Robustness and Performance Analysis of Cyclic Interconnected Dynamical Networks

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Small Steps Towards Network Versions of Hard limits

- Thermodynamics (Carnot)
- Control (Bode)
- Communications (Nyquist-Shannon)
- Computation (Turing/Gödel)
- ...

No network version

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Robustness in Interconnected Dynamical Networks

Robustness in dynamical networks can be assessed w.r.t.

- Information structure in the controller array, or
- Underlying structure of networks
  - Additive external stochastic disturbances (as inputs)
  - Subsystem failure
  - Elimination of coupling in dynamics between two subsystems
  - Specific structures in networks, e.g., autocatalytic networks
  - Uncertainty in model
  - …
We consider the steady-state variance of the state of the network as a robustness measure. Consider a linear time-invariant system

\[
\dot{x}(t) = Ax(t) + \omega(t),
\]

where \( A \) is a Hurwitz matrix and \( \omega(t) \) is a zero-mean continuous time white random process in \( \mathbb{R}^n \).

We consider the steady-state variance of the state of the network as a robustness measure

\[
H_2^2 := \lim_{t \to \infty} E[x(t)^T x(t)]
\]
Theorem: (main result)

Consider the following linear dynamics

\[ \dot{x} = Ax + \omega, \]

where \( A \) is Hurwitz and \( \omega(t) \in \mathbb{R}^n \) is a unit variance white stochastic process. Then we have

\[ - \sum_{i=1}^{n} \frac{1}{2 \text{Re}\{\lambda_i(A)\}} = H^2 = \lim_{t \to \infty} \mathbb{E}[x(t)^T x(t)] = - \sum_{i=1}^{n} \frac{1}{\lambda_i(A_s)}, \]

where \( A_s = A^T + A \).

\[ A^T A = A A^T \]
Class of Matrices

All matrices

Normal matrices

Symmetric matrices
Ex. First Order Consensus problems

- Directed and undirected graphs

- Undirected graphs

- Undirected graphs and few directed graphs

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Structure of Autonomous Cyclic Networks

\[
\begin{align*}
\dot{x}_i &= -a_i x_i + u_i + \omega_i, \\
y_i &= c_i x_i,
\end{align*}
\]

for \( i = 1, 2, \ldots, n \) and \( a_i, c_i > 0 \).
Structure of Autonomous Cyclic Networks

\[
\begin{align*}
\dot{x}_1 &= -a_1 x_1 y_n + \omega_1, \\
\dot{x}_2 &= -a_2 x_2 + y_1 + \omega_2, \\
\vdots \\
\dot{x}_n &= -a_n x_n + y_{n-1} + \omega_n, 
\end{align*}
\]
Abstract form of Autonomous Cyclic Networks

\[ \dot{x}(t) = Ax(t) + \omega(t), \]

\[ A = \begin{bmatrix} -a_1 & 0 & \ldots & 0 & -c_n \\ c_1 & -a_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & -a_{n-1} & 0 \\ 0 & 0 & \ldots & c_{n-1} & -a_n \end{bmatrix} \]
Stability of Autonomous Cyclic Networks

\[ a := \left( a_1 a_2 \cdots a_n \right)^{\frac{1}{n}} \]

\[ c := \left( c_1 c_2 \cdots c_n \right)^{\frac{1}{n}} \]

\[ q := \frac{a}{c} \]

\[
A = \begin{bmatrix}
-a_1 & 0 & \ldots & 0 & -c_n \\
c_1 & -a_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & -a_{n-1} & 0 \\
0 & 0 & \ldots & c_{n-1} & -a_n \\
\end{bmatrix}
\]
Stability of Autonomous Cyclic Networks

\[ a := \left( a_1 a_2 \cdots a_n \right)^{\frac{1}{n}} \]

\[ c := \left( c_1 c_2 \cdots c_n \right)^{\frac{1}{n}} \]

\[ q > \cos\left( \frac{\pi}{n} \right) \quad \text{A is Hurwitz} \]

\[
A = \begin{bmatrix}
-a_1 & 0 & \ldots & 0 & -c_n \\
-c_1 & -a_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & -a_{n-1} & 0 \\
0 & 0 & \ldots & c_{n-1} & -a_n
\end{bmatrix}
\]
Stability of Autonomous Cyclic Networks

When \( a = a_1 = \cdots = a_n \)

\[ q > \cos\left(\frac{\pi}{n}\right) \quad \iff \quad A \text{ is Hurwitz} \]

\[
A = \begin{bmatrix}
-a_1 & 0 & \ldots & 0 & -c_n \\
0 & -a_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & -a_{n-1} & 0 \\
0 & 0 & \ldots & c_{n-1} & -a_n
\end{bmatrix}
\]
Proposition:

Consider the cyclic interconnected network driven by a white random process (CN), where \( q > \cos\left(\frac{\pi}{n}\right) \), then for the steady state variance of the state we have

\[
- \sum_{i=1}^{n} \frac{1}{2 \text{Re}\{\lambda_i(A)\}} \leq H^2 \leq - \sum_{i=1}^{n} \frac{1}{\lambda_i(A_s)}
\]
Cyclic Networks of Identical Subsystems

\[-\sum_{i=1}^{n} \frac{1}{2\text{Re}\{\lambda_i(A)\}} \leq H^2 \leq -\sum_{i=1}^{n} \frac{1}{\lambda_i(A_s)}\]

When \( a = a_1 = \cdots = a_n \)

\[= \begin{cases} 
\frac{n \tan \frac{\beta}{2}}{2c \sin \frac{\beta}{n}}, & q < 1 \\
\frac{n^2}{2c}, & q = 1 \\
\frac{n \tanh \frac{\beta}{2}}{2c \sinh \frac{\beta}{n}}, & q > 1 
\end{cases}\]

\[\beta := \begin{cases} 
\arccos(q)n, & q \leq 1 \\
\text{arcosh}(q)n, & q > 1 
\end{cases}\]

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Cyclic Networks of Identical Subsystems

\[-\sum_{i=1}^{n} \frac{1}{2\text{Re}\{\lambda_i(A)\}} \leq H^2 \leq -\sum_{i=1}^{n} \frac{1}{\lambda_i(A_s)}\]

When \(a = a_1 = \cdots = a_n\)
\(c = c_1 = \cdots = c_n\)

\[H^2 = \begin{cases} 
\frac{n \tan \frac{\beta}{2}}{2c \sin \frac{\beta}{n}} & , \quad q < 1 \\
\frac{n^2}{2c} & , \quad q = 1 \\
\frac{n \tanh \frac{\beta}{2}}{2c \sinh \frac{\beta}{n}} & , \quad q > 1
\end{cases}\]

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Approximation of the Robustness Measure

$H$ can be approximated by

$$H \approx \begin{cases} 
\sqrt{\frac{\tan \frac{\beta}{2}}{2c\beta}} n , & q < 1 \\
\sqrt{\frac{1}{2c}} n , & q = 1 \\
\sqrt{\frac{\tanh \frac{\beta}{2}}{2c\beta}} n , & q > 1 
\end{cases}$$
How this measure scale with network size?

![Graph showing the relationship between robustness measure ($H$) and number of subsystems ($n$). The graph illustrates that the robustness measure increases with the number of subsystems, with decreasing β where $q < 1$ and increasing β where $q > 1$.](image)

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Cyclic Dynamical Networks with Control Inputs

\begin{align*}
\dot{x}_i &= -f_i(x_i) + u_i, \\
y_i &= g_i(x_i), \quad \text{for } 1 \leq i \leq n - 1, \\
\dot{x}_n &= -f_n(x_n) + u_n - u, \\
y_n &= d_n u,
\end{align*}
Cyclic Dynamical Networks with Control Inputs

\[
\begin{align*}
\dot{x}_1 &= -f_1(x_1) + d_n u, \\
\dot{x}_2 &= -f_2(x_2) + g_1(x_1), \\
\vdots \\
\dot{x}_n &= -f_n(x_n) + g_{n-1}(x_{n-1}) - u + \delta, \\
y &= x_n, 
\end{align*}
\]

(CNC)
Cyclic Dynamical Networks with Control Inputs

\[
\begin{align*}
\dot{x}_1 &= -f_1(x_1) + d_n u, \\
\dot{x}_2 &= -f_2(x_2) + g_1(x_1), \\
\vdots \\
\dot{x}_n &= -f_n(x_n) + g_{n-1}(x_{n-1}) - u + \delta, \\
y &= x_n,
\end{align*}
\]

(CNC)

\[
\begin{align*}
a &= \left( f_1'(0)f_2'(0) \cdots f_{n-1}'(0) \right)^{\frac{1}{n-1}} \\
c &= \left( g_1'(0)g_2'(0) \cdots g_{n-1}'(0)d_n \right)^{\frac{1}{n-1}} \\
qu &= \frac{a}{c}
\end{align*}
\]
Hard Limits on $L_2$-gain Disturbance Attenuation

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u + p(x)\delta, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \\
y &= h(x), \quad \delta \in L^2(0, T)
\end{align*}
\]

The $L_2$-gain form $\delta$ to $y$ is less than or equal to $\gamma$

\[
\int_0^T |y(t)|^2 dt \leq \gamma^2 \int_0^T |\delta(t)|^2 dt, \\
\forall T > 0 \text{ and zero initial state}
\]

Finding a stabilizing state feedback which minimizes $\gamma$

$\gamma^*$: the best achievable $L_2$-gain

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Hard Limits on $L_2$-gain Disturbance Attenuation

Theorem:

Consider the cyclic networks (CNC), when $q < 1$ then there exists a hard limit on the best achievable disturbance attenuation, $\gamma^*$, such that the regional state feedback $L_2$-gain disturbance attenuation problem with stability is solvable for each $\gamma > \gamma^*$ and not for $\gamma < \gamma^*$. And the hard limit function can be written as

$$\gamma^* \geq H = \frac{1}{f'_n(0) + c - a}.$$
Motivating Examples: Glycolysis Pathway

Glycolysis pathway is a central energy producer in a living cell.

Autocatalysis

Allosteric Regulation

Catalyzing Enzymes

Cell Consumption of ATP

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Motivating Examples: Glycolysis Pathway

Cell Consumption of ATP
Motivating Examples: Glycolysis Pathway

Cell Consumption of ATP

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Ex. A minimal Autocatalytic Pathway Model

Cell Consumption of ATP

\[ s + \alpha y \xrightarrow{f} x \xrightarrow{k_x} (\alpha + 1)y + x' \]
\[ y \xrightarrow{k_y,\delta} \emptyset \]
Ex. A minimal Autocatalytic Pathway Model

\[
\begin{align*}
\dot{x} &= -k_x x + \frac{1}{\alpha} u \\
\dot{y} &= -k_y y + (\alpha + 1) k_x x - u + \delta,
\end{align*}
\]
Ex. A minimal Autocatalytic Pathway Model

\[
\begin{align*}
\dot{x} &= -k_x x + \frac{V_y q}{1 + \gamma y^h} \\
\dot{y} &= -k_y y + (\alpha + 1)k_x x - \frac{\alpha V_y q}{1 + \gamma y^h} + \delta,
\end{align*}
\]

Feedback designed by Nature

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The glycolysis mechanism is more robust if $k_x$ and $k_y$ are large.

$$H(\alpha, k_x, k_y) = \frac{\alpha}{k_x + \alpha k_y}$$

large $k_x$ and $k_y$ require either a more efficient or a higher level of enzymes.
Conclusion

- We exploit structural properties of networks of interconnection systems in order to characterize their robustness properties and fundamental limits.

- We obtain the new tight bounds on the $H_2$ norm of the network, which measures the expected steady-state dispersion of the state of the entire network.

- We show that the introduced robustness measure depends on characteristics of the underlying digraph of the network as well as the size of the network.
Previous and Ongoing Work

- **Fundamental limitations due to structures of networks**

- **Fundamental limitations of feedback control laws in cyclic dynamical networks**


- **Fundamental limits on robustness measures of networks with linear dynamics**

- **Fundamental limits on improving robustness of networks by “smart” rewiring**
  Preparing the draft!

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