

In class we showed that the potential difference between points a and b can be expressed as a line integral of the electric field,

$$V_b - V_a = - \int_a^b \mathbf{E} \cdot d\mathbf{l}. \quad (1)$$

Most of the line integrals needed in Physics 21 can be evaluated by inspection. Math 23 will address more complicated examples later in the semester. This document gives a brief summary of the “correct” method to evaluate these integrals.

Let C denote the curve (in 3d space) from point a to point b over which you want to integrate some function $f(x, y, z)$. That is, you wish to evaluate

$$I = \int_C f(x, y, z) dl. \quad (2)$$

You must parameterize the curve C by finding functions $x(t)$, $y(t)$, and $z(t)$ such that as t varies from t_a to t_b , $x(t)$, $y(t)$, and $z(t)$ trace out the coordinates of the points on C from the beginning at a to the ending at b . Then

$$I = \int_{t_a}^{t_b} f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt. \quad (3)$$

The square root factor is the ratio of the arc length dl along the curve C to the differential dt along the t axis.

The integral in Eq. (2) doesn't look exactly like the one in Eq. (1). However, if we define $\hat{\mathbf{T}}$ to be a unit vector tangent to the curve C and pointing in the direction one traverses as one moves from a to b , then we can replace $d\mathbf{l}$ by $\hat{\mathbf{T}} dl$, giving

$$- \int_a^b \mathbf{E} \cdot d\mathbf{l} = \int_a^b (-\mathbf{E}) \cdot \hat{\mathbf{T}} dl, \quad (4)$$

where $(-\mathbf{E}) \cdot \hat{\mathbf{T}}$ is a scalar function just like $f(x, y, z)$, and we can apply the result shown in Eq. (3).

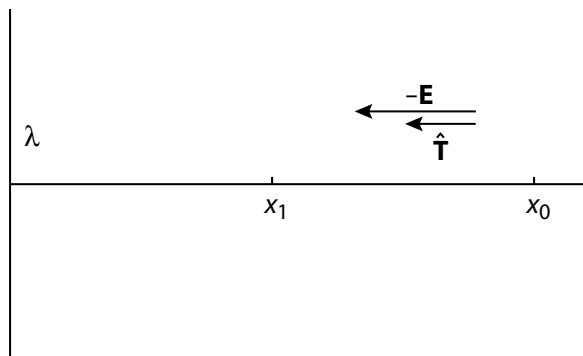
We will evaluate the integral in Eq. (1) for the case of the electric field of an infinitely long line of charge,

$$\mathbf{E} = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{r} \hat{\mathbf{r}}, \quad (5)$$

where $\hat{\mathbf{r}}$ is a unit vector that points directly away from the line. Let's consider a simple situation where the line of charge is along the y axis, and we want to find the potential difference $V_b - V_a$ between a point a at $x = x_0$ and another point b at $x = x_1$ closer to the line of charge. The diagram shows the field $(-\mathbf{E})$ and the unit tangent vector $\hat{\mathbf{T}}$ at a representative point between x_0 and x_1 .

The dot product $(-\mathbf{E}) \cdot \hat{\mathbf{T}}$ just gives the magnitude E of \mathbf{E} , since $\hat{\mathbf{T}}$ is a unit vector that points in the same direction as $(-\mathbf{E})$. Since we are considering a path on the x axis,

$$(-\mathbf{E}) \cdot \hat{\mathbf{T}} = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{x}. \quad (6)$$



A simple parameterization of the line from x_0 to x_1 is

$$x(t) = x_0 + t(x_1 - x_0), \quad y(t) = z(t) = 0, \quad (7)$$

where t runs from $t = 0$ to $t = 1$. According to Eq. (3), the line integral is

$$I = \frac{\lambda}{2\pi\epsilon_0} \int_0^1 \frac{1}{x(t)} \left| \frac{dx}{dt} \right| dt. \quad (8)$$

Using $x(t)$ from Eq. (7) and

$$\frac{dx}{dt} = x_1 - x_0, \quad (9)$$

we have

$$I = \frac{\lambda}{2\pi\epsilon_0} \int_0^1 \frac{1}{x_0 + t(x_1 - x_0)} |x_1 - x_0| dt \quad (10)$$

$$= \frac{\lambda(x_0 - x_1)}{2\pi\epsilon_0} \int_0^1 \frac{1}{x_0 + t(x_1 - x_0)} dt \quad (11)$$

$$= \frac{\lambda(x_0 - x_1)}{2\pi\epsilon_0} \frac{1}{x_1 - x_0} \ln(x_0 + t(x_1 - x_0)) \Big|_{t=0}^{t=1} \quad (12)$$

$$= -\frac{\lambda}{2\pi\epsilon_0} \ln \frac{x_1}{x_0} \quad (13)$$

$$= \frac{\lambda}{2\pi\epsilon_0} \ln \frac{x_0}{x_1}. \quad (14)$$

This result has the correct sign. On physical grounds, we expect the potential should be higher closer to the (positively charged) wire. If we (as an external agent) move a (positive) test charge closer to the wire (that is, to $x_1 < x_0$), we have to exert force to overcome the repulsive Coulomb force. When we do this, we raise the potential energy of the test charge.

We have to work fairly hard to get the right sign rigorously for this example. An alternative is to take the back-of-the-envelope approach of integrating

$$\frac{1}{2\pi\epsilon_0} \int_{x_0}^{x_1} \frac{\lambda}{r} dr = -\frac{\lambda}{2\pi\epsilon_0} \ln \frac{x_0}{x_1}. \quad (15)$$

Eq. (15) gives the wrong sign because the upper limit of the integral is smaller than the lower limit, and that introduces an unwanted sign change. You have to reverse the sign on physical grounds to get the right answer. The “correct” method accounts for the fact that the direction of the line integral is from x_0 towards x_1 .