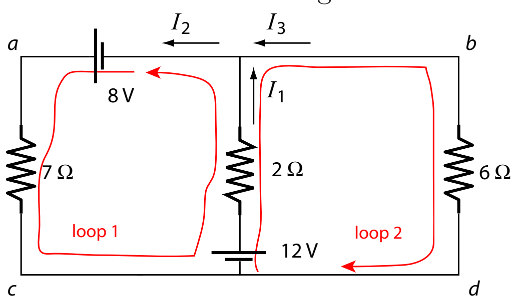
Physics 21
Fall, 2011

Solution to Hour Exam #1

The graders for the problems were:
1 Tupa, 2 Faust, 3 Beels, 4 Malenda, 5 Glueckstein
For questions about the grading, see the grader by Oct. 26.

Problem 1. Consider the following circuit:



(a) (13 pts.) Write Kirchhoff's loop and junction (or node) equations needed to determine the currents I_1, I_2, and I_3. Use the currents and their directions specified by the arrows in the diagram. **You must write these equations as defined in Physics 21.** You should have three equations for three unknowns. **Draw clearly and label on the diagram the loop used to determine each loop equation.**

(b) (3 pts.) Determine the currents I_1, I_2, and I_3 (including the correct sign) by explicit solution, by hand, of the equations you determined in part (a). **You must show your work.**

(c) (2 pts.) Use the currents you calculated in part (b) to find the potential difference $V_c - V_b$ between the corner points labelled c and b on the diagram, using the path through corner point a. Show your work, including the potential change across any circuit elements on this path.

(d) (2 pts.) Repeat part (c), but take the path through corner point d. Show your work, including the potential change across any circuit elements on this path. You should get the same value for $V_c - V_b$ that you found in part (c).

a) $\boxed{\text{node} \quad I_1 + I_3 = I_2}$

loop 1 $\quad 8 - 7I_2 + 12 - 2I_1 = 0$

loop 2 $\quad 12 - 2I_1 + 6I_3 = 0$

$$\boxed{\begin{array}{ll} \text{loop 1} & 2I_1 + 7I_2 = 20 \\ \text{loop 2} & 2I_1 - 6I_3 = 12 \end{array}}$$

b) subst I_2 in loop 1

$2I_1 + 7(I_1 + I_3) = 9I_1 + 7I_3 = 20$

$6\,[\,9I_1 + 7I_3 = 20\,]$

$7\,[\,2I_1 - 6I_3 = 12\,]$

$54I_1 + 42I_3 = 120$

$14I_1 - 42I_3 = 84$

$68I_1 = 204$

$\boxed{I_1 = 3A}$

from loop 2, $6I_3 = 2I_1 - 12 = -6$

$\boxed{I_3 = -1A}$

from node eq $\boxed{I_2 = 2A}$

c) $V_c - V_b = +8 - 7I_2$

$= 8 - 14$

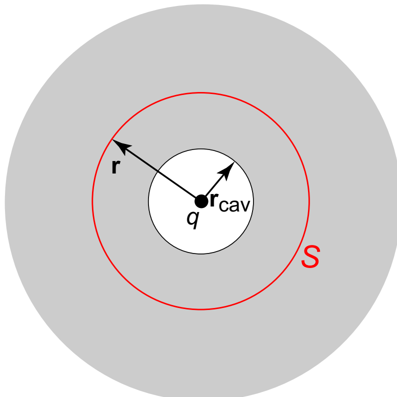
$= -6\ V$

d) $V_c - V_b = 6I_3$

$= 6(-1)$

$= -6\ V$

Problem 2. A point charge q is located at the center of a spherical cavity of radius r_{cav} inside an insulating, spherical charged solid:



For this problem, $q = -6.1\,\mu\text{C}$, $r_{\text{cav}} = 3.3\,\text{cm}$, the charge density in the solid is $\rho = 1.5 \times 10^{-3}\,\text{C/m}^3$, and you are to use Gauss' Law to calculate the electric field inside the solid at a distance $r = 5.6\,\text{cm}$ from the center of the cavity.

(a) (3 pts.) Write Gauss' Law.

(b) (3 pts.) What is the shape and location of the Gaussian surface S that you will use? Draw and label S neatly on the diagram to the left.

(c) (6 pts.) What is the total charge enclosed by S?

(d) (6 pts.) Find the magnitude of the electric field at distance $r = 5.6\,\text{cm}$ from the point charge q.

(e) (2 pts.) What is the direction of the electric field? Justify your answer.

a) $\int_S \vec{E}\cdot d\hat{A} = \frac{Q_{encl}}{\varepsilon_0}$

b) S is a sphere of radius r centered on q

c) $Q_{encl} = q + \frac{4}{3}\pi(r^3 - r_{cav}^3)\rho$

$q = -6.1\times10^{-6}\,C$

$\frac{4}{3}\pi(.056^3 - .033^3)\,1.5\times10^{-3} = 8.776\times10^{-7}\,C$

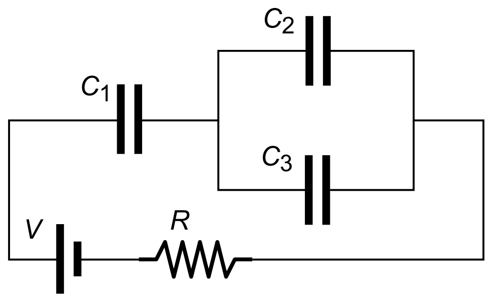
$Q_{encl} = -5.22\times10^{-6}\,C$

d) $4\pi r^2 E = \frac{Q_{encl}}{\varepsilon_0} \Rightarrow E = \frac{1}{4\pi\varepsilon_0}\frac{Q_{encl}}{r^2}$

$E = 9\times10^9\,\frac{5.22\times10^{-6}}{.056^2} = 1.5\times10^7\,\frac{N}{C}$

e) direction is inward toward q since Q_{encl} is negative

Problem 3. A battery with $V = 12$ V is connected to a circuit with capacitors $C_1 = 3.0$ mF, $C_2 = 3.5$ mF, and $C_3 = 2.5$ mF as shown.



(a) (2 pts.) What is the equivalent (or effective) capacitance of the three capacitors shown?

(b) (12 pts.) Assume the circuit has been connected for a very long time. Find the charge on each capacitor and the potential difference across each capacitor. Identify the charges as Q_1, Q_2, and Q_3, and the potential differences as V_1, V_2, and V_3. Explain carefully the steps you take to determine your answer.

(c) (6 pts.) Suppose it takes the capacitors 15 s after the battery is connected to become 99% charged. What is R?

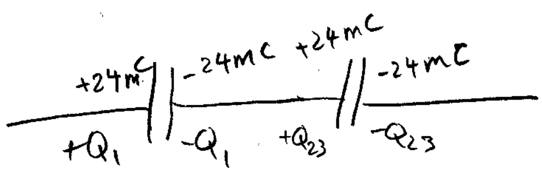
a) $C_{23} = C_2 + C_3 = 6.0 \text{ mF}$

$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_{23}} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

$\boxed{C_{eq} = 2 \text{ mF}}$ ✓

b) $Q = CV = 2 \times 10^{-3} \times 12 \text{ V}$

$= 24 \text{ mC}$



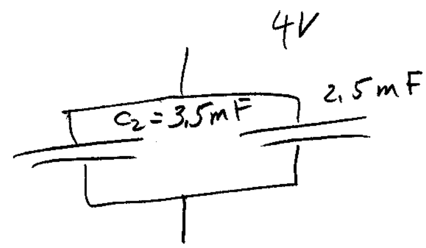
$Q_1 = 24 \text{ mC}$

$V_1 = \frac{Q_1}{C_1} = \frac{24 \text{ mC}}{3 \text{ mF}} = 8 \text{ V}$

$\boxed{V_1 = 8 \text{ V}}$

Therefore voltage across caps in ∥ is $12 - 8 = 4$ V

$\boxed{V_2 = V_3 = 4 \text{ V}}$



$Q_2 = C_2 \times 4 \text{V}$

$= 3.5 \text{ mF} \times 4 \text{ V}$

$\boxed{Q_2 = 14 \text{ mC}}$

$Q_3 = C_3 \times 4 \text{V}$

$= 2.5 \times 4$

$\boxed{Q_3 = 10 \text{ mC}}$

as expected $Q_2 + Q_3 = 24$ mC

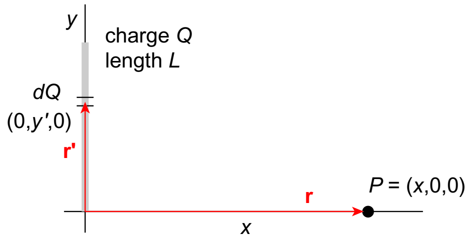
c) $1 - e^{-t/RC} = 0.99$

$e^{-t/RC} = .01 \Rightarrow e^{t/RC} = 100$

$\frac{t}{RC} = \ln 100$

$R = \frac{t}{C \ln 100} = \frac{15}{.002 \ln 100} = 1.6 \text{ k}\Omega$

Problem 4. For this problem you are to find the electric field at the point P on the x axis due to the charge distribution on the y axis. A total charge of Q is uniformly spread out on a thin wire of length L. The lower end of the wire is at the origin.



(a) (3 pts.) What is the linear charge density λ on the wire?

(b) (4 pts.) On the diagram shown, draw the vectors $\mathbf{r}$ and $\mathbf{r}'$ that correspond to the field point (P) and the charge point (where dQ is), respectively. Give the vector $(\mathbf{r} - \mathbf{r}')$ in terms of its components.

(c) (6 pts.) Write an expression for the electric field $d\mathbf{E}$ at the point P due to the element of charge dQ on the wire. Show how to write dQ in terms of the variable of integration.

(d) (7 pts.) Integrate to find ONLY THE y COMPONENT of the electric field $\mathbf{E}$ at the point P due to the wire. Your answer should be in terms of Q, L, x, and physical constants.

a) $\lambda = \frac{Q}{L}$

b) $\vec{r} = x\hat{\imath} \qquad \vec{r}\,' = y\hat{\jmath}$

$\vec{r} - \vec{r}\,' = x\hat{\imath} - y\hat{\jmath}$

c) $d\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{dQ\,(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$

$dQ = \lambda\, dy = \frac{Q}{L}\, dy$

d) $dE_y = \frac{1}{4\pi\varepsilon_0} \underbrace{\frac{Q}{L}\, dy}_{dQ} \frac{-y}{(x^2+y^2)^{3/2}}$

$dE_y = -\frac{1}{4\pi\varepsilon_0} \frac{Q}{L} \frac{y\, dy}{(x^2+y^2)^{3/2}}$

$E_y = \int_0^L dE_y =$

$= -\frac{1}{4\pi\varepsilon_0} \frac{Q}{L} \int_0^L \frac{y\, dy}{(x^2+y^2)^{3/2}}$

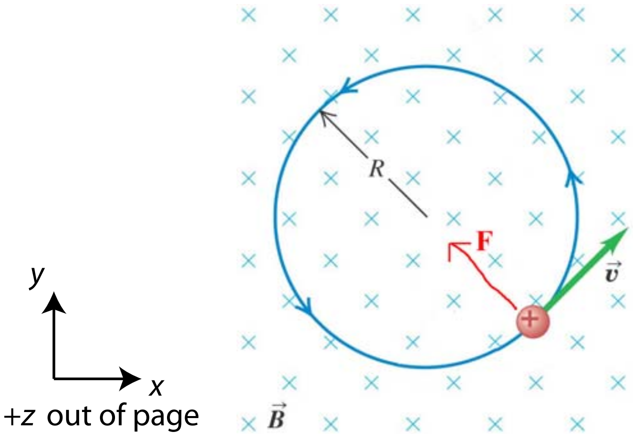
$= -\frac{1}{4\pi\varepsilon_0} \frac{Q}{L} (-) \frac{1}{(x^2+y^2)^{1/2}}$

$E_y = \frac{1}{4\pi\varepsilon_0} \frac{Q}{L} \frac{1}{\sqrt{x^2+y^2}} \Big|_0^L$

$= \frac{1}{4\pi\varepsilon_0} \frac{Q}{L} \left[\frac{1}{\sqrt{x^2+L^2}} - \frac{1}{x} \right]$

$$\boxed{E_y = -\frac{1}{4\pi\varepsilon_0} \frac{Q}{L} \left[\frac{1}{x} - \frac{1}{\sqrt{x^2+L^2}} \right]}$$

Problem 5. A particle moves in a circular orbit in a uniform magnetic field $\mathbf{B}$ in the $-z$ direction (into the page). The orbit is confined to the xy plane. The charge and mass of the particle are $q = 3.20 \times 10^{-19}$ C and $m = 6.75 \times 10^{-26}$ kg, respectively, and the magnitude of $\mathbf{B}$ is $B = 0.500$ T.



(a) (5 pts.) At some time t_0 the particle is at the point shown on the diagram, and its instantaneous velocity is

$$\mathbf{v} = (3.42 \times 10^4 \text{ m/s})\,\hat{\mathbf{i}} + (3.08 \times 10^4 \text{ m/s})\,\hat{\mathbf{j}}.$$

Give the components of the magnetic force $\mathbf{F}$ on the particle at the time t_0. Draw an arrow on the diagram that shows the direction of $\mathbf{F}$.

(b) (5 pts.) Find the radius R of the circular orbit.

(c) (5 pts.) How much time does it take for the particle to make one revolution?

(d) (5 pts.) Through what potential difference would the particle have to be accelerated from rest to acquire the speed it has?

a) $\vec{F} = q\,\vec{v} \times \vec{B}$

$= q\,(v_x \hat{x} + v_y \hat{y}) \times B(-\hat{z})$

use $\hat{x} \times (-\hat{z}) = -\hat{x} \times \hat{z} = \hat{y}$

$\hat{y} \times (-\hat{z}) = -\hat{x}$

$\vec{F} = -q v_y B \hat{x} + q v_x B \hat{y}$

$= -3.2 \times 10^{-19} \quad 30800 \times 0.5\, \hat{x}$

$+ 3.2 \times 10^{-19} \quad 34200 \times 0.5\, \hat{y}$

$\vec{F} = -4.93 \times 10^{-15}$ N $\hat{x}$

$+ 5.47 \times 10^{-15}$ N $\hat{y}$

b) $v = \sqrt{v_x^2 + v_y^2} = \sqrt{34200^2 + 30800^2}$

$v = 4.60 \times 10^4$ m/s

$R = \dfrac{mv}{qB} = \dfrac{6.75 \times 10^{-26} \times 46000}{3.2 \times 10^{-19} \quad 0.5}$

$= 1.94 \times 10^{-2}$ m $= 19.4$ mm

c) $T = \dfrac{2\pi R}{v}$

$= \dfrac{2\pi \quad 0.0194 \text{ m}}{4.6 \times 10^4 \text{ m/s}}$

$= 2.65 \times 10^{-6}$ s

$= 2.65$ μs

d) $qV = \frac{1}{2} m v^2$

$V = \dfrac{mv^2}{2q}$

$= \dfrac{6.75 \times 10^{-26} \quad 46000^2}{2 \times 3.2 \times 10^{-19}}$

$= 223$ V