LINEAR ALGEBRA $\underline{\text{VI}}$ – LINEAR MODELS AND DECISIONS

LINEAR TRANSFORMATIONS

$L: V \rightarrow V$ IS A LINEAR TRANSFORMATION ON V iff $L(\alpha_1 v_1 + \ldots + \alpha_N v_N) = \alpha_1 L(v_1) + \ldots + \alpha_N L(v_N)$

$$\therefore L(v) = L\left(\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_N\end{pmatrix}\right) = L(\alpha_1 v_1 + \ldots + \alpha_N v_N) \; [\text{FOR } \langle v_1, \ldots, v_N\rangle \text{ A BASIS}] = \alpha_1 L(v_1) + \ldots + \alpha_N L(v_N)$$

THUS ANY TRANSFORMATION IS COMPLETELY SPECIFIED BY WHAT IT DOES TO THE BASIS VECTORS

FOR v_i A BASIS VECTOR, $L(v_i) = m_{1i} v_1 + \ldots + m_{Ni} v_N$, A LINEAR COMBINATION OF THE BASIS VECTORS

THAT IS, $L(v_i) = \begin{pmatrix} m_{1i} \\ \vdots \\ m_{Ni}\end{pmatrix}$

$$\text{THUS, } \{L(v_1) \ldots L(v_N)\} = \left\{\begin{pmatrix} m_{11} \\ \vdots \\ m_{N1}\end{pmatrix} \cdots \begin{pmatrix} m_{1N} \\ \vdots \\ m_{NN}\end{pmatrix}\right\} = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1N} \\ m_{21} & m_{22} & \cdots & m_{2N} \\ \vdots & \vdots & & \vdots \\ m_{N1} & m_{N2} & \cdots & m_{NN}\end{pmatrix}$$

COMPLETELY SPECIFIES THE LINEAR TRANSFORMATION L

FURTHERMORE,

$$L(v) = L\left(\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_N\end{pmatrix}\right) = \alpha_1 L(v_1) + \ldots + \alpha_N L(v_N) = \begin{pmatrix} m_{11} & \cdots & m_{1N} \\ \vdots & & \vdots \\ m_{N1} & \cdots & m_{NN}\end{pmatrix}\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_N\end{pmatrix} = \alpha_1\begin{pmatrix} m_{11} \\ \vdots \\ m_{N1}\end{pmatrix} + \ldots + \alpha_N\begin{pmatrix} m_{1N} \\ \vdots \\ m_{NN}\end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 m_{11} + \alpha_2 m_{12} + \ldots + \alpha_N m_{1N} \\ \vdots \\ \alpha_1 m_{N1} + \alpha_2 m_{N2} + \ldots + \alpha_N m_{NN}\end{pmatrix} = \begin{pmatrix} \left\langle \text{1ST ROW}; m_{11} \ldots m_{1N}, \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_N\end{pmatrix}\right\rangle \\ \vdots \\ \left\langle \text{NTH ROW}; m_{N1} \ldots m_{NN}, \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_N\end{pmatrix}\right\rangle\end{pmatrix}$$ FOR THE EUCLIDEAN INNER PRODUCT

$\therefore$ A LINEAR TRANSFORMATION CAN BE COMPUTED IN TERMS OF ITS REPRESENTATION AS $\begin{pmatrix} m_{11} & \cdots & m_{1N} \\ \vdots & & \\ m_{N1} & \cdots & m_{NN}\end{pmatrix}$ IN TERMS OF ITS EFFECTS OF THE (FIXED) BASIS VECTORS

SUCH A REPRESENTATION IS CALLED A MATRIX, AND IS SYMBOLIZED $M = (m_{ij})$

$(L_1 + L_2)(v) \equiv L_1(v) + L_2(v)$ BY THIS DEFINITION, LINEAR TRANSFORMATIONS (THUS MATRICES) FORM A GROUP

IT INDUCES $A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}) = (c_{ij}) = C$ FOR MATRICES A, B, C

SIMILARLY $(\alpha L)(v) \equiv \alpha(L(v))$ BY THIS DEFINITION, LINEAR TRANSFORMATIONS AND MATRICES ARE RELATED TO THE UNDERLYING FIELD, AND BECOME THEMSELVES A VECTOR SPACE

$\alpha A = \alpha(a_{ij}) = (\alpha a_{ij}) = (b_{ij}) = B$ FOR MATRICES A, B

$(L_1 L_2)(v) \equiv L_1(L_2(v))$ BY THIS DEFINITION, LINEAR TRANSFORMATIONS AND MATRICES ACQUIRE A MULTIPLICATION — TOGETHER WITH ADDITION AS ABOVE, THEY FORM A RING {IDENTITY AND INVERSES NOT YET DEFINED}

THIS INDUCES, FOR MATRICES A, B, C, $AB = (a_{ij})(b_{ij}) = (c_{ij}) = C$

HOW IS c_{ij} TO BE COMPUTED?

$$C = (c_{ij}) = (\langle i\text{th ROW OF } A, j\text{th COLUMN OF } B\rangle)$$

AN ISOMORPHISM BETWEEN RING/VECTOR SPACE OF LINEAR TRANSFORMATIONS AND OF MATRICES