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Cyclic-cubes and wrap-around butterflies

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Abstract

Recently a new family of Cayley graphs of fixed even node degree called the Cyclic-cubes were proposed as interconnection networks by Fu and Chau (1998). We show that these networks are simply wrap-around k-ary butterflies. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Interconnection networks form the backbone of distributed memory parallel architecture. One of the desirable properties of these networks is their symmetry. Symmetry often implies simpler mapping algorithms as well as better fault tolerance. Since symmetry is an inherent property of Cayley graphs [2], there have been many attempts to base new interconnection networks on Cayley graphs. Fixed node degree is also a desirable property of interconnection networks. It lowers hardware costs as well as allows unbounded size architectures using the same processors.

Fu and Chau have recently proposed a class, G_n^k, of Cayley graphs with fixed even node degrees [1]. They call these graphs the Cyclic-cubes. We show in this paper that this class of graphs is the same as the wrap-around k-ary butterflies. This identification will help visualization of these graphs and their mappings. It will also enable porting of results developed for binary butterflies regarding tree mappings [3], search algorithms [4], data selection and broadcast [5], etc.

There are nk^n nodes in G_n^k, each labeled with a cyclic permutation of n symbols $t_0, t_1, \ldots, t_{n-1}$ in that order. In addition, each symbol is attached with a *rank* between 0 and $k-1$ written as the superscript of the symbol. Examples of node labels of G_4^3 are $(t_1^0 t_2^2 t_3^0 t_0^1)$, $(t_2^1 t_3^1 t_0^2 t_1^2)$ and $(t_0^1 t_1^2 t_2^0 t_3^2)$. To specify the edges of the graph, define functions $f^{(i)}, f^{(-i)}: G_n^k \to G_n^k$, $0 \leqslant i < n$ as follows: [1]

$$f^{(i)}\left(t_e^{j_e} \ldots t_{n-1}^{j_{n-1}} t_0^{j_0} t_1^{j_1} \ldots t_{e-1}^{j_{e-1}}\right) = \left(t_{e+1}^{j_{e+1}} \ldots t_{n-1}^{j_{n-1}} t_0^{j_0} t_1^{j_1} \ldots t_e^{j_e+i}\right),$$

$$f^{(-i)}\left(t_e^{j_e} \ldots t_{n-1}^{j_{n-1}} t_0^{j_0} t_1^{j_1} \ldots t_{e-1}^{j_{e-1}}\right) = \left(t_{e-1}^{j_{e-1}-i} \ldots t_{n-1}^{j_{n-1}} t_0^{j_0} t_1^{j_1} \ldots t_{e-2}^{j_{e-2}}\right).$$

(Note that the operations on the rank of a symbol are performed modulo k.) Then $u \in G_n^k$ is connected to

$$f^{(0)}(u), f^{(1)}(u), \ldots, f^{(k-1)}(u), \quad f^{(0)}(u), f^{(-1)}(u), \ldots, f^{(-(k-1)))}(u).$$

[1] In [1] authors use ranks 1 through k and call the functions $f^{(0)}$ and $f^{(-0)}$ as g and g^{-1}, respectively. Our notation simplifies the representation.

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Clearly each of these edges are bidirectional. In particular, if $f^{(i)}(u) = v$ then $f^{(-i)}(v) = u$. Further, for $n \geqslant 3$, all the destinations of u are distinct. Therefore the node degree of G_n^k is $2k$. G_n^2 is identical to the graph defined in [6]. It has been shown in [1] that G_n^k is a Cayley graph and supports Hamilton cycle, and embedding of hypercube and mesh.

2. Isomorphism to the wrap-around k-ary butterfly network

Previously Chen and Lau [7] have shown that G_n^2 defined in [6] is identical to a binary butterfly. We will now prove that the graph G_n^k is the same as the wrap-around k-ary butterfly B_n^k. The definition of the k-ary butterfly network can be found in [8].

Definition 1. The n-dimensional wrap-around k-ary butterfly, $n \geqslant 3$, $k \geqslant 2$, is defined to be a graph with nk^n nodes labeled $\langle l, w\rangle$ where $l \in \mathbb{Z}_n$ and $w \in \mathbb{Z}_k^n$, and having edges [2]

$\langle l, w_0 w_1 \ldots w_{n-1}\rangle \to \langle l-1, w'_0 w'_1 \ldots w'_{n-1}\rangle$,

where $w'_i = w_i$ for $i \neq (l-1)$, and $w'_{l-1} \in \mathbb{Z}_k$,

$\langle l, w_0 w_1 \ldots w_{n-1}\rangle \to \langle l+1, w'_0 w'_1 \ldots w'_{n-1}\rangle$,

where $w'_i = w_i$ for $i \neq l$, and $w'_l \in \mathbb{Z}_k$.

Thus the $2k$ nodes connected to any node $\langle l, w\rangle$ are obtained by either decreasing its first index l by 1 (modulo n) and replacing the $(l-1)$th component, w_{l-1}, of the k-ary representation of w, by an integer between 0 and $k-1$ or by increasing l by 1 (modulo n) and replacing w_l by an integer between 0 and $k-1$.

B_n^k can be visualized as a $k^n \times n$ array of nodes with node $\langle l, w\rangle$ placed in the lth column and the wth row. Clearly, nodes in any column have connections only to the nodes in the neighboring columns (except for the wrap-around links between the $(n-1)$th column and the 0th column). To simplify the drawing the wrap-around links, 0th column is redrawn following the $(n-1)$th column. B_3^3 is shown in Fig. 1.

[2] Here, $\mathbb{Z}_n$ denotes integers 0 through $n-1$ and $\mathbb{Z}_k^n$ denotes n copies of $\mathbb{Z}_k$. Thus $w \in \mathbb{Z}_k^n$ may be expressed as $w_0 w_1 \ldots w_{n-1}$ where $w_i \in \mathbb{Z}_k$. The string $w_0 w_1 \ldots w_{n-1}$ may be thought of as the representation of integer w in radix k system.

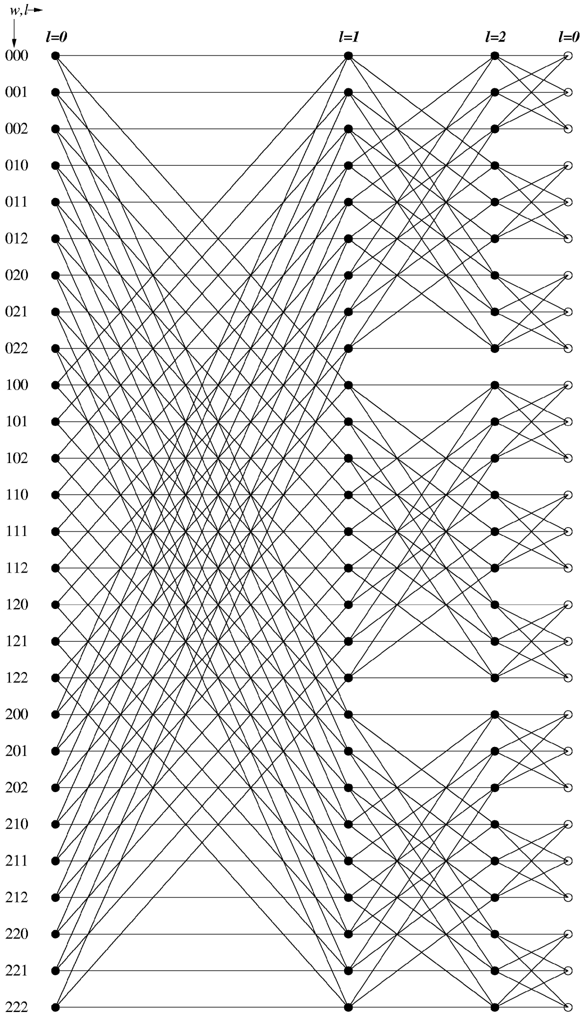

Fig. 1. Wrap-around k-ary butterfly B_3^3.

To demonstrate the isomorphism between G_n^k and B_n^k, define a function $\psi : G_n^k \to B_n^k$ as

$$\psi\left(t_e^{j_e} \ldots t_{n-1}^{j_{n-1}} t_0^{j_0} t_1^{j_1} \ldots t_{e-1}^{j_{e-1}}\right) = \langle e,\ j_0 j_1 \ldots j_{n-1}\rangle. \quad (1)$$

Clearly ψ is one-one and onto. We now show that it also preserves the graph connectivity. Let

$$u = \left(t_e^{j_e} \ldots t_{n-1}^{j_{n-1}} t_0^{j_0} t_1^{j_1} \ldots t_{e-1}^{j_{e-1}}\right).$$

Its image $\psi(u)$ is given in (1). The connectivity in G_n^k shows that u is connected to $f^{(i)}(u)$, $0 \leqslant i \leqslant (n-1)$. Now,

$$\begin{aligned}
&\psi\bigl(f^{(i)}(u)\bigr)\\
&\quad=\psi\bigl(f^{(i)}\bigl(t_e^{j_e}\dots t_{n-1}^{j_{n-1}}t_0^{j_0}t_1^{j_1}\dots t_{e-1}^{j_{e-1}}\bigr)\bigr)\\
&\quad=\psi\bigl(t_{e+1}^{j_{e+1}}\dots t_{n-1}^{j_{n-1}}t_0^{j_0}t_1^{j_1}\dots t_e^{j_e+i}\bigr)\\
&\quad=\langle e+1, j_0 j_1\dots j_{e-1}\ j_e+i\ j_{e+1}\dots j_{n-1}\rangle. \qquad (2)
\end{aligned}$$

By comparing the second indices of $\psi(u)$ in (1) and $\psi(f^{(i)}(u))$ in (2), one can see that all their components match, except the eth component, j_e+i, which takes a value between 0 and $(n-1)$ since the rank operations are always modulo n. Therefore from the definition of the connectivity of B_n^k, $\psi(u)$ and $\psi(f^{(i)}(u))$, $0\leqslant i\leqslant(n-1)$ are connected. Further, since $f^{(-i)}$ is simply the inverse of $f^{(i)}$, $\psi(f^{(-i)}(u))$, and $\psi(u)$, $0\leqslant i\leqslant(n-1)$ are also connected.

This proves that the even degree Cayley graph G_n^k is isomorphic to the wrap-around k-ary butterfly B_n^k.

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