. If the solution curve ever leaves this region, it must be by crossing L, since the rest of the boundary is part of the solution curve, which can't cross itself.

So if the curve eventually leaves the region, there exists a first parameter value d, $d > c$, where the solution curve meets L. The point $s(d)$ must be to the right of $s(c)$. Then, inside the region bounded by the simple closed curve C, we have another, which has clockwise orientation:

– the arc of the solution curve from $s(c)$ to $s(d)$;

– the segment of the straight line L back from $s(d)$ to $s(c)$.

But by our geometric Lemma 3, no such simple closed curve is possible.

We have now finished showing that, under the subcase (1.2) assumption, the solution curve can not go below the line $y = y_L$ for large parameter values, so in particular can't have the origin as limit. This contradicts the hypotheses of the theorem, so case (1.2) can not occur.

This completes the examination of case (1).

(2) $s(t)$ is tangent to K infinitely often as $t \to +\infty$.

Since K has only finitely many semibranches, $s(t)$ must be tangent to some one of them, say $\{\eta(u) = (\eta_1(u), \eta_2(u))\}$, infinitely often. Since $s(t) \to (0, 0)$, infinitely many of these points of tangency lie within any given compact neighborhood of $(0, 0)$. Hence, by Lemma 2, the solution curve is a subset of the branch. Then, by the last assertion of Lemma 1, which says that the tangent direction of the branch has a limit, so does the tangent direction of the solution curve. Actually, it follows from the discussion given following the definition of the inflection set K that $s(t)$ moves along a straight line in this case.

(3) The solution $s(t)$ crosses some semibranch $k(u)$ of K infinitely often, with infinitely many reversals of orientation as $u \to 0$.

Reverse the sign of the parameter u, if necessary, so that u is non-negative along this semibranch. The assumption for this case implies that there is a monotonic decreasing sequence $u_i \to 0$ such that

$$(-1)^i(\eta_1' A - \eta_2' B) > 0,$$

the functions $A,\ B$ and η_1', η_2' being evaluated at $(\eta_1(u_i), \eta_2(u_i))$. By continuity, there must be a sequence ν_i of coordinates of other "in-between" points of the semibranch, $\nu_i \to 0$ monotonically, such that

$$u_1 > \nu_1 > u_2 > \nu_2 > u_3 > \nu_3., ...,$$

and such that $(\eta_1' A - \eta_2' B) = 0$ at every ν_i. But this is just another way of writing the free-parameter form of our differential equation, and so $k(u)$ is a [free-parameter] solution of it. Since $s(t)$ meets $k(u)$, and there can be only one solution through the meeting point, $s(t)$ actually lies in K, contradicting the hypothesis of this case.

(4) The solution $s(t)$ crosses some semibranch $k(u)$ of K infinitely often; and for all crossings with u close enough to 0, the orientation of the crossing is the same.

We have already taken care of the case that there are also infinitely many tangential meetings. So we can suppose that there are none (i.e., none left once u is sufficiently close to 0). Then there are infinitely many successive crossings with the same orientation. It isn't hard to show that this is possible only if the solution spirals about the origin.

This concludes the proof of the theorem.

In conclusion, we suggest a generalization of our theorem to implicit analytic differential equations of the form $f(x, y, x', y') = 0$, $x, y \in N$, see [KHABI] , making use of the theorem proved above and a theorem of [KHABI]. This theorem states that there exists an analytic manifold M, a differential equation (*) on M which is locally of the form (1) on M, and a proper analytic map $h : M \to N$, having the property that every solution of $f = 0$, is the image under h of a solution of (*). Other information on singular sets of implicit $C^{(2)}$ differential equations may be found in [KHABII].

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