Physics 31
Spring, 2012

Solution to HW #3

Problem A The wave function of a particle is given by

$$\psi(x) = N \exp\left[-\left(\frac{x}{3.1915\,\text{Å}}\right)^2\right]$$

(When you evaluate this function, take all distances x to be in Å.)

(a) What is the correct numerical value of the normalization constant N? *Hint:* You must explicitly evaluate N. Use the general result $\int_{-\infty}^{\infty} \exp(-ax^2)\,dx = \sqrt{\pi/a}$.

(b) Make a plot of $\psi(x)$ and $\left|\psi(x)\right|^2$. Indicate numerical values.

(c) Estimate the probability that a measurement of the particle's position gives a value between $x = 1.45$ Å and $x = 1.55$ Å.

(d) Use the sketch you made in part (b) to estimate the uncertainty Δx in the particle's position. *Hint:* Do not do any integrals to answer this part of the question.

(a) We evaluate N for an arbitrary value of the scaling constant b (not just 3.1915 Å) from the normalization condition

$$1 = \int_{-\infty}^{\infty} \left|\psi(x)\right|^2 \, dx = N^2 \int_{-\infty}^{\infty} \exp\left[-2\left(\frac{x}{b}\right)^2 dx\right],$$

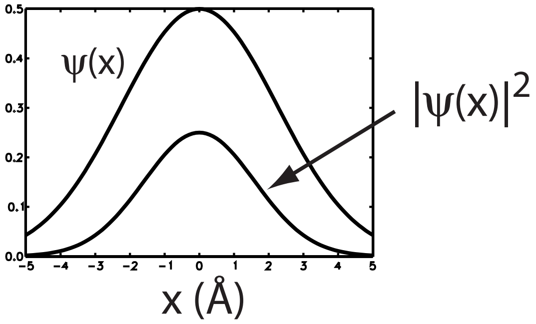
The value of the integral is $\sqrt{\pi/a}$, with $a = 2/b^2$. Hence

$$1 = N^2 b \sqrt{\pi/2} \Rightarrow N = \left(\frac{2}{\pi b^2}\right)^{1/4}.$$

Substituting $b = 3.1915$ Å leads to

$$N = 0.5000\,\text{Å}^{-1/2}$$

(b) $\psi(x)$ and $\left|\psi(x)\right|^2$ look like this:



(c) We can estimate the probability P as

$$P = \left|\psi(1.5\,\text{Å})\right|^2 \times 0.1\,\text{Å}$$
$$= 0.25 \exp\left[-2\left(\frac{1.5\,\text{Å}}{3.1915\,\text{Å}}\right)^2\right] \times 0.1 \approx 1.6\%.$$

(d) A rough guess of Δx from the FWHM (full width at half maximum) of the graph of $\left|\psi(x)\right|^2$ might be 4 Å.

Problem B The relativistic momentum is given by

$$p = mv\gamma = \frac{mv}{\sqrt{1 - v^2/c^2}}.$$

Find an expression for the velocity v in terms of the momentum p and the mass m. Verify that your expression reduces properly to $v = p/m$ in the nonrelativistic limit.

Squaring both sides leads to

$$p^2 = \frac{m^2 v^2}{1 - v^2/c^2} \quad \Rightarrow \quad p^2\left(1 - v^2/c^2\right) = m^2 v^2.$$

By expanding the terms on the left and rearranging, we can solve for v^2:

$$p^2 = \left(m^2 + p^2/c^2\right) v^2 \quad \Rightarrow \quad v^2 = \frac{p^2}{m^2\left[1 + p^2/(m^2c^2)\right]}$$

Taking the square root leads to

$$v = \frac{p/m}{\sqrt{1 + p^2/(m^2c^2)}}$$

This expression clearly goes to the classical limit $v = p/m$ in the non-relativistic limit, for which $p \ll mc$. One can also show that as $p \to \infty$, the velocity $v \to c$.

3–4 Find the de Broglie wavelength of the 40 keV electrons used in a certain electron microscope.

Starting from the nonrelativistic formula $\text{KE} = p^2/(2m_e)$, we have

$$p = \sqrt{2(9.11 \times 10^{-31}\,\text{kg})(40000\,\text{eV})(1.602 \times 10^{-19}\,\text{J/eV}}$$
$$= 1.08 \times 10^{-22}\,\text{kg m/s}$$
$$v = p/m_e = 1.08 \times 10^{-22}\,\text{m/s} \sim 0.4c$$
$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34}}{1.08 \times 10^{-22}} = 6.13\,\text{pm}.$$

Since the ratio of the electron's speed to the speed of light is not negligible, let's check the relativistic formulas. Using $m_e = 0.511\ \text{MeV}/c^2$, we note that the total relativistic energy E of the electron is

$$E = m_e c^2 + \text{KE} = 0.511\ \text{MeV} + 40.0\ \text{keV} = 0.551\ \text{MeV}.$$

We can solve the E–p relation for pc:

$$E = \sqrt{(pc)^2 + (m_e c^2)^2} \Rightarrow pc = \sqrt{E^2 - (m_e c^2)^2}$$
$$pc = \sqrt{(0.551\ \text{MeV})^2 - (0.511\ \text{MeV})^2} = 0.2061\ \text{MeV}$$
$$p = \frac{(0.2061 \times 10^6\ \text{eV}) \times (1.602 \times 10^{-19}\,\text{J/eV})}{3 \times 10^8\,\text{m/s}}$$
$$p = 1.10 \times 10^{-22}\,\text{kg m/s}$$

With this p, we find

$$\lambda = h/p = 6.02\ \text{pm},$$

so the relativistic effect is small.

3–5 By what percentage will a nonrelativistic calculation of the de Broglie of a 100 keV electron be in error?

For the relativistic calculation, follow the method of problem 4. The kinetic and total energy of the electron are 0.1 MeV and 0.611 VMeV, respectively.

$$\begin{aligned} pc &= \sqrt{(0.611\text{ MeV})^2 - (0.511\text{ MeV})^2} = 0.3350\text{ MeV} \\ p &= \frac{(0.3350\times10^6\text{ eV})\times(1.602\times10^{-19}\text{ J/eV})}{3\times10^8\,\text{m/s}} \\ p &= 1.789\times10^{-22}\,\text{kg m/s} \\ \lambda &= h/p = 3.70\times10^{-12}\,\text{m}. \end{aligned}$$

For the classical result, $p = \sqrt{2m_e(KE)}$. We get

$$\begin{aligned} p &= \sqrt{2(9.11\times10^{-31}\,\text{kg})(10^5\,\text{eV})(1.602\times10^{-19}\,\text{J/eV})} \\ p &= 1.708\times10^{-22}\,\text{kg m/s} \\ \lambda &= h/p = 3.88\times10^{-12}\,\text{m}. \end{aligned}$$

The nonrelativistic result is about 4.6% high.

3–9 Green light has a wavelength of about 550 nm. Through what potential difference must an electron be accelerated to have this wavelength?

If the de Broglie wavelength of the electron is $\lambda = 550\,\text{nm}$, then the electron's momentum p must be

$$p = \frac{h}{\lambda} = \frac{6.626\times10^{-34}}{550\times10^{-9}\,\text{m}} = 1.205\times10^{-27}\,\text{kg m/s}.$$

This value of p is much smaller than the momenta obtained in problems 3–4 and 3–5, and in those problems the effects of relativity were small. So we can use the nonrelativistic formula $KE = p^2/(2m_e)$:

$$\begin{aligned} KE &= \frac{(1.205\times10^{-27}\,\text{kg m/s})^2}{2(9.11\times10^{-31}\,\text{kg})} = 7.97\times10^{-25}\,\text{J} \\ &\quad\div 1.602\times10^{-19}\,\text{J/eV} \\ &= 4.97\times10^{-6}\,\text{eV} \end{aligned}$$

The electron's kinetic energy in eV is numerically equal to the desired potential difference in volts, so $\Delta V \sim 5\,\mu\text{V}$.

3–32 Compare the uncertainties in the velocities of an electron and a proton confined in a 1.00 nm box.

$$\Delta x \Delta p = \Delta x(m\Delta v) = \frac{\hbar}{2} \Rightarrow \Delta v = \frac{\hbar}{2m\Delta x}.$$

Substituting the electron mass and $\Delta x = 1\times10^{-9}$ m gives

$$\Delta v = 6\times10^4\,\text{m/s}.$$

Δv for the proton will be smaller by the mass ratio $m_p/m_e \approx 1830$. The value of Δv for the proton will be about 30 m/s.

3–37 A marine radar operating at a frequency of 9400 MHz emits groups of electromagnetic waves 0.0800 μs in duration. The time needed for the reflections of these groups to return indicates the distance to the targets. (a) Find the length of each group and the number of waves it contains. (b) What is the approximate minimum bandwidth (that is, spread of frequencies) the radar receiver must be able to process?

The wavelength is

$$\lambda = \frac{c}{\nu} = \frac{3\times10^8}{9400\times10^6} = 0.0319\,\text{m},$$

and the pulse length is

$$0.0800\,\mu\text{s}\times c = 24\,\text{m},$$

so that the number of wavelengths in the pulse train is

$$\frac{24}{0.0319} = 752.$$

To get the bandwidth, we use the energy-time form of the uncertainty principle,

$$\Delta E\,\Delta t \sim \frac{\hbar}{2}.$$

Using $E = h\nu$, we have

$$h\Delta\nu\,\Delta t \sim \frac{h}{4\pi} \Rightarrow \Delta\nu\,\Delta t \sim \frac{1}{4\pi}.$$

The bandwidth is $\Delta\nu$, which will be about

$$\Delta\nu \sim \frac{1}{4\pi\Delta t} = \frac{1}{4\pi\times0.0800\,\mu\text{s}} \sim 1\,\text{MHz}$$

One can get the same answer working from the position-momentum form of the uncertainty principle. Start with

$$\Delta x\,\Delta p \sim \tfrac{1}{2}\hbar = \frac{h}{4\pi},$$

and use $p = \hbar k = 2\pi\nu\hbar/c = h\nu/c$ to obtain

$$\Delta x\,\Delta p = \Delta x\,\frac{h}{c}\Delta\nu \sim \frac{h}{4\pi} \Rightarrow \Delta\nu = \frac{c}{4\pi\Delta x}.$$

Using $\Delta x = 24\,\text{m}$ gives the same $\Delta\nu$.