LINEAR MODELS AND DECISIONS

15) $E(Y) = E(X\beta+\varepsilon) = X\beta + 0 = X\beta$

$E(\hat{Y}) = E[X(X'X)^{-1}X'Y] = X(X'X)^{-1}X'E(Y) = X(X'X)^{-1}X'X\beta = X\beta$

$E(\hat{\varepsilon}) = E(Y-\hat{Y}) = E(Y) - E(\hat{Y}) = 0$

$E(\hat{\beta}) = E((X'X)^{-1}X'Y) = (X'X)^{-1}X'E(Y) = (X'X)^{-1}X'X\beta = \beta$

$V(Y) = E[(Y-E(Y))(Y-E(Y))'] = E[(Y-X\beta)(Y-X\beta)'] = E[\varepsilon\varepsilon'] = E[(\varepsilon-0)(\varepsilon-0)'] = E[(\varepsilon-E(\varepsilon))(\varepsilon-E(\varepsilon))'] = V(\varepsilon) = \sigma^2 I$

$E(YY') = E[(X\beta+\varepsilon)(X\beta+\varepsilon)'] = E[(X\beta+\varepsilon)(\beta'X'+\varepsilon')] = E[X\beta\beta'X' + X\beta\varepsilon' + \varepsilon\beta'X' + \varepsilon\varepsilon']$

$= E[X\beta\beta'X'] + E[X\beta\varepsilon'] + E[\varepsilon\beta'X'] + E[\varepsilon\varepsilon'] = X\beta\beta'X' + X\beta E(\varepsilon)' + E(\varepsilon)\beta'X' + E(\varepsilon\varepsilon')$

$= X\beta\beta'X' + 0 + 0 + E(\varepsilon\varepsilon') = X\beta\beta'X' + V(\varepsilon)$

$V(\hat{Y}) = E[(\hat{Y}-E(\hat{Y}))(\hat{Y}-E(\hat{Y}))'] = E[(\hat{Y}-X\beta)(\hat{Y}-X\beta)'] = E[(\hat{Y}-X\beta)(\hat{Y}'-\beta'X')]$

$= E[\hat{Y}\hat{Y}' - \hat{Y}\beta'X' - X\beta\hat{Y}' + X\beta\beta'X'] = E[X(X'X)^{-1}X'YY'X(X'X)^{-1}X' - X(X'X)^{-1}X'Y\beta'X' - X\beta Y'X(X'X)^{-1}X' + X\beta\beta'X']$

$= X(X'X)^{-1}X'E(YY')X(X'X)^{-1}X' - X(X'X)^{-1}X'E(Y)\beta'X' - X\beta E(Y)'X(X'X)^{-1}X' + X\beta\beta'X'$

$= X(X'X)^{-1}X'[X\beta\beta'X' + V(\varepsilon)]X(X'X)^{-1}X' - X(X'X)^{-1}X'X\beta\beta'X' - X\beta\beta'X'X(X'X)^{-1}X' + X\beta\beta'X'$

$= X(X'X)^{-1}X'[X\beta\beta'X' + V(\varepsilon)]X(X'X)^{-1}X' - X\beta\beta'X' - X\beta\beta'X' + X\beta\beta'X'$

$= X(X'X)^{-1}X'X\beta\beta'X'X(X'X)^{-1}X' + X(X'X)^{-1}X'V(\varepsilon)X(X'X)^{-1}X' - X\beta\beta'X'$

$= X\beta\beta'X' + X(X'X)^{-1}X'V(\varepsilon)X(X'X)^{-1}X' - X\beta\beta'X' \qquad = X(X'X)^{-1}X'V(\varepsilon)X(X'X)^{-1}X'$

$= X(X'X)^{-1}X'\sigma^2 I X(X'X)^{-1}X' = \sigma^2 X(X'X)^{-1}X'X(X'X)^{-1}X' = \sigma^2 X(X'X)^{-1}X'$

$V(\hat{\beta}) = E(\hat{\beta}\hat{\beta}') - E(\hat{\beta})E(\hat{\beta})' = E[(X'X)^{-1}X'YY'X(X'X)^{-1}] - \beta\beta' = (X'X)^{-1}X'E(YY')X(X'X)^{-1} - \beta\beta'$

$= (X'X)^{-1}X'[X\beta\beta'X' + V(\varepsilon)]X(X'X)^{-1} - \beta\beta' = (X'X)^{-1}X'X\beta\beta'X'X(X'X)^{-1} + (X'X)^{-1}X'V(\varepsilon)X(X'X)^{-1} - \beta\beta'$

$= \beta\beta' + (X'X)^{-1}X'V(\varepsilon)X(X'X)^{-1} - \beta\beta' \qquad = (X'X)^{-1}X'V(\varepsilon)X(X'X)^{-1}$

$= (X'X)^{-1}X'\sigma^2 I X(X'X)^{-1} = \sigma^2 (X'X)^{-1}X'X(X'X)^{-1} = \sigma^2 (X'X)^{-1}$

NOTE THAT $V(\hat{Y})$ COULD HAVE BEEN COMPUTED SIMILARLY, i.e., BY TAKING $E(\hat{Y}\hat{Y}') - E(\hat{Y})E(\hat{Y})'$ AS THE FIRST STEP