MECHANICAL WAVES Part I

In Physics 11, you used $\vec{f}=m\vec{a}$ to solve for the motion of a few particles. By combining $f=ma$ with the relation $\sum$*torques = change in angular momentum,* you solved for the motion of a rigid body in which all the particles that make up the body were restrained to move together because the object was rigid.

In reality, there are no rigid bodies. Especially, the particles that make up a liquid or gas move relative to one another. The study of these general motions is called *continuum mechanics*. One special and important motion of the particles in a continuous material is the transport of a disturbance called a *wave*. We will now develop the equations and terminology that describe wave motion.

Our treatment here is limited to waves in linear material in which all waves travel at the same constant speed.

I. A Wave Pulse

The waves shown in Figs. 1 and 2 are moving to the right. The motion in Fig. 1 is called a *transverse* wave because the string particles move perpendicularly to the direction that the wave is moving. The motion in Fig. 2, on the other hand, is called a *longitudinal* wave because the particles of the spring are moving right and left, which are parallel to the direction of the wave motion.

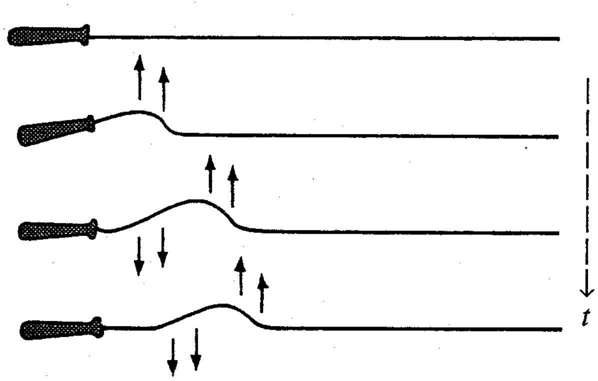

Fig. 1. Transverse Wave

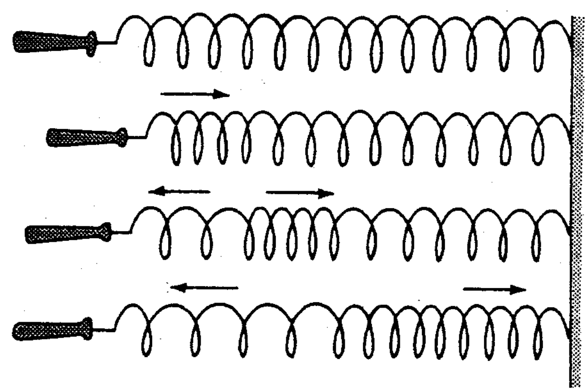

Fig. 2. Longitudinal Wave

Ia. The Equation for a Wave

Consider, as an example, the motion in Fig. 1 when the end of the string at $x = 0$ is moved up and down so the displacement $y_0(t)$ of the string at $x = 0$ varies in time as

$$y_0(t) = A_0 e^{-at^2}. \qquad (1)$$

When we say that the wave is traveling to the right with a velocity v, we mean that the displacement of the string at $x = \Delta x = vt$ is

$$y(x,t) = A_0 e^{-a(t - \frac{x}{v})^2}, \qquad (2)$$

as illustrated in Fig. 3. That is, the displacement at a distance $x = \Delta x$ has the same form as at $x = 0$ except that it is delayed by the time it takes to travel from $x = 0$ to $x = \Delta x$. If the wave travels in the negative x-direction, replace $(t - \frac{x}{v})$ in Eq. (2) by $(t + \frac{x}{v})$.

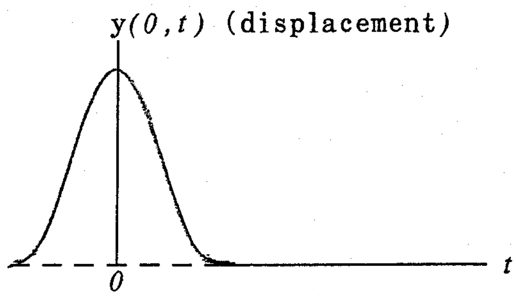

Fig. 3a. The displacement at $x = 0$.

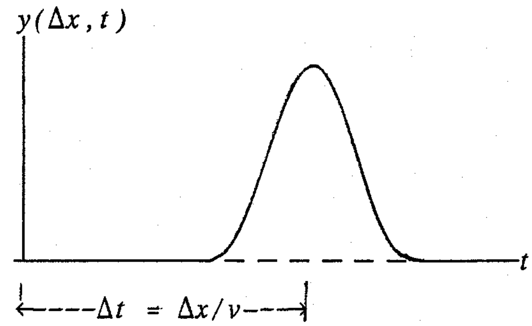

Fig. 3b. The displacement at $x = \Delta x$.

Note that the displacement of a one-dimensional wave, as in Eq. (2), is a function of two independent variables, here x and t. Also, if we set $x = 0$ in Eq. (2) we obtain $y_o(t)$ in Eq. (1), and so we will write $y(0,t)$ in place of $y_o(t)$.

Example 1. At $x = 0$, the string in Fig. 1 is moved up and down so that $y(0,t) = 2.0\ \sin(80t)$ in SI units. What is the displacement $y(x,t)$ for any value of x if the wave travels to the right at 40 m/s?

Solution: Replace t by $(t - \frac{x}{40})$ in the expression for $y(0,t)$; that is $y(x,t) = 2.0\ \sin[80(t - x/40)] = 2.0\ \sin(80t - 2x)$.

Ib. The Velocity of a Wave.

Experiment: In Experiment 5 of Physics 22, you measured the speed of a wave pulse as it traveled back and forth along a guitar wire. Data from that experiment is shown in Fig. 4. We will explain later in these notes why the pulse is reflected up-side-down when it reaches the ends of the wire. Thus, the time between two up pulses is the time to travel the length of the wire twice, about 2.0 meters.

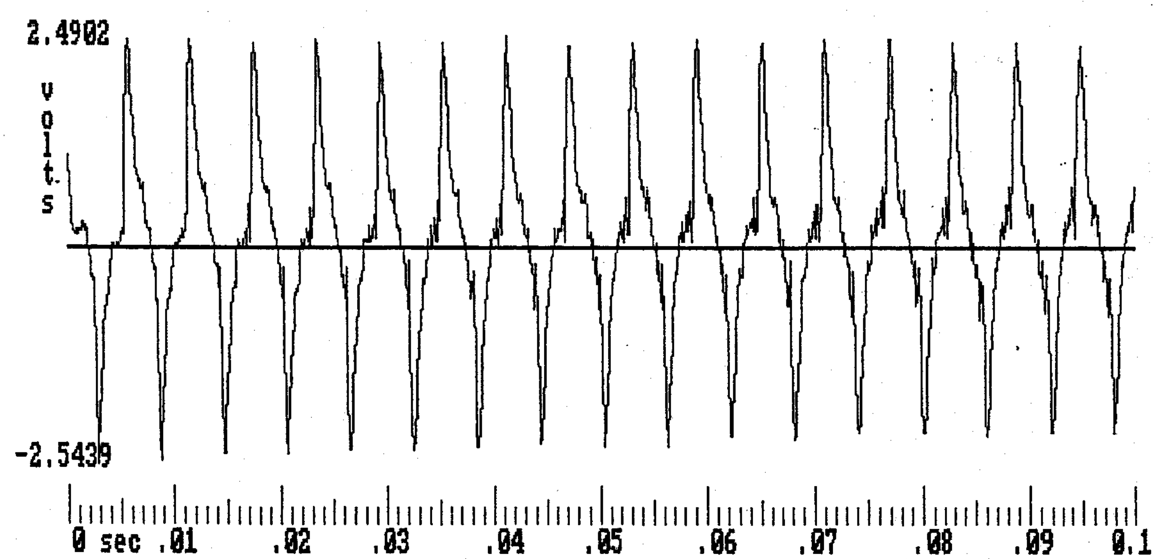

Fig. 4. Data from experiment 5 in Physics 22. When the wire moves up and down as the pulse passes through a magnetic field, a Faraday voltage is induced in the wire.

From Fig. 4, we determine that the time for the pulse to travel *2.00* m (the time between two up pulses) is *0.00593* s, and so the velocity of the wave is

$$v = \frac{2.00}{0.00593} = 337 \text{ m/s}.$$

When we return to the study of electromagnetic waves (consisting of changing electric and magnetic fields), we will describe how the velocity of these waves is measured. The result is about 3.0×10^8 m/s.

Formulas for Wave Velocities: Using $F = ma$ and a knowledge of the contact forces between neighboring particles that make up the material in which mechanical waves travel, we can determine the velocity of waves in terms of the properties and states of these materials. To illustrate this procedure, consider the pulse traveling in the string shown in Fig. 1. A drawing of the pulse is shown in Fig. 5 as observed by someone traveling with the wave. To this observer, the wave is stationary and the string is moving in the opposite direction along the shape of the pulse.

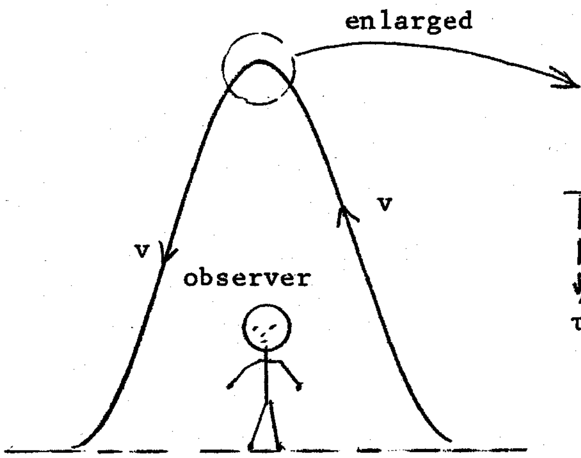

Fig. 5. A pulse on a string as as seen by an observer moving with the wave.

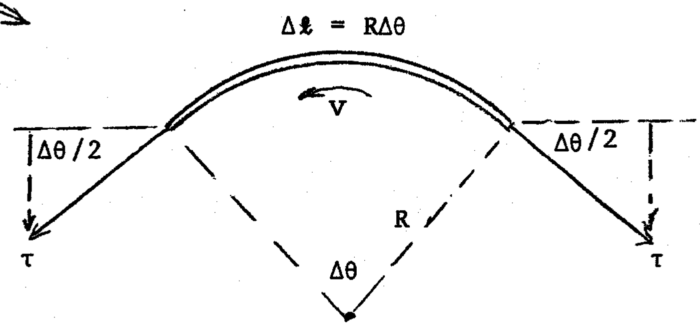

Fig. 6. Force diagram for the piece of string at the top of the pulse. The string is moving with a speed v relative to the moving observer, where v is also the wave speed

In order to use $F = ma$ to analyse the motion of the string, we need F, m and a. Ignoring the weight of the piece of string shown in Fig.6, the sum of forces on this piece is $2\tau \sin(\Delta\theta/2)$ down, where $\Delta\theta$ is the angle subtended from the center of curvature for the arc of the piece of string and τ is the tension in the string. For small angles, $\sin(\Delta\theta/2) \approx \Delta\theta/2$, and so the force is $\tau(\Delta\theta)$ down.

The mass of the piece of string is $\Delta m = \rho(\text{length}) = \rho(R\Delta\theta)$, where ρ is the mass (kg) per length (meter) of the string material.

For the small piece of string shown in Fig. 6, we consider it to be a circle of material moving with a speed v, which is also the wave speed that we wish to determine. For circular motion, the acceleration is $a = v^2/R$.

We now have expressions for F, Δm, and a. By substituting into $F = ma$,

we obtain
$$\tau(\Delta\theta) = (\rho R\Delta\theta)\frac{v^2}{R}$$

or
$$v = \sqrt{\tau/\rho} \qquad (3)$$

for the speed of the wave, where τ is the tension and ρ is the mass per length of the string material.

Expressions for the speeds of different mechanical waves have been obtained in similar ways using $F = ma$. In general the result is of the form

$$v = \sqrt{\frac{restoring\ force}{inertial\ term}}\ .$$

The velocity of sound waves in air with pressure as the restoring force was first analysed by Newton, who incorrectly assumed that the pressure was given by $p = nRT/V$, with the Kelvin temperature T constant. Others recognized that the motions are so fast that no heat flow takes place and so the work changes the air thermal energy. By assuming an adiabatic process to determine the pressure from

$$pV^{\gamma} = constant,$$

where $\gamma = (specific\ heat\ at\ constant\ pressure)/(specific\ heat\ at\ constant\ V)$ they obtained an expression for the speed of sound waves in a gas that can be written

$$v = \sqrt{\frac{\gamma RT}{M}}\ , \qquad (4)$$

with M = molecular mass of the gas molecules. For air, $\gamma \approx 1.4$ and $M \approx 28.8 \times 10^{-3}$ kg/mole.

Example 2: What is the speed of sound in air when the temperature is 300 K?

Solution:

$$v = \sqrt{1.4(8.314)300)/(28.8 \times 10^{-3})} = 348 \text{ m/s}.$$

II. Sinusoidal Waves

In example 1, the end of a string was vibrated so that $y(0,t) = A\sin(\omega t)$, with $\omega = 80$ rad/s. The wave traveled in the x-direction with a speed $v = 40$ m/s and the resulting wave was

$$y(x,t) = A\sin[\omega(t-x/v)]. \qquad (5)$$

We consider such waves in detail because (i) there are many common and important examples of such waves and (ii) wave trains of other shapes can be analysed as a sum of sinusoidal waves.

To start, set $x = 0$ in Eq. (5) and plot $y(0,t) = A\sin(\omega t)$ in Fig. 7 as a function of time and set $t = 0$ in Eq. (5) and plot $y(x,0)$ as a function of x in Fig.8. Note in Fig. 7 that $y(0,t)$ oscillates with a frequency $f = \omega/(2\pi)$ and $y(0,t)$ is periodic in a time T, called the period, with $T = 1/f$.

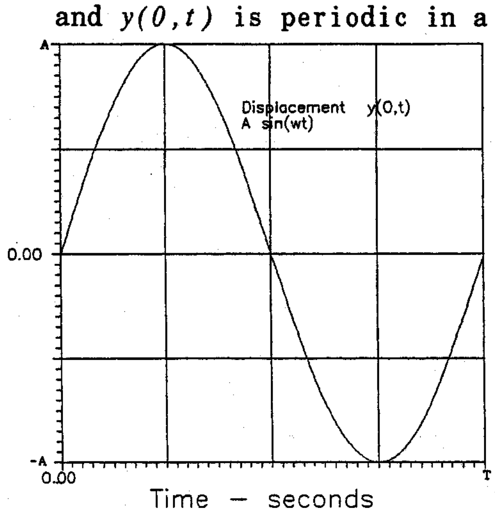

Fig. 7. $y(0,t) = A\sin(\omega t)$

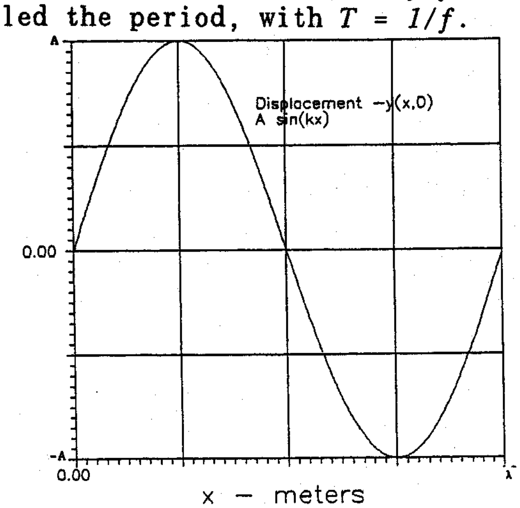

Fig. 8. $y(x,0) = -A\sin(\omega x/v)$

In Fig. 8, we notice that the function $y(x,0)$ oscillate in x and repeats its form in a distance $\lambda = vT = v/f$. The variable λ is called the wavelength. For easy reference, let us summarize the above relations with the expressions

$$v = \lambda/T = f\lambda, \qquad \text{Eq.(6a)}$$

$$T = 1/f, \qquad \text{Eq.(6b)}$$

$$\omega = 2\pi f = 2\pi/T \qquad \text{Eq.(6c)}$$

$$\text{and } k = 2\pi/\lambda \qquad \text{Eq.(6d)}$$

and rewrite Eq. (5) as

$$y(x,t) = A\ \sin[\omega(t-x/v)] \qquad \text{Eq.(7a)}$$

$$= A\ \sin[2\pi(ft - x/\lambda)] \qquad \text{Eq.(7b)}$$

$$= A\ \sin(\omega t-kx). \qquad \text{Eqs.(7c)}$$

III. Energy Transported by Waves

The string in Fig.1 would exert a force on your hand if you were holding the right end of the string when the pulse arives. As your hand reacts and moves up, work is done. That is, the wave transports energy.

We limit our analysis of this energy transport to sinusoidal waves. Consider the wave on a string given by Eq.(7c). The derivative with respect to time of Eq.(7c) is the velocity of the piece of string at the point x and the time t. If ρ is the mass per length (meter) of the string, then ρ(dx) is the the mass of a piece of string of length dx. Continuing, the kinetic energy of this piece is

$$dk = \frac{1}{2}(dm)v^2 = \frac{1}{2}(\rho dx)\left[\frac{\partial y}{\partial t}\right]^2 = \frac{1}{2}(\rho dx)A^2\omega^2[\cos(\omega t-kx)]^2 = k(dx), \qquad (8)$$

where

$$k(x,t) = \frac{1}{2}\rho[A\omega\ \cos(\omega t-kx)]^2 \qquad (9)$$

is the kinetic energy per length at time t. A graph of Eq.(9) is shown in Fig.9 for one wavelength λ.

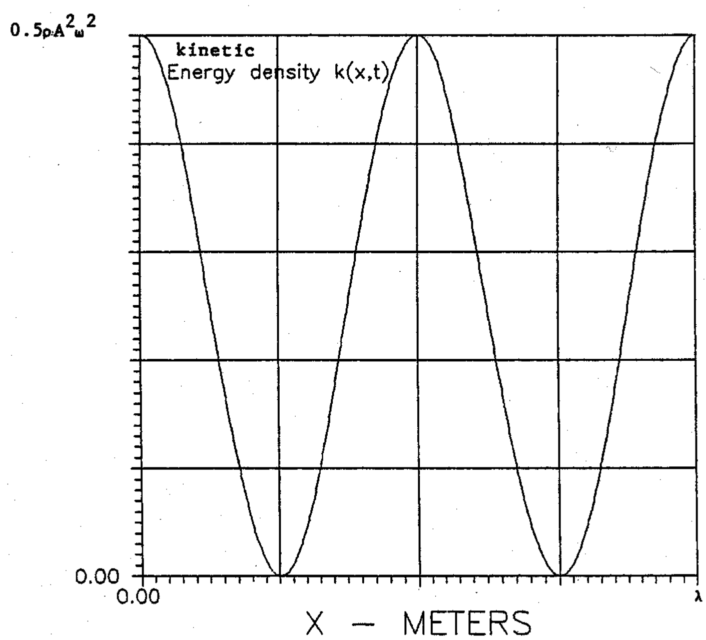

Fig. 9. Distribution of the kinetic energy of pieces of the string at $t = 0$ $k(x,0)$ (see Eq.(9)), plotted for one wavelength.

The total kinetic energy in one wavelength is equal to the area under this curve,

$$K = \int_0^\lambda k(x,0)dx = \frac{1}{2}\rho A^2\omega^2\int_0^\lambda [\cos(kx)]^2 dx = \frac{1}{4}\rho A^2\omega^2\lambda. \qquad (10)$$

Each piece of string oscillates up and down in simple harmonic motion, which you studied in Physics 11. The sum of the kinetic and potential energies is conserved as each piece of string moves with the kinetic energy a maximum at the middle of the oscillation where the potential energy is zero. The kinetic energy goes to zero at the maximum displacement where the potential energy is equal to what the maximum of the kinetic energy was at the midpoint. Thus, the potential energy of one wavelength of the string is equal to the kinetic energy in a wavelength, as given by Eq. (10).

Consequently, the sum of the kinetic and potential energies in a wavelength is

$$K + V = \frac{1}{2}\rho A^2\omega^2\lambda \ . \qquad (11)$$

During the time of one period of oscillation, this energy travels one wavelength λ, or, using wave velocity $v = \lambda/T$,

$$\text{Average power } \langle P\rangle = \frac{K + V}{T} = \frac{1}{2}\rho A^2\omega^2\frac{\lambda}{T} = \frac{1}{2}\rho A^2\omega^2 v \qquad (12)$$

is the average power transported by the wave.

We will refer to this expression when we determine the power transported by an electromagnetic wave, such as sunlight.

Mechanical Waves - Part II Sum of Waves

I. Three-dimensional Waves

Before considering the sum of waves, we generalize the treatment of of one-dimesional waves treated in Part I, to waves that spread out in three dimensions, as is the case shown in Fig. 1 for a sound wave generated by an oscillating point source S. The circles, called wavefronts in the figure, represent the maximums of the waves that are spreading out in all directions.

Because The power transported by the wave is the same for all distances r from the source, and the area of a wave front varies as r^2 and the power depends on the amplitude squared (see Eq.13 of Part I), we conclude that the amplitude of the three-dimensional wave varies as A/r, where A is a constant. Thus, the three-dimensional sinusoidal wave has the form

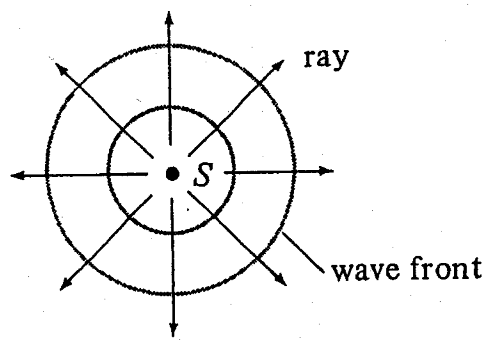

Fig.1. Sound wave in 3-D.

$$y(r,t) = \frac{A}{r}\sin(\omega t - kr). \qquad (1)$$

Compare this expression to Eq.(7c) for a 1-D wave in Part I of these notes.

II. Interference of Two Sound Waves.

Consider, as an example, the two sound sources in Fig. 2 that oscillate in phase with an angular frequency ω. A listener located as shown in the figure hears the sum of both sounds. For simplicity, assume that the intensity of both received sounds are equal. Then the received wave is

$$y(t) = A\sin(\omega t - kr_1) + A\sin(\omega t - kr_2). \quad (2)$$

Referring to the trigonometric identity

$$\sin\alpha + \sin\beta = 2\cos\left[\frac{\alpha-\beta}{2}\right]\sin\left[\frac{\alpha+\beta}{2}\right],$$

Eq.(2) can be rewritten as

$$y(t) = 2A\cos\left[0.5k(r_2-r_1)\right]\sin\left[\omega t - 0.5k(r_1+r_2)\right]. (3)$$

From the argument of the sine term, we conclude that this received signal oscillates at the same angular frequency ω as the frequency of the sources. However, the amplitude of this signal (the part of Eq.(3) before the sine term,

$$2A\cos\left[0.5k(r_2-r_1)\right] = 2A\cos\left\{\frac{\pi(r_2-r_1)}{\lambda}\right\}, \quad (4)$$

varies depending on the difference (r_2-r_1) in the distances from the two sources to the listener. In particular. the net amplitude is zero if (r_2-r_1) is $\lambda/2$, $3\lambda/2$, $5\lambda/2$, *etc*. because $\cos(\pi/2) = \cos(3\pi/2) = etc. = 0$.

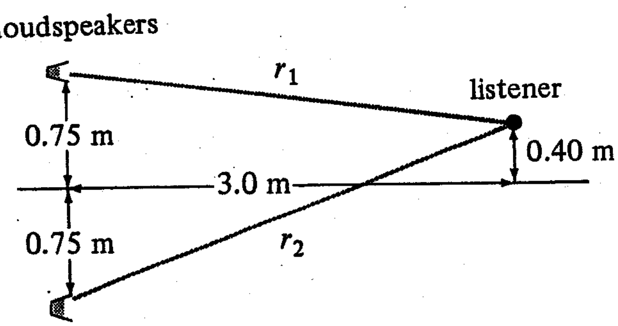

Fig.2. The listener hears the sum of the two waves.

On the other hand, the amplitude of the received signal is $2A$ if (r_2-r_1) is 0, λ, 2λ, *etc*. because $\cos(0) = -\cos(\pi) = \cos(2\pi) = etc. = 1$.

In summary, the received signal is zero if the distances from the two signals differ by $(2n+1)\lambda/2$ and is a maximum if the difference in distances is $n\lambda$, where $n = 0,1,2,3,4,etc.$

Example 1:

A listener in Fig.2 standing on the center line would hear a maximum sound. The intensity of the sound decreases as the listener moves to either side, reaching the first minimum at the position in the figure. What is the frequency of the sound? The speed of sound is 340 m/s.

Solution:

Refer to Fig.3 and use the Pythagorean theorem to determine r_1 and r_2 :

$$r_2 = \sqrt{(.75 + .40)^2 + (3.0)^2} = 3.2129 \text{ m}$$

$$r_1 = \sqrt{(.75 - .40)^2 + (3.0)^2} = 3.0203 \text{ m}$$

and so $r_2 - r_1 = 0.1926$ m. For this point to be the first minimum $r_2 - r_1 = \lambda/2$, or $\lambda = 0.3851$ m. Then, $f = (340 \text{ m/s})/\lambda = 883$ Hz.

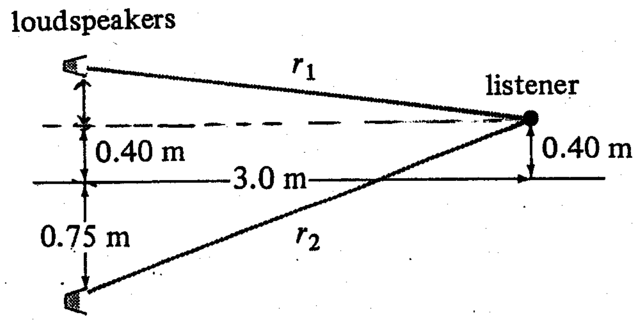

Fig.3. Example 1.

II. Reflection of Waves

When a wave reaches a boundary of the media in which it is traveling, part of the wave is reflected and part is transmitted into the adjoining material. In this section, we limit our analysis to situations in which the wave is totally reflected.

In the example of a wave traveling in a string that is fixed at its end, as shown in Fig. 4, the wave is totally reflected upside down.

To explain why the reflected wave is inverted if the end is fixed, let $y_i(x,t)$ be the incident wave and $y_r(x,t)$ be the reflected wave and the total wave is

$$y(x,t) = y_i(x,t) + y_r(x,t).$$

Since the string is fixed at $x = L$, $y(L,t) = 0$ or $y_r(L,t) = - y_i(L,t)$. That is, the reflected wave starts upside down and travels in the opposite direction to the incident direction, as shown in Fig. 4.

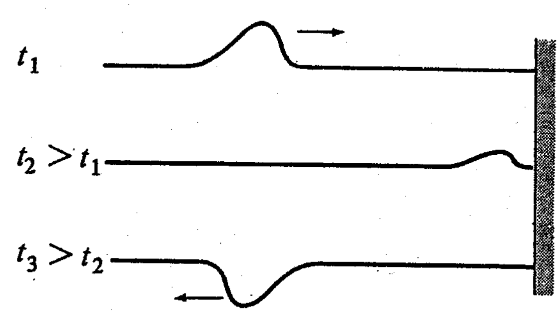

Fig.4. Reflection from a fixed end

Another situation in which a wave is totally reflected occurs when a wave reaches a free end, as is illustrated in Fig.5. At a free end, no force is applied, and so the slope of the relected wave at the end is the same as that of the incident wave there. We conclude that the wave is not inverted as shown in Fig. 5.

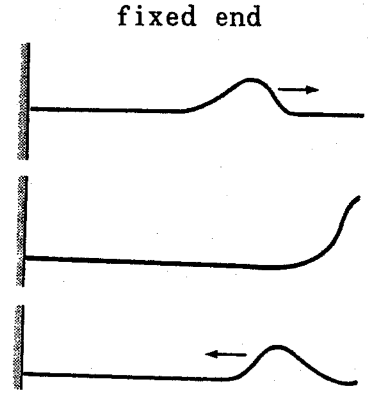

Fig. 5. Reflection from a free end.

III. Standing Waves

As an example of reflection from a fixed end, let the incident wave

$$y_i(x,t) = A\sin(\omega t+kx) \text{ (incident)} \qquad (5)$$

be a wave traveling in the negative x-direction toward a fixed end at $x = 0$. Assume that the wave reaches the end at $t = 0$. Then the reflected wave is

$$y_r(x,t) = -A\sin(\omega t-kx) \qquad (6)$$

and the total displacement is

$$y(x,t) = y_i(x,t) + y_r(x,t) = A[\sin(\omega t+kx) - \sin(\omega t-kx)]$$

$$y(x,t) = 2A\sin(kx)\cos(\omega t)\ . \qquad (7)$$

We used the trigonometric identity

$$\sin\alpha - \sin\beta \equiv 2\sin\left[\frac{\alpha-\beta}{2}\right]\cos\left[\frac{\alpha+\beta}{2}\right]$$

to obtain Eq. (7).

To interpret Eq. (7), we identify the term $2A\sin(kx)$ as the amplitude of the oscillations of the piece of string at the position x. The angular frequency of these oscillations is ω, as specified in the cosine term. The plot in Fig.6 of $\pm 2A\sin(kx)$ is called the envelope of the so-called standing wave in Eq.(7) because it shows the limits of the amplitude of the oscillations of each piece of string.

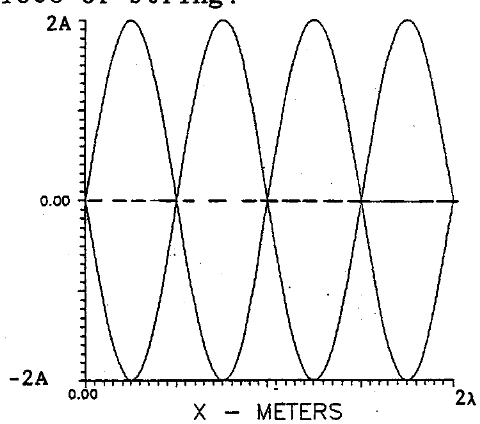

Fig. 6. The amplitude of the standing wave, $2A\sin(kx)$, in Eq.(7).

The locations where the envelope is zero are called *nodes*; the string does not move at the nodes. The string on either side of a node are oscillating *180* degrees out of phase, *i.e.*, when one side is moving up, the string on the other side is moving down. This motion is called a *standing wave* because the envelope pattern in Fig.6 does not move or change with time.

The motion given by Eq.(7) is unrealistic because it requires that the reflected wave never reaches the other end of the string where it would be reflected again. That is, Eq.(7) applies to strings that are infinitely long. However, it also applies to a finite string whose length is such that its other end is fixed at a node of the pattern in Fig. 6, which does not move. Since the nodes in Fig. 6 are a half wavelength $\lambda/2$ apart, this standing wave pattern can occur if the length of the string is $n\lambda/2$, where $n = 1, 2, 3,$ *etc*.

Three standing wave patterns for a string fixed at both ends are shown in Fig. 7. For these standing waves, the wavelengths are $2L$, L, and $2L/3$, where L is the length of the string. The speed of the waves on a string, as given by Eq. (3) of part I of these notes, depends on the tension and the mass density of the string. Assuming these are specified, the speed of the waves are the same for all the patterns in Fig. 7

From $f = v/\lambda$, we conclude that the three lowest frequencies of vibration are $v/2L$, v/L, and $f = 3v/2L$. The lowest frequency of vibration, $v/2L$, is

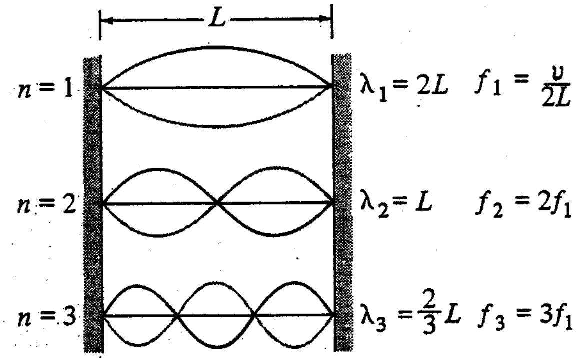

Fig.7. Three Standing Waves for a string fixed at both ends.

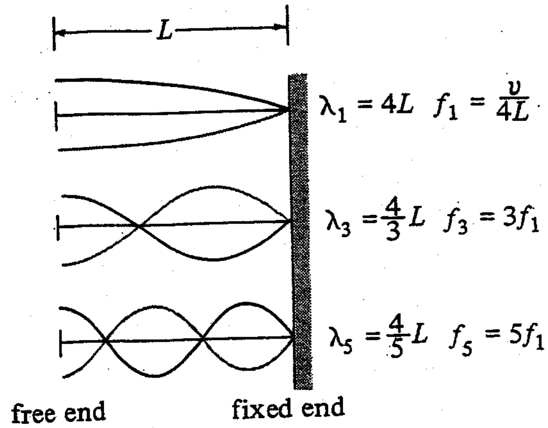

Fig.8. Standing Waves with one end fixed and one end free.

called the *fundamental* or the *first harmonic*. The frequencies of the higher harmonics are multiples of the fundamental, as given in Fig.7.

In Fig.8, we show three standing waves for a system with one end fixed and the other end free. In this case, the wavelengths are $4L$, $4L/3$, and $4L/5$, respectively. The frequencies of the three lowest modes are given in Fig.8. An example of such a system is the sound vibrations in an organ pipe with one end open and the other end closed.

Example 2:

A steel piano wire that is *1.0* m long has a mass per unit length of *0.006* kg/m. It is fixed at both ends. The tension in the wire is adjusted so that the fundamental frequency is *500* Hz. (a) What is the wave speed on the wire? (b) What is the tension in the wire? (c) What is the frequency of the second harmonic? (d) The wire vibrating at its fundamental produces a sound wave that travels at *340* m/s in air. What is the wavelength of this sound wave?

Solution:

(a) Referring to the top drawing in Fig.7, we conclude that the wavelength is *2.0* m. From $v = f\lambda = (500 \text{ m/s})(2.0 \text{ m}) = 1{,}000$ m/s.

(b) According to Eq.(3) of part I of these notes, $v = \sqrt{\tau/\rho} = 1{,}000$ m/s, $\tau = (1000)^2(0.006 \text{ kg/m}) = 6{,}000$ Newtons.

(c) From Fig. 7, $f_2 = 2f_1 = 1{,}000$ Hz.

(d) Since $v = f\lambda = 340 \text{ m/s} = (500 \text{ Hz})\lambda$, or, $\lambda = 0.68$ m.