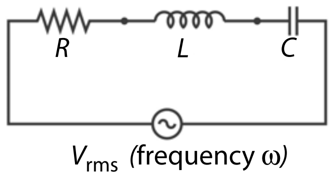
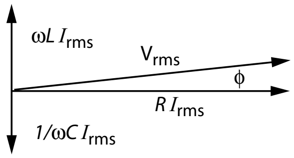
Physics 21
Fall, 2004

Solution, Practice Exam #2

Problem 1. For the following circuit:





(a) Write the loop equation for this circuit.

$$V - IR - L\frac{dI}{dt} - \frac{Q}{C} = 0$$

(b) If $R = 20\,\Omega$, $L = 0.025\,\mathrm{H}$, and $C = 50\,\mu\mathrm{F}$ ($50 \times 10^{-6}\,\mathrm{F}$), find the frequency ω for which the current lags the voltage across the inductor by 30°. *Hint:* Make sure $\omega > 0$.

OOPS, the question SHOULD HAVE SAID, find the frequency ω for which the applied voltage lags the voltage across the inductor by 30°.

From the diagram, the phase angle ϕ can be written

$$\tan\phi = \frac{\omega L - \frac{1}{\omega C}}{R}.$$

If the applied voltage $V \sim \sin(\omega t + \phi)$ lags the voltage across the inductor by 30°, then $\phi = 60°$ and

$$\tan\phi = \sqrt{3} = \frac{\omega L - \frac{1}{\omega C}}{R}$$

We transform this into a quadratic equation for ω

$$L\omega^2 - \sqrt{3}R\omega - 1/C = 0,$$

whose solution is

$$\omega = \frac{\sqrt{3}R \pm \sqrt{3R^2 + 4L/C}}{2L}.$$

We must take the plus sign so that $\omega > 0$; the result is $\omega = 1.82\,\mathrm{kH}$. A negative value of ω would reverse the directions of the vertical vectors on the phasor diagram, and change the phase relations.

(c) If $V_{\mathrm{rms}} = 100\,\mathrm{V}$, what is the rms current? What is the peak current?

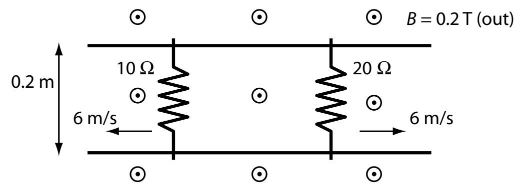
From the phasor diagram,

$$V_{\mathrm{rms}} = \sqrt{\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2}\, I_{\mathrm{rms}}$$

Substituting $\omega = 1.82\,\mathrm{kH}$ and the other values gives $V_{\mathrm{rms}} = 40.0 I_{\mathrm{rms}}$. For $V_{\mathrm{rms}} = 100\,\mathrm{V}$,

$$I_{\mathrm{rms}} = 2.5\,\mathrm{A} \qquad I_{\mathrm{peak}} = \sqrt{2} I_{\mathrm{rms}} = 3.5\,\mathrm{A}$$

Problem 2. A magnetic field $B = 0.2$ T points out of the paper. The $10\,\Omega$ resistor moves to the left with a speed of 6 m/s, and the $20\,\Omega$ resistor moves to the right at 6 m/s. The top and bottom wires have no resistance, and the loop is 0.2 m high.



(a) What is the current induced in the circuit?

Use Faradays's Law:

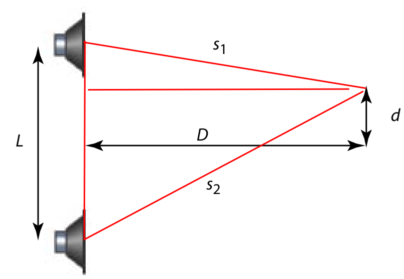
$$\mathcal{E} = -\frac{d\Phi}{dt} = -BL(6\,\mathrm{m/s} + 6\,\mathrm{m/s})$$
$$= (-0.2\,\mathrm{T})(0.2\,\mathrm{m})(12.0\,\mathrm{m/s}) = -0.48\,\mathrm{V}$$

The current is 0.48 $\mathrm{V}/(10\,\Omega + 20\,\Omega) = 16$ mA.

(b) What direction does it flow in?

The current is clockwise, by Lenz's Law.

Problem 3. Two identical speakers driven at frequency ω are separated by a distance $L = 3.0$ m. When a microphone in centered between the speakers a distance $D = 4.0$ m in front of them, the sound is loudest. When the microphone is moved $d = 0.5$ m as shown, the first minimum in volume is heard. What is the frequency of the sound? Use 345 m/s for the speed of sound in air.



When the microphone is moved up 0.5 m, the distance to the top speaker is

$$s_1 = \sqrt{1^2 + 4^2} = 4.123\,\mathrm{m},$$

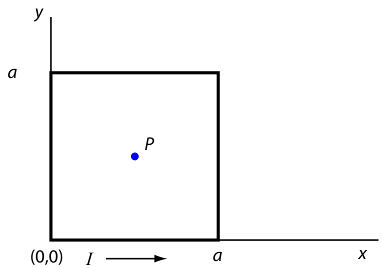
and the distance to the lower speaker is

$$s_2 = \sqrt{2^2 + 4^2} = 4.472\,\mathrm{m},$$

so the difference in lengths is 0.349 m, which must be one half of a wavelength. Hence $\lambda = 0.698\,\mathrm{m}$. Using $\lambda f = v$,

$$f = v/\lambda = \frac{345\,\mathrm{m/s}}{0.698\,\mathrm{m}} = 494\,\mathrm{Hz}$$

Problem 4. Consider a current loop that has the form of a square of side a:



(a) Explain why the current in each side of the square makes an identical contribution to the magnetic field at the point P at the center of the square.

The square is symmetric about its midpoint, so each side is equivalent. The contribution to the field at P from each side is perpendicular to the page, as well, so no components cancel.

(b) Find the magnetic field at P, using the Biot-Savart Law. *Hint:* When you set up the integral over the bottom side of the square, change variables to $u = x - \frac{a}{2}$ in order to transform the integral to one on the equation sheet.

We use

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

where $d\mathbf{l} = dx\,\hat{\mathbf{i}}$, the coordinate vector $\mathbf{r}$ of the field point P is $\mathbf{r} = (a/2)\hat{\mathbf{i}} + (a/2)\hat{\mathbf{j}}$, and the location of the current point is $\mathbf{r}' = x\hat{\mathbf{i}}$. Then

$$\mathbf{r} - \mathbf{r}' = \left(\frac{a}{2} - x\right)\hat{\mathbf{i}} + \frac{a}{2}\hat{\mathbf{j}}$$

and

$$d\mathbf{l} \times (\mathbf{r} - \mathbf{r}') = dx\,\frac{a}{2}\hat{\mathbf{k}}$$

The total $\mathbf{B}$, including an extra factor of four for the four sides of the square, is

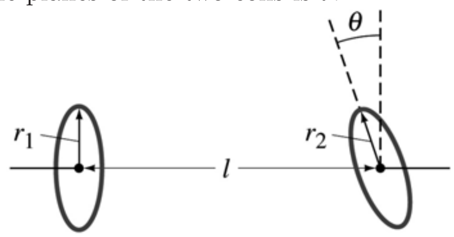
$$\mathbf{B} = 4 \times \frac{\mu_0 I}{4\pi} \int_0^a \frac{(a/2)\,dx}{[(a/2 - x)^2 + (a/2)^2]^{3/2}}\hat{\mathbf{k}}$$

The suggested change of variable yields

$$\mathbf{B} = 4 \times \frac{\mu_0 I}{4\pi} \int_{-a/2}^{a/2} \frac{(a/2)\,du}{[u^2 + (a/2)^2]^{3/2}}\hat{\mathbf{k}}$$

$$\mathbf{B} = \frac{\mu_0 I}{\pi} \frac{2\sqrt{2}}{a}$$

Problem 5. Consider two small circular loops of radii r_1 and r_2, which are separated by a distance l that is large compared to r_1 and r_2. The line joining their centers is perpendicular to the plane of coil 1, and the angle between the planes of the two coils is θ.



(a) Show that the mutual inductance M is given approximately by

$$M = \frac{\mu_0}{2\pi} \frac{A_1 A_2 \cos\theta}{l^3},$$

where the A's are the areas of the loops. State clearly the approximations you make. *Hint:* The magnitude of the field of a coil of radius r, on axis, a distance x from the plane of the coil, is

$$B = \frac{\mu_0 I r^2}{2(r^2 + x^2)^{3/2}}.$$

Since the loops are small, we assume that the magnetic field due to the first loop is nearly constant at the second loop, and vice versa. The magnetic field of the first loop at a distance l is

$$B = \frac{\mu_0 I r_1^2}{2(r_1^2 + l^2)^{3/2}} \approx \frac{\mu_0 I r_1^2}{2l^3} \qquad \text{(since } l \gg r_2\text{)}.$$

The flux of this field through the second loop is $BA_2\cos\theta$, or

$$\Phi = \frac{\mu_0 I r_1^2}{2l^3} A_2 \cos\theta = \frac{\mu_0 I \pi r_1^2}{2\pi l^3} A_2 \cos\theta.$$

Since $A_1 = \pi r_1^2$, we have

$$\Phi = \frac{\mu_0 I}{2\pi} \frac{A_1 A_2 \cos\theta}{l^3}$$

$$\frac{d\Phi}{dt} = \left(\frac{\mu_0}{2\pi} \frac{A_1 A_2 \cos\theta}{l^3}\right) \frac{dI}{dt}$$

The expression in the large parenthesis is M, as required.

(b) If $r_1 = 5$ mm, $r_2 = 8$ mm, $\theta = 60°$, and $l = 1.5$ m, find M.

Converting to meters and substituting, one gets

$$M = 4.7 \times 10^{-16}\,\text{H}.$$