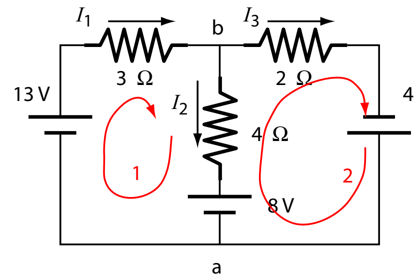
Physics 21 Fall, 2004

Solution, Practice Exam #1

Problem 1.



(a) Write the loop and node equations needed to determine the currents I_1, I_2, and I_3 in the circuit shown.

$$\begin{aligned} &\text{node:} && I_1 = I_2 + I_3 \\ &\text{loop 1:} && 13 - 3I_1 - 4I_2 - 8 = 0 \\ &\text{loop 2:} && 8 + 4I_2 - 2I_3 + 4 = 0 \end{aligned}$$

(b) Determine the currents by explicit solution of the equations. Rearrange loop 1 and loop 2 equations:

$$\begin{aligned} &\text{loop 1:} && 3I_1 + 4I_2 = 5 \\ &\text{loop 2:} && 4I_2 - 2I_3 = -12 \end{aligned}$$

Use the node equation to eliminate I_3 from the loop 2 equation; multiply the resulting equation by 1.5:

$$\begin{aligned} 3I_1 + 4I_2 &= 5 \\ -3I_1 + 9I_2 &= -18 \end{aligned}$$

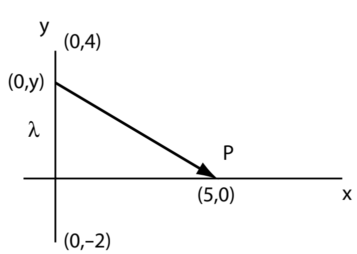
Add to eliminate I_1, solve to get I_2. Substitute back to get I_1 then I_3. Results are

$$I_1 = 3.0\,\text{A}, \quad I_2 = -1.0\,\text{A}, \quad I_3 = 4.0\,\text{A}$$

(c) Determine the value of the potential difference between point a and point b ($V_b - V_a$) in the above circuit along the path through the 4 V battery and the 2 Ω resistor.

$$V_b = V_a - 4 + 2I_3 = V_a - 4 + 8 \Rightarrow V_b - V_a = 4\,\text{V}$$

Problem 2. A line charge density of $\lambda = 6.0 \times 10^{-7}$ C/m extends along the y-axis from $y' = 2.0$ m to $y' = 4.0$ m, as shown.



(a) What is the vector $(\mathbf{r} - \mathbf{r}')$ from a point $(0, y')$ on the line charge to a point P at $(x, y) = (5.0, 0.0)$ m?

$$\mathbf{r} - \mathbf{r}' = 5\hat{\mathbf{i}} - y\hat{\mathbf{j}}$$

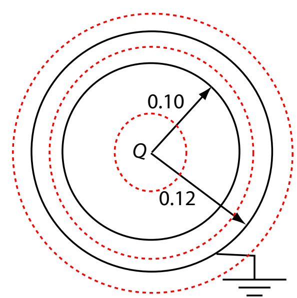
(b) Determine the electric field at the point P due to the line charge.

$$d\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{dQ}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{r} - \mathbf{r}') = \frac{1}{4\pi\epsilon_0} \frac{(5\hat{\mathbf{i}} - y\hat{\mathbf{j}})}{(5^2 + y^2)^{3/2}}\, dQ$$

Now get E_x and E_y separately, using $dQ = \lambda\, dy$:

$$\begin{aligned} E_x &= \frac{5\lambda}{4\pi\epsilon_0} \int_{-2}^{4} \frac{dy}{(5^2 + y^2)^{3/2}} = \frac{5\lambda}{4\pi\epsilon_0} \left. \frac{y}{5^2\sqrt{5^2 + y^2}} \right|_{-2}^{4} \\ &= \frac{5\lambda}{4\pi\epsilon_0} \left[\frac{4}{25\sqrt{41}} + \frac{2}{25\sqrt{29}} \right] = 1.08 \times 10^3\,\text{N/C} \\ E_y &= -\frac{\lambda}{4\pi\epsilon_0} \int_{-2}^{4} \frac{y\, dy}{(5^2 + y^2)^{3/2}} = \frac{\lambda}{4\pi\epsilon_0} \left. \frac{1}{\sqrt{5^2 + y^2}} \right|_{-2}^{4} \\ &= \frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{41}} - \frac{1}{\sqrt{29}} \right] = -159\,\text{N/C} \end{aligned}$$

Problem 3. A point charge Q is placed at the center of a spherical metal shell with inner and outer radii of 0.10 m and 0.12 m, respectively, as shown.



(a) Use Gauss's Law to determine Q if the electric field is $60{,}000/r^2$ N/C pointing radially inward for $r < 0.1$ m.

Choose a spherical surface with radius $r < 0.1$ m. We have

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{encl}}}{\epsilon_0},$$

where the integral is $E \times 4\pi r^2$ and $Q_{\text{encl}} = Q$:

$$-\frac{60,000}{r^2} 4\pi r^2 = -60,000 \times 4\pi = \frac{Q}{\epsilon_0}.$$

The minus sign is because the field points inward and $d\mathbf{A}$ point outward. Solving for Q yields

$$Q = -60,000 \times 4\pi\epsilon_0 = -6.67 \times 10^{-6}\,\text{C}$$

(b) The metal shell is connected to ground (*i.e.*, the electric field is zero outside the shell.) Use Gauss's Law to determine the charge (not charge density) at $r = 0.10$ m.

To find the surface charge $Q_{0.10}$ at $r = 0.10$ m, choose a surface with $r > 0.10$ m but inside the metal shell, where $\mathbf{E} = 0$. Then the surface integral is zero and

$$0 = \frac{Q_{\text{encl}}}{\epsilon_0} = \frac{Q + Q_{0.10}}{\epsilon_0} \Rightarrow Q_{0.10} = -Q = 6.67\,\mu\text{C}$$

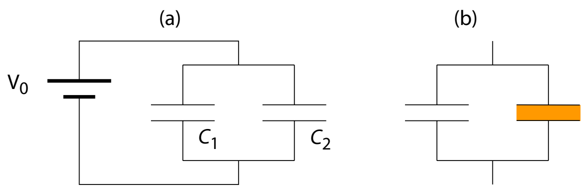
(c) Use Gauss's Law to determine the charge at $r = 0.12$ m.

Draw surface at $r > 0.12$ m. Given that $\mathbf{E}$ outside the grounded shell is zero, we again have the surface integral zero:

$$0 = \frac{Q_{\rm encl}}{\epsilon_0} = \frac{Q + Q_{0.10} + Q_{0.12}}{\epsilon_0} \Rightarrow Q_{0.12} = 0$$

The outer surface charge $Q_{0.12} = 0$ since $Q + Q_{0.10}$ is already zero.

Problem 4. Two capacitors C_1 and C_2 are connected in parallel, and each acquires a charge when connected to a source of voltage V_0. The voltage source is then disconnected and a dielectric ($K = 5.0$) is inserted to fill the space between the plates of C_2.



(a) If $C_1 = 4.0\,\mu$F, $C_2 = 1.0\,\mu$F, and $V_0 = 9.0$ V, find the initial charge on each capacitor (before the dielectric is inserted).

$$Q_1^0 = C_1 V_0 = 4\,\mu\text{F} \times 9\,\text{V} = 36\,\mu\text{C}$$
$$Q_2^0 = C_2 V_0 = 1\,\mu\text{F} \times 9\,\text{V} = \ \ 9\,\mu\text{C}$$

(b) What is the charge on each capacitor after V_0 is disconnected and the dielectric is inserted?

The total charge on both capacitors is $45\,\mu$C; it will be distributed to equalize the voltages. Let Q_1 and Q_2 be the charges on C_1 and C_2, respectively:

$$V = \frac{Q_1}{C_1} = \frac{Q_2}{KC_2} \Rightarrow \frac{45 - Q_2}{4} = \frac{Q_2}{5}$$

Solve for Q_2:

$$4Q_2 = 5(45 - Q_2) \Rightarrow Q_2 = 25$$

The charge on C_2 is 25 μC and on C_1 is 20 μC.

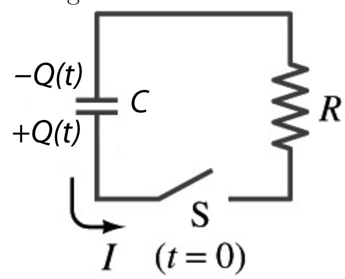
(c) What is the voltage across each capacitor after the dielectric is inserted?

The voltages across the two capacitors will be the same:

$$V = \frac{25\,\mu\text{C}}{5\,\mu\text{F}} = \frac{20\,\mu\text{C}}{4\,\mu\text{F}} = 5\,\text{V}$$

Problem 5.

The capacitor shown in the diagram is charged to a voltage V_0. At $t = 0$, the switch is closed, allowing the capacitor to discharge.



(a) What is the charge Q_0 on the capacitor before the switch is closed?

$$Q_0 = CV_0$$

(b) Write the loop equation for this circuit for $t \geq 0$.

$$\frac{Q}{C} - IR = 0$$

(c) What is the relation between the charge on the capacitor $Q(t)$ and the current I? *Hint:* Be careful with the sign.

Charge flowing out of the capacitor in the direction indicated for the current corresponds to a decrease in $Q(t)$; hence

$$I = -\frac{dQ}{dT}$$

(d) Use the results of part (c) to transform the loop equation into a differential equation for $Q(t)$.

$$\frac{Q}{C} + R\frac{dQ}{dt} = 0 \Rightarrow \frac{dQ}{dt} = -\frac{Q}{RC}$$

(e) Show that the solution

$$Q(t) = CV_0 \exp\left(-\frac{t}{RC}\right)$$

satisfies the loop equation.

Use

$$I = -\frac{dQ}{dt} = -CV_0 \exp\left(-\frac{t}{RC}\right)\left(-\frac{1}{RC}\right) = \frac{Q}{RC}$$

and substitute. It works.

(f) If $R = 10\,\Omega$ and $C = 5\,\mu$F, how long will it take the charge on the capacitor to decay to 1% of its value at $t = 0$.

$$\exp(-t/RC) = 0.01 \Rightarrow \frac{t}{RC} = \ln 100$$
$$t = RC \ln 100 = 10\,\Omega \times 5\,\mu\text{F} \times 4.605 = 2.3 \times 10^{-4}\,\text{s}$$